

THE PRIMITIVE p -FROBENIUS GROUPS

P. FLEISCHMANN, W. LEMPKEN, AND PHAM HUU TIEP

(Communicated by Ronald M. Solomon)

ABSTRACT. Let p be a fixed prime. A finite primitive permutation group G with every two-point stabilizer $G_{\alpha,\beta}$ being a p -group is called a *primitive p -Frobenius group*. Using our earlier results on *p -intersection subgroups*, we give a complete classification of the primitive p -Frobenius groups.

1. INTRODUCTION

Let Σ_Ω denote the symmetric group on a finite set Ω and $G \leq \Sigma_\Omega$ a transitive permutation group. Suppose that every two-point stabilizer $G_{\alpha,\beta}$ is trivial; then a classical result of Frobenius states that G is a semidirect product $K : G_\alpha$, where the normal subgroup K consists precisely of the identity together with all elements $g \in G$ that do not fix any point of Ω . Moreover the point stabilizer G_α acts on K (via conjugation) in such a way that no nontrivial element $g \in G_\alpha$ has a nontrivial fixed point in K . A permutation group with these properties is called a *Frobenius group* with *Frobenius kernel* K .

As a consequence of a celebrated theorem of J. G. Thompson, the Frobenius kernel K is nilpotent and G_α is a *semiregular group*, i.e. G_α has a faithful action on a suitable vector space V such that $C_V(g) = 0$ for every $g \in G_\alpha \setminus \{id\}$ (notice that V can be chosen as a characteristically simple subgroup of a Sylow group of K). If moreover G is primitive, then $K \cong \mathbb{Z}_p^\ell =: V$ and G_α acts semiregularly on V .

In the course of his investigation of finite near fields, Zassenhaus [9] obtained a complete classification of finite semiregular groups. Putting all these results together, one has a complete picture of primitive Frobenius groups.

In this paper we finish our investigation of the following p -local variant of the situation above:

Definition 1.1. Let p be a fixed prime. We define $\mathcal{F}(p)$ to be the set of all (faithful) finite primitive permutation groups with every 2-point stabilizer $G_{\alpha,\beta}$ being a p -group. An element of $\mathcal{F}(p)$ will be called a *primitive p -Frobenius group*.

Furthermore, $\mathcal{F}^a(p)$ denotes the subset of $\mathcal{F}(p)$ consisting of those groups with abelian socle, and we set $\mathcal{F}^{na}(p) := \mathcal{F}(p) \setminus \mathcal{F}^a(p)$.

In [3], we defined a proper subgroup $X < G$ to be a *p -intersection subgroup* of G if and only if X is not a p -group but for any $g \in G \setminus X$ the intersection $X \cap X^g$ is a p -group. The set of p -intersection subgroups of G will be denoted by $\mathcal{I}_p(G)$. The

Received by the editors June 19, 1996 and, in revised form, November 5, 1996.
1991 *Mathematics Subject Classification*. Primary 20B15.

main result of [3] is a classification of the elements in $\mathcal{I}_p(G)$ for all almost simple groups. Now we will apply this result to obtain a complete classification of the primitive p -Frobenius groups.

Remarks. (i) It is clear that $\mathcal{F}(p)$ includes the class \mathcal{F}^∞ of all finite primitive Frobenius groups, which is the intersection $\bigcap_{p \text{ prime}} \mathcal{F}(p)$.

(ii) Every finite group G with $O_p(G) = 1$ admits a faithful transitive permutation representation with every $G_{\alpha,\beta}$ being a p -group: take the action of G on G/P , $P \in \text{Syl}_p(G)$. Thus it is the primitivity condition that makes the class $\mathcal{F}(p)$ interesting.

(iii) It is immediate from the definition, that $G \in \mathcal{F}(p)$ implies that either G_α is a p -group or $G_\alpha \in \mathcal{I}_p(G)$.

Our notation will be as follows: Ω always denotes a finite set and G a finite group with $\pi(G)$ the set of prime divisors of $|G|$. For $\alpha \in \Omega$ and $G \leq \Sigma_\Omega$ we write G_α for $\text{Stab}_G(\alpha)$. For any subgroup $S \leq G$ we set $\text{Aut}_G(S) := N_G(S)/C_G(S)$. Also, $X < \cdot Y$ means that X is a maximal subgroup of a group Y . The socle of G , $\text{soc}(G)$, is the product of all minimal normal subgroups of G . A subgroup $G \leq \Sigma_\Omega$ is called *primitive* (resp. *regular*) if it is transitive and $G_\alpha < \cdot G$ (resp. $G_\alpha = 1$). The symmetric, resp. alternating, group on n symbols is denoted by Σ_n , resp. \mathcal{A}_n .

Observe that if $G \in \mathcal{F}^a(p)$ then $\text{soc}(G) = V = \mathbb{F}_\ell^n$ is elementary abelian (the prime ℓ may differ from p) and G is the semi-direct product of V and any point stabilizer G_α , $a \in \Omega$. In this case, one can identify Ω with V , $a \in \Omega$ with the zero vector of V , then embed G_0 in $GL(V)$ so that the action of $G_\alpha = G_0$ on Ω and the linear action of G_0 on V are compatible.

Let us recall the following definition of [2]: A pair (G, V) consisting of a finite group G and a finite-dimensional $\mathbb{F}G$ -module V over some field \mathbb{F} is called *p' -semiregular* if every nontrivial p' -element of G acts without any fixed points on $V \setminus \{0\}$. G is called *p' -semiregular* if (G, V) is p' -semiregular for a suitable V .

Now the following statement is immediate:

Proposition 1.2. *Suppose $\text{soc}(G) = V$ is elementary abelian. Then $G \in \mathcal{F}^a(p)$ if and only if the pair (G_0, V) is p' -semiregular, and V is a faithful irreducible G_0 -module. \circ*

In [2] all p' -semiregular pairs (G_0, V) have been determined.

The present paper provides a proof of the following complete classification of the groups in $\mathcal{F}(p)$:

Theorem 1.3. *Let G be an element of $\mathcal{F}(p)$ and put $S := \text{soc}(G)$. Then precisely one of the following three cases occurs:*

(i) *G has a regular normal subgroup V . In this case $V = S$ is elementary abelian, $G \in \mathcal{F}^a(p)$ and $G = V : G_0$ with p' -semiregular pair (G_0, V) as described in [2] (and V is a faithful irreducible G_0 -module).*

(ii) *G has no regular normal subgroup and G_α is nilpotent. In this case $p = 2$, $G \in \mathcal{F}^{na}(2)$, $G_\alpha \in \text{Syl}_2(G)$ and $S = O^2(G) = S_1 \times \dots \times S_k$ is the unique minimal normal subgroup of G ; moreover, $S_1 \cong \dots \cong S_k \cong L_2(q)$ with $q = 2^n \pm 1 > 5$ a prime or $q = 9$. Furthermore, $[G : S]$ and k are powers of 2.*

(iii) *G has no regular normal subgroup and a point stabilizer G_α is an element of $\mathcal{I}_p(G)$. In this case S is simple, i.e. G is almost simple. Furthermore, the tuples (S, G, p, G_α) are as listed in Table I below.*

TABLE I. p -intersection maximal subgroups in almost simple groups

S	G	p	G_α
\mathcal{A}_5	$G = S$	3	\mathcal{A}_4
\mathcal{A}_5	$\mathcal{A}_5 \leq G \leq \Sigma_5$	2	$N_G(\mathbb{Z}_3)$
\mathcal{A}_6	$\mathcal{A}_6 \leq G \leq \text{Aut}(\mathcal{A}_6)$	2	$N_G(3^2)$
\mathcal{A}_6	$PGL_2(9), M_{10}, \text{Aut}(\mathcal{A}_6)$	2	$N_G(\mathbb{Z}_5)$
\mathcal{A}_n, n prime $\notin \{7, 11, 17, 23\}$	$G = S$	p with $\frac{n-1}{2} = p^f$	$\mathbb{Z}_n : \mathbb{Z}_{\frac{n-1}{2}}$
$\mathcal{A}_n, 5 \leq n \in \mathcal{F}$	Σ_n	2	$\mathbb{Z}_n : \mathbb{Z}_{n-1}$
$L_2(7) \cong L_3(2)$	$G = S$ $G = PGL_2(7)$	2 2	$\Sigma_4^{(i)}, i = 1, 2$ $\mathcal{N}(X), \mathcal{N}(T_1) \cong D_{12}$
$L_2(7)$	$G = S$	3	B
$L_2(11)$	$G = S$ $PGL_2(11)$	2 2	$N(T_{\text{cox}})$ $N(T_1) \cong D_{20}$ $N(T_{\text{cox}}) \cong D_{24}$
$L_2(11)$	$G = S$	5	B
$L_2(2^a), 2^a \geq 4$	$G = S$	$p = 2^a - 1 \in \mathcal{M}$	B
$L_2(q), q \notin \mathcal{F} \cup \{4, 7, 9, 11\}$	$\pi(G/S) \subseteq \{2\}$	2	$N(T_1)$
$L_2(q), q \notin \mathcal{M} \cup \{4, 5, 9, 11\}$	$\pi(G/S) \subseteq \{2\}$	2	$N(T_{\text{cox}})$
$L_2(2^\ell)$	$2 < G/S = \ell$ prime	2	$N(T_{\text{cox}})$
$L_2(r), r = 2p^x + 1$ a prime	$G = S$	$p > 2$	B
$L_2(3^m), \begin{cases} 3^m = 2p^x + 1 \\ m (p-1) \\ m \text{ an odd prime} \end{cases}$	$G = S$	$p > 2$	B
$L_2(r), r \in \mathcal{F}$	$G/S \leq \mathbb{Z}_2$	2	B
$PSU_3(q), q \in \{3, 5, 9\}$	$G/S \leq \mathbb{Z}_2$	2	B
${}^2G_2(3)' \cong L_2(8)$	$G/S \leq \mathbb{Z}_3$	2	$B, N_G(B)$
${}^2B_2(q), q = 2^l, l = 2a + 1 > 1$	$G = S$ $G = S$ $ G/S = l$ prime	$p = 2^l - 1 \in \mathcal{M}$ 2 2	B $D_{2(q-1)}, X_\pm = \mathbb{Z}_{q \pm 2\sqrt{q}+1} : \mathbb{Z}_4$ $X_\pm : \mathbb{Z}_l$ with $5 \parallel X_\pm $
$L_3(4)$	$G/S \leq F \cong \mathbb{Z}_2$	2	$N(\mathbb{Z}_3^2 : Q_8)$
$L_3(4)$	$\Delta \leq G/S \leq \mathbb{Z}_2 \times \mathbb{Z}_2$	2	$N(\mathbb{Z}_3^2 : Q_8)$
${}^3D_4(q)$	$\pi(G/S) \subseteq \{2\}$	2	$N(T_5), T_5 = q^4 - q^2 + 1$
$L_p(q), PSU_p(q)$	$\pi(G/S) \subseteq \{p\}$	p	$N(T_{\text{cox}})$
M_{11}	$G = S$	2	$3^2 : Q_8 \cdot 2$
J_1	$G = S$	2	$\Sigma_3 \times D_{10}$
M_{23}	$G = S$	11	$23 : 11$
BM	$G = S$	23	$47 : 23$
$M(??)$	$G = S$	29	$59 : 29$

Conversely, each of the groups mentioned in (i), (ii) and (iii) is a member of $\mathcal{F}(p)$, where in case (ii) we have to assume in addition that $\text{Aut}_G(S_i) \cong PGL_2(7)$ if $q = 7$ and $\text{Aut}_G(S_i) \cong PGL_2(9), M_{10}$ or $\text{Aut}(\mathcal{A}_6)$ if $q = 9$.

In Table I B denotes a Borel subgroup; T_1, T_{cox} denote split- and Coxeter tori, respectively, $L_2(q) := PSL_2(q), N(H) := N_G(H)$; if $G > S, \mathcal{N}(X) := \{N_G(X) \mid X \in \mathcal{I}_p(S), \text{ maximal in } S\}$. Here ?? means either $59 : 29 < L_2(59) < \cdot M$

or $59 : 29 < \cdot M$. The existence of $L_2(59)$ in M is not settled yet. \mathcal{F} , resp. \mathcal{M} , denotes the set of Fermat, resp. Mersenne, primes.

Due to the isomorphisms $\mathcal{A}_5 \cong L_2(4) \cong L_2(5)$ and $\mathcal{A}_6 \cong L_2(9)$ these groups and their automorphic decorations are listed only in the alternating groups' section. For $L_3(4)$, F (resp. $\Delta \cong \mathbb{Z}_2$) is generated by field (resp. graph) automorphisms.

Note that for any $X \in \mathcal{I}_p(G)$ there is a subgroup $H \leq G$ such that $X \in \mathcal{I}_p(H)$ and X is maximal in H . Hence the results above give also a complete classification of p -intersection subgroups occurring in arbitrary finite groups. In particular we obtain the following result which can be viewed as the p -local version of Zassenhaus' classification of Frobenius complements:

Theorem 1.4. *Let G be a finite group, p a prime and $X < G$ a p -intersection subgroup. Then either X is p' -semiregular or X is a solvable group (occurring as G_α in Table I up to a suitable normal p -subgroup of X).*

2. PREREQUISITES

The reader is referred to [2] for a complete list of p' -semiregular groups; here we restrict ourselves to listing the perfect ones. \mathcal{R} is the set of all primes r such that $r = 2^a \cdot 3^b + 1$ for $a \geq 2, b \geq 0$, and $(r + 1)/2$ is a prime.

R. Guralnick and R. Wiegand [5] also classified p' -semiregular groups G such that the underlying field of the corresponding G -modules has characteristic p , and pointed out a very interesting connection of these groups with multiplicative structures of Galois field extensions. The authors' proof in [2] is independent of that in [5] and in addition also describes the relevant G -modules.

Theorem 2.1. *Let G be a perfect finite group and (G, V) a p' -semiregular pair for a faithful irreducible $\mathbb{F}G$ -module V . Then one of the following holds:*

- (i) $G \cong SL_2(p^a)$ for some $a \geq 1$ with $p^a > 3$.
- (ii) $G \cong {}^2B_2(2^{2a+1})$ for some $a \geq 1$ with $p = 2$.
- (iii) $G \cong {}^2B_2(2^{2a+1}) \times SL_2(2^{2b+1})$ with $a, b \geq 1, \gcd(2a + 1, 2b + 1) = 1$ and $p = 2$.
- (iv) $G \cong SL_2(r)$ with $r \in \mathcal{R} \cup \{7, 17\}$ and $p = 3$.
- (v) $G \cong SL_2(5)$ and $p \geq 7$.
- (vi) $G = ES$, where $E = O_2(G) \cong 2_-^{1+4}, S \cong SL_2(5), E \cap S = Z(G) \cong \mathbb{Z}_2$ and $p = 2$.

Conversely, if (G, p) satisfies any of the conditions (i) – (vi), then there exists a faithful absolutely irreducible G -module V such that (G, V) is p' -semiregular.

Proof. Let (G, V) be p' -semiregular for a faithful irreducible $\mathbb{F}G$ -module V and $\text{char } \mathbb{F} = \ell$. First suppose that ℓ divides $|G|$. Then it is clear that $\ell = p$ and the irreducibility of V forces $O_p(G) = 1$. By Theorem 4.1 of [2] (cf. also [5]), (G, p) is as listed in (i) – (iv). Next suppose that ℓ does not divide $|G|$. Then Theorem 5.6 of [2] and the irreducibility of V force (G, p) to satisfy one of the conditions (iv) – (vi) or (i) with $p^a = 4, 5, 9$. The existence of p' -semiregular pairs for the groups G listed has also been established in [2]. ◻

Example 2.2. (i) (Zassenhaus) $\ell^2 : SL_2(5) \in \mathcal{F}^\infty$ for all primes $\ell \equiv \pm 1 \pmod{10}$ and $\ell^4 : SL_2(5) \in \mathcal{F}^\infty$ for all primes $\ell \equiv \pm 3 \pmod{10}$.

- (ii) $p^{2a} : SL_2(p^a) \in \mathcal{F}(p)$ for any prime p .
- (iii) $3^6 : SL_2(13) \in \mathcal{F}(3)$ (Hering's group).
- (iv) $7^8 : (2_-^{1+4} \setminus \mathcal{A}_5) \in \mathcal{F}(2)$ and $7^4 : SL_2(9), 5^{12} : SL_2(13) \in \mathcal{F}(3)$.

The following result will be used in the next section.

Theorem 2.3. *Let G be a finite group with a nilpotent maximal subgroup H .*

(i) (Thompson; see [4], Thm. 10.3.2) *If H has odd order then G is solvable.*

(ii) (Baumann; see [1]) *If G is non-solvable then $O^2(G/F(G))$ is a direct product of simple groups isomorphic to $L_2(q)$ with primes q of the form $2^n \pm 1$ or $q = 9$.*

We will also need the following result, which is an easy consequence of the classification of finite simple groups:

Lemma 2.4. *Let \mathcal{E} be a nonabelian finite simple group and let $\alpha \in \text{Aut}(\mathcal{E})$ be an element whose order is coprime to $|\mathcal{E}|$. Then $C_{\mathcal{E}}(\alpha)$ is not nilpotent.*

Proof. See [3]. \circ

3. REDUCTION TO THE SIMPLE SOCLE CASE

Proposition 3.1. *Let $G \in \mathcal{F}^{na}(p)$. Then G does not contain any regular normal subgroup. In particular, $S := \text{soc}(G)$ is the unique minimal normal subgroup of G , $C_G(S) = 1$, and $G/S \cong G_\alpha/S_\alpha$.*

Moreover, for any point stabilizer G_α one of the following is true:

(i) $p = 2$ and $G_\alpha \in \text{Syl}_2(G)$;

(ii) $G_\alpha \in \mathcal{I}_p(G)$.

Proof. 1) Suppose first that G_α is a p -group. As G_α is maximal and nilpotent, 2.3 implies $p = 2$ and so $G_\alpha \in \text{Syl}_2(G)$.

2) Suppose next that $1 \neq R \triangleleft G$ with $R \cap G_\alpha = 1$. We can assume $R \leq \text{soc}(G)$ and R is not solvable (since $G \in \mathcal{F}^{na}(p)$). Suppose in addition that G_α is not a p -group. Then we can find $x \in G_\alpha$ of prime order $q \neq p$. Now for any $1 \neq y \in R$, $G_\alpha \cap G_{y(\alpha)}$ is a p -group. In particular, $x^y \notin G_\alpha$, so $y^x \neq y$ and x acts fixed-point-freely on $R \setminus \{1\}$. By a well-known theorem of Thompson (cf. [4], Thm. 10.2.1) R must be nilpotent, a contradiction. Thus G_α must be a p -group. By 1), $G_\alpha \in \text{Syl}_2(G)$, and $|R| = [G : G_\alpha]$ is odd. So R (and $G = R \cdot G_\alpha$) is solvable, again a contradiction.

3) The claims concerning S now follow immediately. Furthermore, if G_α is a p -group, then (i) is fulfilled due to 1); otherwise one arrives at (ii). \circ

Corollary 3.2. *Let $G \in \mathcal{F}(p)$. Then $G \in \mathcal{F}^a(p)$ if and only if G contains a regular normal subgroup. \circ*

Proposition 3.3. *Suppose that $G \in \mathcal{F}^{na}(p)$ and $\text{soc}(G)$ is not simple. Then conclusion (ii) of Theorem 1.3 holds.*

Proof. A basic tool for studying finite permutation groups is the reduction theorem first stated by O’Nan and Scott (see [8]). Here we are using an expanded version of this theorem given in [7]. Because of 3.2, the primitive permutation group G under question has no regular normal subgroups (and $\text{soc}(G)$ is not simple). In this case, the O’Nan-Scott theorem says that G is either a *simple diagonal action* or a *product action* group; cf. [7]. We shall use the notation given there. In particular, $B = \text{soc}(G) = S_1 \times \dots \times S_k$ with $S_1 \cong \dots \cong S_k \cong T$ for a non-abelian finite simple group T .

1) Suppose G is a simple diagonal action group, that is, case III(a) of [7] occurs. Then for some $\alpha \in \Omega$ one has $B_\alpha = \{(a, a, \dots, a) \mid a \in T\}$. In particular, taking $g := (a, a, \dots, a)$, $h := (a, 1, \dots, 1)$ for a non-identity p' -element $a \in T$, one sees

that $g = g^h \in B_\alpha \cap B_{h(\alpha)}$. Meanwhile $h \notin B_\alpha$, contrary to the assumption that $G \in \mathcal{F}(p)$.

We have shown that G is a product action group, i.e., case III(b) of [7] occurs. In this case, Ω can be identified with Γ^ℓ for some finite set Γ and some ℓ dividing k . Furthermore, if H denotes $Aut_G(S_1 \times S_2 \times \dots \times S_{k/\ell})$ (after suitably reindexing the S_i 's if necessary), then H acts primitively on Γ , with socle K isomorphic to $T^{k/\ell}$. Moreover, H is of type II or III(a) (in the notation of [7]). Finally, G can be embedded in $W = H \wr \Sigma_\ell$, and the action of G on Ω is induced by the natural product action of W on Ω (cf. [7]).

2) Next we consider the situation when H is of type III(a). Then $\ell < k$. Arguing as in 1), one sees that $K_\gamma \cap K_{\gamma'}$ is not a p -group for some $\gamma, \gamma' \in \Gamma, \gamma \neq \gamma'$. For the distinct points $\alpha := (\gamma, \dots, \gamma), \alpha' := (\gamma', \dots, \gamma')$ in Ω , one has $B_\alpha = (K_\gamma)^\ell, B_{\alpha'} = (K_{\gamma'})^\ell$. In particular, $B_\alpha \cap B_{\alpha'}$ is not a p -group, a contradiction.

Thus H must be of type II, i.e., $k = \ell$.

3) At this point we show that H_γ is a p -group.

First observe that B_α is a p -group, with $\alpha = (\gamma, \dots, \gamma)$. Indeed, suppose $s = (s_1, \dots, s_k) \in B_\alpha$ is not a p -element. Without loss we may suppose $s_1 \in T$ is not a p -element. Then $s = s^t \in B_\alpha \cap B_{t(\alpha)}$ with $t := (s_1, 1, \dots, 1)$. This implies that $t \in B_\alpha$. Now taking $u := (1, g_2, \dots, g_k)$, one has $t = t^u \in B_\alpha \cap B_{u(\alpha)}$ for any $g_i \in T$. We conclude that all $(1, g_2, \dots, g_k)$ are contained in B_α , contradicting the equality $B_\alpha = (K_\gamma)^k$.

Suppose there is an element $x \in H_\gamma$ whose order is a prime r different from p . Every element in W can be canonically written in the form $(g_1, \dots, g_k)\pi$ for $g_i \in \Sigma_\Gamma$ and $\pi \in \Sigma_k$. Then G_α contains an element $g = (x, g_2, \dots, g_k)\pi$ with $\pi(1) = 1$. For any $y \in C_K(x)$ and $h := (y, 1, \dots, 1)$, we have $g = g^h \in G_\alpha \cap G_{h(\alpha)}$, yielding $h \in B_\alpha$. Due to the above observation $C_K(x)$ is a p -group. Choose $Q \in Syl_r(H)$ with $x \in Q$. Then $Q \cap K \triangleleft Q$ and $Z(Q) \cap (Q \cap K) \leq C_K(x)$. So $1 = Z(Q) \cap (Q \cap K)$, which implies that $1 = Q \cap K$. But $Q \cap K \in Syl_r(K)$, hence $(r, |K|) = 1$. Now 2.4 applied to K and x provides a contradiction.

4) We have proved that H_γ is a p -group. If p is odd, then the maximality of H_γ in H together with 2.3 implies that H is solvable, a contradiction. So we conclude that $p = 2$. We claim that G_α is a 2-group. For, suppose that $g = (g_1, \dots, g_k)\pi \in G_\alpha$ has order r , an odd prime, and $\alpha = (\gamma, \dots, \gamma)$. Observe that $G_\alpha \leq W_\alpha = H_\gamma \wr \Sigma_k$. Since $g_i \in H_\gamma$ has order a power of 2, we conclude that π has order r . In particular, we may suppose that π permutes the groups S_1, \dots, S_r cyclically. Choose $c \in K \setminus K_\gamma$. Then for $y := (c, 1, \dots, 1) \in B$ we have $[y^{g^i}, y^{g^j}] = 1$ for all $i, j = 1, 2, \dots, r$. Hence $\tilde{y} := yy^g y^{g^2} \dots y^{g^{r-1}} \in C_B(g)$. From this it follows that $g = g^{\tilde{y}} \in G_\alpha \cap G_{\tilde{y}(\alpha)}$, yielding $\tilde{y} \in B_\alpha$. But in this case c belongs to K_γ , contrary to the choice of c . Consequently, G_α is a 2-group, and so $G_\alpha \in Syl_2(G)$.

Applying 2.3 to the maximal subgroup G_α of G , we come to conclusion (ii) of Theorem 1.3. \circ

The following is one of the main results in [3]; here it classifies the elements in $\mathcal{F}^{na}(p)$ with simple socle:

Theorem 3.4. *Let $S \trianglelefteq G \leq Aut(S)$ with S a finite nonabelian simple group, and suppose that $X \in \mathcal{I}_p(G)$ is maximal in G . Then $(S, G, p, X = G_\alpha)$ is one of the tuples listed in Table I.*

Proof. See [3]. \circ

4. PROOF OF THEOREM 1.3

By 3.2 and the remarks in the introduction we can assume that $G \in \mathcal{F}^{na}(p)$.

First we suppose that G_α is nilpotent. Applying 2.3, we see that $F(G) = 1$ (as $G \in \mathcal{F}^{na}(p)$) and $S := \text{soc}(G) = O^2(G)$ is the direct product of k isomorphic (since S is characteristically simple) simple groups $S_1 \cong \dots \cong S_k \cong L_2(q)$ with $q = 2^n \pm 1$ a prime or $q = 9$. If $k = 1$, then a direct inspection of maximal subgroups of G with $L_2(q) \leq G \leq \text{Aut}(L_2(q))$ shows that $p = 2$ and $G_\alpha \in \text{Syl}_2(G)$, that is, conclusion (ii) of 1.3 holds. If $k > 1$, then, applying 3.3, one again obtains that $p = 2$ and $G_\alpha \in \text{Syl}_2(G)$. Moreover, due to 3.1, $G/S \cong G_\alpha/S_\alpha$ and hence $[G : S]$ is a 2-power. But G/S acts transitively on the set $\{S_1, \dots, S_k\}$; therefore k is a 2-power. Thus (ii) is fulfilled.

Now suppose that (ii) does not hold. Then $G_\alpha \in \mathcal{I}_p(G)$. In this case 3.3 shows that $\text{soc}(G)$ is simple. Applying 3.4, we arrive at (iii). \circ

ACKNOWLEDGEMENTS

The authors are grateful to Prof. R. Solomon for helpful discussions, and to the referee for his valuable comments that greatly improved the exposition of the paper.

REFERENCES

- [1] B. Baumann, Endliche nichtauflösbare Gruppen mit einer nilpotenten maximalen Untergruppe, *J. Algebra* **38** (1976), 119 – 135. MR **53**:13391
- [2] P. Fleischmann, W. Lempken and P.H. Tiep, Finite p' -semiregular groups, *J. Algebra* **188** (1997), 547 – 579. CMP 97:08
- [3] P. Fleischmann, W. Lempken and P.H. Tiep, The p -intersection subgroups in quasi-simple and almost simple finite groups, submitted.
- [4] D. Gorenstein, “*Finite Groups*”, Harper and Row, 1968. MR **38**:229
- [5] R. Guralnick and R. Wiegand, Galois groups and the multiplicative structure of field extensions, *Trans. Amer. Math. Soc.* **331** (1992), 563 – 584. MR **93c**:12004
- [6] B. Huppert, “*Endliche Gruppen*”, Springer, 1967. MR **37**:302
- [7] M. Liebeck, C. Praeger and J. Saxl, On the O’Nan-Scott theorem for finite primitive permutation groups, *J. Austral. Math. Soc. (A)* **44** (1988), 389 – 396. MR **89a**:20002
- [8] L. L. Scott, Representations in characteristic p , *Santa Cruz Conference on Finite Groups*, pp. 318 – 331, Proc. Sympos. Pure Math., vol. **37**, Amer. Math. Soc., Providence, R.I., 1980. MR **82e**:20052
- [9] H. Zassenhaus, Über endliche Fastkörper, *Abhandl. Math. Sem. Hamburg. Univ.* **11** (1936), 187-220.

(P. Fleischmann and W. Lempken) INSTITUTE FOR EXPERIMENTAL MATHEMATICS, UNIVERSITY OF ESSEN, ELLERNSTR. 29, 45326 ESSEN, GERMANY

(Pham Huu Tiep) DEPARTMENT OF MATHEMATICS, OHIO STATE UNIVERSITY, COLUMBUS, OHIO 43210

E-mail address: tiep@math.ohio-state.edu