

## CONTINUITY OF K-THEORY: AN EXAMPLE IN EQUAL CHARACTERISTICS

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(Communicated by Thomas Goodwillie)

ABSTRACT. If  $k$  is a perfect field of characteristic  $p > 0$ , we show that the Quillen K-groups  $K_i(k[[t]])$  are uniquely  $p$ -divisible for  $i = 2, 3$ . In fact, the Milnor K-groups  $K_n^M(k((t)))$  are uniquely  $p$ -divisible for all  $n > 1$ . This implies that  $K(A) \rightarrow \operatorname{holim}_{\overline{n}} K(A/\mathfrak{m}^n)$  is 4-connected after profinite completion for  $A$  a complete discrete valuation ring with perfect residue field.

Let  $A$  be a complete discrete valuation ring with maximal ideal  $\mathfrak{m}$ . Let

$$K^{top}(A) = \operatorname{holim}_{\overline{n}} K(A/\mathfrak{m}^n)$$

We say that K-theory is *continuous* (at  $A$ ) if it commutes with the (inverse) limit, in the sense that

$$K(A)^\wedge \rightarrow K^{top}(A)^\wedge$$

is an equivalence, where  $X \rightarrow X^\wedge$  denotes profinite completion.

This question of continuity has acquired new relevance since the fibers of

$$K(A/\mathfrak{m}^n)^\wedge \rightarrow K(A/\mathfrak{m})^\wedge$$

are now better understood, and have been shown by McCarthy [Mc] to agree with the corresponding fibers in topological cyclic homology. Hence we are in a position where we sometimes can calculate  $K^{top}(A)$ .

One situation where we have an affirmative answer is the theorem of Suslin and Panin [Su], [P], which says that if  $A$  is a Henselian discrete valuation ring with maximal ideal  $\mathfrak{m}$ , then

$$K(A)^\wedge_\ell \rightarrow \operatorname{holim}_{\overline{n}} K(A/\mathfrak{m}^n)^\wedge_\ell$$

is an equivalence for all primes  $\ell$  different from the characteristic of (the field of fractions of)  $A$ . So, if  $A$  is of characteristic zero, then K-theory is continuous at  $A$ .

This theorem was used critically in Bökstedt and Madsen's calculation [BM] of the K-theory of the  $p$ -adic integers in order to get the correspondence with topological cyclic homology (here the situation was a bit special, as a similar statement holds for  $TC$ ).

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Received by the editors October 17, 1996.

1991 *Mathematics Subject Classification*. Primary 11S70; Secondary 13J05, 19D45, 19D50.

*Key words and phrases*. Continuity of K-theory, complete discrete valuation ring, ring of formal power series, Milnor K-theory.

The author was supported by the Danish research academy.

The limiting condition in Suslin/Panin's result is that  $\ell$  has to be different from the characteristic of  $A$ . If  $\ell \neq \text{char}(A/\mathfrak{m})$  then a result of Gabber [Ga] tells us furthermore that  $K(A)^\wedge_\ell \rightarrow K(A/\mathfrak{m}^n)^\wedge_\ell \rightarrow K(A/\mathfrak{m})^\wedge_\ell$  are equivalences, and so the situation is really rather degenerate. However, at the characteristic of  $A$ , K-theory is largely unknown, except for some rather old results in dimensions  $\leq 2$ .

In dimension 2, much insight can be deduced from a generators-and-relations presentation of  $K_2(A)$  (e.g. [DS]). It is the hope that a presentation of  $K(A)$  as in [D] can shed light on the general situation. In this paper we choose a different approach, in that we try to study the lower K-groups directly by means of Milnor K-theory, and then use Merkur'ev and Suslin's results [MS] to get information about the K-groups themselves.

We take results showing that  $K(A)^\wedge \rightarrow K^{\text{top}}(A)^\wedge$  is somewhat connected for evidence that  $K(A)$  may be continuous.

In this note we shall prove

**Theorem 1.** *If  $A$  is a complete discrete valuation ring with perfect residue field, then*

$$K(A)^\wedge \rightarrow K^{\text{top}}(A)^\wedge$$

*is 4-connected.*

*Note added in proof* (Aug. 26, 1997). It appears that Thomas Geisser has shown that  $K_i^{\text{ind}}(F)$  is uniquely  $p$ -divisible for any field  $F$  of characteristic  $p > 0$ . Using a slight modification of lemma 6, the observations in this note then show that  $K(A)^\wedge \rightarrow K^{\text{top}}(A)^\wedge$  is an equivalence for the rings in theorem 1.

*Remark 2.* Note that our definition of  $K_i^{\text{top}}(A) = \pi_i K^{\text{top}}(A)$  differs slightly from, e.g., the notion in [W], as we are using the homotopy limit. This is connected to the inverse limit of the homotopy groups by the short exact sequence

$$0 \rightarrow \varprojlim^{(1)} K_{i+1}(A/\mathfrak{m}^n) \rightarrow K_i^{\text{top}}(A) \rightarrow \varprojlim K_i(A/\mathfrak{m}^n) \rightarrow 0.$$

We will be using completion of spectra as in [B]. Note that completions commute with homotopy inverse limits. Recall that if  $X$  is a spectrum, then there is an exact sequence

$$0 \rightarrow \text{Ext}(\mathbf{Z}[1/p]/\mathbf{Z}, \pi_n X) \rightarrow \pi_n(X^\wedge_p) \rightarrow \text{Hom}(\mathbf{Z}[1/p]/\mathbf{Z}, \pi_{n-1} X) \rightarrow 0.$$

*Remark 3.* The reason we have to profinitely complete everything before we ask for an equivalence, is that  $K(A) \rightarrow K^{\text{top}}(A)$  is certainly not an equivalence integrally. For instance, if  $A$  is the  $p$ -adic integers,  $\mathbf{Z}^\wedge_p$ , then  $F = \mathbf{Q}^\wedge_p$  is uncountable, and hence by [M, 11.10]  $K_2(F)$  is uncountable. By the localization sequence

$$0 = K_2(\mathbf{F}_p) \rightarrow K_2(A) \rightarrow K_2(F) \rightarrow K_1(\mathbf{F}_p) \rightarrow 0$$

we get that  $K_2(A)$  is uncountable too. But, as each  $K_3(\mathbf{Z}/p^k\mathbf{Z})$  is a finite group, we get that the  $\varprojlim^{(1)}$  term vanishes, and  $K_2^{\text{top}}(A) \cong \varprojlim K_2(\mathbf{Z}/p^k\mathbf{Z})$ , which by [M, p. 180] and [W] is trivial.

A similar consideration rules out integral continuity for the case of equal characteristics too.

*Proof of theorem 1.* Let  $F$  be the field of fraction of  $A$ , let  $k$  be the residue field, and let  $p$  be the characteristic of  $k$ .  $F$  has characteristic either zero or  $p$ . The characteristic zero part is taken care of by the Suslin/Panin result mentioned above.

The only remaining piece is the case where  $\text{char}(F) = \text{char}(k) = p > 0$ . Then  $A$  must be isomorphic to the ring of formal power series  $k[[t]]$  [Se, II, 4.2]. This is stated separately as proposition 7 below, and the proof will occupy the rest of the paper.  $\square$

First we prove some useful lemmas regarding the  $p$ -divisibility of the groups  $K_i(A)$ . This is important in this context, since we have by [He] that  $\pi_i K^{\text{top}}(A) \hat{=} \hat{=} 0$  for  $i > 1$ .

**Lemma 4.** *Let  $A = k[[t]]$  be the ring of formal power series in a perfect field  $k$  of characteristic  $p > 0$ , and let  $F = k((t)) = k[[t]][t^{-1}]$  be its field of fractions. Let  $n > 1$ . Then  $K_n(A)$  is uniquely  $p$ -divisible if and only if  $K_n(F)$  is.*

*Proof.* As  $k$  is perfect, we get by [K] or [Hi] that  $K_n(k)$  is uniquely  $p$ -divisible for  $n > 0$ . By [Ge, 1.3] the localization sequence breaks up into short exact sequences

$$0 \rightarrow K_n(A) \rightarrow K_n(F) \rightarrow K_{n-1}(k) \rightarrow 0.$$

By hypothesis, multiplication by  $p$  is an isomorphism on two of the three groups, and hence also on the third.  $\square$

For a field  $F$ , note that  $K_1(F)$  is the multiplicative group  $F^* = F - \{0\}$ . Milnor K-theory,  $K^*(F)$ , is defined as the graded ring represented as the quotient of the tensor algebra  $T_*(F^*) = \bigoplus_{n=0}^{\infty} (F^*)^{\otimes n}$  by the homogeneous ideal generated by the elements  $x \otimes (1 - x) \in F^* \otimes F^*$ . The product in K-theory then defines a map of graded rings  $K_*^M(F) \rightarrow K_*(F)$  which is an isomorphism for  $* < 3$ .

**Lemma 5.** *Let  $F = k((t))$ , where  $k$  is a perfect field of characteristic  $p$ . Then  $K_n^M(F)$  is uniquely  $p$ -divisible for  $n > 1$ .*

*Proof.* Since  $F$  is of characteristic  $p$ , the main result of Izboldin [I] gives that  $K_n^M(F)$  has no  $p$ -torsion, so we just have to show that it is  $p$ -divisible. We do this by showing that  $K_2^M(F)$  is  $p$ -divisible. This is enough, for the surjection  $(F^*)^{\otimes n} \rightarrow K_n^M(F)$  factors through  $K_2^M(F) \otimes (F^*)^{\otimes n-2}$ , which is  $p$ -divisible, and so  $K_n^M(F)$  must be  $p$ -divisible for all  $n > 1$ .

Consider the generator  $\{x, y\} \in K_2^M(F)$ , where  $x, y \in F^*$ . We will show that it has a  $p$ th root (cf. [M, A.14]). This is clear if there is a  $z \in F$  such that  $y = z^p$ , for then  $\{x, y\} = \{x, z^p\} = \{x, z\}^p$ .

On the other hand, suppose  $y$  has no  $p$ th root in  $F$ , and consider the inseparable extension  $F \subseteq F(t^{1/p})$  of degree  $p$ . Note that, as  $k$  is perfect, the norm map  $F(t^{1/p})^* \rightarrow F^*$  given by  $z \mapsto z^p$  is surjective ( $at^{-N} \prod_{i=0}^{\infty} (1 - a_i t^i)$  is hit by  $a^{1/p} t^{-\frac{N}{p}} \prod_{i=0}^{\infty} (1 - a_i^{1/p} t^{\frac{i}{p}})$ ). In particular,  $y$  has a  $p$ th root in  $F(t^{1/p})$ . This means that  $F \subseteq F[z]/(z^p - y) \subseteq F(t^{1/p})$ , and since the first inclusion is not an isomorphism, the latter must be, since  $p$  is prime.

So,  $F \subseteq E = F[z]/(z^p - y)$  is a field extension of degree  $p$ , and  $x$  is in the image of the norm map, and hence [M, 14.3] applies to show that  $\{x, y\}$  has a  $p$ th root in  $K_2(F)$ .  $\square$

**Lemma 6.** *If  $A = k[[t]]$ , where  $k$  is perfect of characteristic  $p > 0$ , then  $K_2(A)$  and  $K_3(A)$  are uniquely  $p$ -divisible.*

*Proof.* We have to show that  $K_2(F)$  and  $K_3(F)$  are uniquely  $p$ -divisible for  $F = k((t))$ . For  $i = 2$ , note that  $K_2^M(F) \cong K_2(F)$  and use the two foregoing lemmas.

Consider the map  $K_3^M(F) \rightarrow K_3(F)$ . Call the kernel  $K$  and the cokernel  $K_3^{\text{ind}}(F)$ , and we are done if both groups turn out to be uniquely  $p$ -divisible. By Merkur'ev and Suslin [MS],  $K_3^{\text{ind}}(F)$  is uniquely  $p$ -divisible. Also,  $K$  is annihilated by  $(3-1)! = 2$ . If  $p = 2$ , the unique 2-divisibility of  $K_3^M(F)$  implies that  $K = 0$ , and if  $p \neq 2$  then  $p = 2i + 1$  acts as the identity on  $K$ . Anyhow,  $K$  is uniquely  $p$ -divisible too.  $\square$

**Proposition 7.** *Let  $k$  be a perfect field. Then*

$$K(k[[t]]) \widehat{\rightarrow} K^{\text{top}}(k[[t]]) \widehat{\rightarrow}$$

*is 4-connected.*

*Proof.* Let  $p$  be the characteristic of  $k$ . First note that  $K_i(k[[t]]) \rightarrow K_i^{\text{top}}(k[[t]])$  is an isomorphism if  $i < 2$ , and the groups are without  $p$ -torsion. If  $\ell$  is a prime different from  $p$ , then  $\pi_i K(k[[t]]) \widehat{\ell} \cong \pi_i K(k[[t]]/t^n) \widehat{\ell} \cong \pi_i K(k) \widehat{\ell}$  by [Ga], and so

$$K(k[[t]]) \widehat{\ell} \xrightarrow{\cong} K^{\text{top}}(k[[t]]) \widehat{\ell} \xrightarrow{\cong} K(k) \widehat{\ell}.$$

By Hesselholt [He],  $K^{\text{top}}(k[[t]]) \widehat{p}$  has vanishing homotopy groups in dimension greater than 1. This is actually only stated for finite fields, but the proof works equally well for perfect fields.

If  $G$  is an Abelian group, we have by [BK, VI] that

$$\text{Ext}(\mathbf{Z}[1/p]/\mathbf{Z}, G) = 0$$

if and only if  $G$  is  $p$ -divisible, and

$$\text{Hom}(\mathbf{Z}[1/p]/\mathbf{Z}, G) = 0$$

if the  $p$ -torsion elements in  $G$  are of bounded order.

So the proposition follows from the exact sequence

$$0 \rightarrow \text{Ext}(\mathbf{Z}[1/p]/\mathbf{Z}, K_i(A)) \rightarrow \pi_i(K(A) \widehat{p}) \rightarrow \text{Hom}(\mathbf{Z}[1/p]/\mathbf{Z}, K_{i-1}(A)) \rightarrow 0,$$

the unique  $p$ -divisibility of  $K_i(k[[t]])$  for  $i = 2, 3$ , and the lack of  $p$ -torsion in  $K_1(k[[t]])$ .  $\square$

#### ACKNOWLEDGMENTS

It is a pleasure to acknowledge helpful conversations with Steffen Bentzen. This note was written during my stay at Aarhus University, and I am grateful to Ib Madsen for inviting me.

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