

ON THE REDUCIBILITY
OF LINEAR DIFFERENTIAL EQUATIONS
WITH QUASIPERIODIC COEFFICIENTS
WHICH ARE DEGENERATE

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ABSTRACT. This paper proves the reducibility of a class of linear differential equations with quasiperiodic coefficients which are degenerate with respect to a small perturbation parameter. Our results generalize some that were obtained by Jorba and Simó.

1. INTRODUCTION AND MAIN RESULTS

Notation and definitions. The function $f(t)$ is called a quasiperiodic function of t with frequencies $\omega_1, \omega_2, \dots, \omega_r$, if there is a function $F(\theta_1, \theta_2, \dots, \theta_r)$, which is 2π -periodic in all its arguments θ_i ($i = 1, 2, \dots, r$), such that $f(t) = F(\omega_1 t, \omega_2 t, \dots, \omega_r t)$. If $F(\theta) = F(\theta_1, \theta_2, \dots, \theta_r)$ ($\theta = (\theta_1, \theta_2, \dots, \theta_r)$) is analytic on a strip $D_\rho = \{\theta \in C^r \mid |\operatorname{Im} \theta_j| \leq \rho, j = 1, 2, \dots, r\}$, we say that $f(t)$ is analytic quasiperiodic in D_ρ . Denote the sup-norm of f on D_ρ by $\|f\|_\rho = \sup_{\theta \in D_\rho} |F(\theta)|$.

An $n \times n$ matrix $Q(t) = (q_{ij}(t))_{1 \leq i, j \leq n}$ is called analytic quasiperiodic on D_ρ with frequencies $\omega_1, \omega_2, \dots, \omega_r$ if q_{ij} ($i, j = 1, 2, \dots, n$) are all analytic quasiperiodic on D_ρ with the frequencies $\omega_1, \omega_2, \dots, \omega_r$. Define a matrix norm of Q by $\|Q\|_\rho = n \times \max_{1 \leq i, j \leq n} \|q_{ij}\|_\rho$. It is easy to see that $\|Q_1 Q_2\|_\rho \leq \|Q_1\|_\rho \cdot \|Q_2\|_\rho$. If Q is a constant matrix, set $\|Q\| = \|Q\|_\rho$ for simplicity. Write the average of $Q(t)$ as $\bar{Q} = (\bar{q}_{ij})_{1 \leq i, j \leq n}$, where $\bar{q}_{ij} = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T q_{ij}(t) dt$; for the existence of the limit, see [1]. Let $A(t)$ be an $n \times n$ quasiperiodic matrix. The differential equations $\dot{x} = A(t)x$, $x \in R^n$, are called reducible if there exists a nonsingular quasiperiodic change of variables $x = \Phi(t)y$ such that $\Phi(t)$ and $\Phi^{-1}(t)$ are quasiperiodic and bounded, and such that it changes the equations to $\dot{y} = By$, where B is a constant matrix.

Problems. In this paper we consider the reducibility of the following linear differential equations:

$$(1.1) \quad \dot{x} = (A + \epsilon Q(t))x, \quad x \in R^n,$$

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where A is an $n \times n$ constant matrix with different eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, $Q(t)$ is an $n \times n$ quasiperiodic matrix of time t with frequencies $\omega_1, \omega_2, \dots, \omega_r$, and ϵ is a small perturbation parameter.

This problem was considered by Jorba and Simó in [2]. They proved that if

$$(1.2) \quad |\langle \omega, k \rangle \sqrt{-1} + \lambda_i - \lambda_j| \geq \frac{\alpha}{|k|^\tau}, \forall 0 \neq k \in Z^r, \forall i, j = 1, 2, \dots, n,$$

and

$$(1.3) \quad \frac{d}{d\epsilon}(\bar{\lambda}_i(\epsilon) - \bar{\lambda}_j(\epsilon))|_{\epsilon=0} \neq 0, \quad i \neq j,$$

where $\alpha > 0$ and $\tau > r - 1$ are constants and $\bar{\lambda}_i(\epsilon)$ ($i = 1, 2, \dots, n$) are eigenvalues of $A + \epsilon Q$, then, for sufficiently small $\epsilon_0 > 0$, there exists a nonempty Cantor subset $E \subset (0, \epsilon_0)$ such that for $\epsilon \in (0, \epsilon_0)$ the equations (1.1) are reducible.

The method used in [2] is a kind of KAM iteration. In the KAM iteration the difficulty is caused by a small divisor $\langle \omega, k \rangle + \lambda_i - \lambda_j$. The small divisor conditions (1.2) are necessary to overcome the difficulty of the small divisor. Since the frequencies ω are invariant in the iteration, which is different from the usual KAM iteration [4], one must adjust the small parameter $\epsilon \in (0, \epsilon_0)$ to guarantee the small divisor conditions. So Jorba and Simó needed the nondegeneracy conditions (1.3) to guarantee existence of the nonempty Cantor subset E , on which all the small divisor conditions in the KAM iterations hold.

If the nondegeneracy conditions (1.3) do not hold, we say $\bar{\lambda}_i(\epsilon) - \bar{\lambda}_j(\epsilon)$ are degenerate. This degenerate case is mentioned in [2], but there is no any result given. Like the motivation of [2], in this paper we want to consider this degenerate case. We will prove a similar result under weaker nondegeneracy conditions. Moreover, in the situation of the usual nondegeneracy conditions (1.3), our result is just the same as that of [2], but the proof is simpler.

Main result.

Theorem. *Suppose $A = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ with $\lambda_i \neq \lambda_j$ for $i \neq j, 1 \leq i, j \leq n$, and $Q(t) = (q_{ij}(t))_{1 \leq i, j \leq n} = \sum_{k \in Z^r} Q_k e^{i\langle k, \omega \rangle t}$ is an analytic quasiperiodic matrix on D_ρ with frequencies $\omega_1, \omega_2, \dots, \omega_r$, where the Fourier coefficients Q_k depend analytically on the small parameter ϵ . Let $Q_0^d = \text{diag}(\bar{q}_{11}, \bar{q}_{22}, \dots, \bar{q}_{nn})$, where \bar{q}_{ii} is the average of $q_{ii}(t)$. Suppose that for $i \neq j, \epsilon(\bar{q}_{ii} - \bar{q}_{jj})$ has one of the following forms:*

$$\mu_1 \epsilon^{l_1} + o(\epsilon^{l_1}), \mu_2 \epsilon^{l_2} + o(\epsilon^{l_2}), \dots, \mu_p \epsilon^{l_p} + o(\epsilon^{l_p}),$$

where $\mu_i \neq 0, i = 1, 2, \dots, p, 1 \leq l_1 < l_2 < \dots < l_p$, and $o(\epsilon^l)$ is of order smaller than ϵ^l as $\epsilon \rightarrow 0$. Suppose

(1) (nonresonance conditions) $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ and $\omega = (\omega_1, \omega_2, \dots, \omega_r)$ satisfy

$$|\langle k, \omega \rangle + \lambda_i - \lambda_j| \geq \frac{\alpha}{|k|^\tau}, \forall 0 \neq k \in Z^r, \forall 1 \leq i, j \leq n,$$

where $\alpha > 0, \tau > r - 1$.

(2) Q is l_p order continuously differentiable with respect to sufficiently small ϵ and

$$\left\| \frac{d^l \tilde{Q}}{d\epsilon^l} \right\|_{D_\rho} \leq M_l, \quad l = 0, 1, 2, \dots, l_p,$$

where $\tilde{Q} = Q - Q_0^d$. Then, there exist $N_1, N_2, \dots, N_p, N_i$ depending on $M_1, M_2, \dots, M_{l_j}, \alpha, \tau, n, l_p$ and $\mu_1 \epsilon^{l_1} + o(\epsilon^{l_1}), \dots, \mu_j \epsilon^{l_j} + o(\epsilon^{l_j})$, where $l_j \leq l_i - 2$, such that if $|\mu_i| > N_i, i = 1, 2, \dots, p$, then for sufficiently small $\epsilon_0 > 0$, there exists a nonempty Cantor subset $E \subset (0, \epsilon_0)$ with positive Lebesgue measure such that for $\epsilon \in E$ the equations (1.1) are reducible, i.e., there exists a nonsingular quasiperiodic transformation $x = \Phi(t)y$ that changes (1.1) to $\dot{y} = By$, where B is a constant matrix. If ϵ_0 is small enough, the relative measure of E in $(0, \epsilon_0)$ is close to 1. Moreover, the quasiperiodic matrix $\Phi(t)$ has the same frequencies as $Q(t)$.

Remark 1. If $l_1 = 1$, we can choose $N_1 = 0$. If $(\bar{q}_{ii} - \bar{q}_{jj})$ take only the form $\mu\epsilon + o(\epsilon)$, this corresponds to the nondegenerate case of [2].

Remark 2. There are many ω and λ satisfying the nonresonance conditions in the theorem. We refer to [2], [3] for detailed discussions about nonresonance conditions or small divisor conditions.

2. PROOF OF THE MAIN RESULT

In this section we prove the theorem by the same idea of [2] with a little modification, which can simplify the proof in the nondegenerate case.

A. Outline of the proof. Write the equations (1.1) as

$$(2.1) \quad \dot{x} = (A^+ + \epsilon \tilde{Q}(t))x,$$

where $A^+ = \text{diag}(\lambda_1^+, \lambda_2^+, \dots, \lambda_n^+) = A + \epsilon Q_0^d, \tilde{Q} = Q(t) - \epsilon Q_0^d$. From [2] we know that under the change of variables $x = (I + \epsilon P(t))y$, where P satisfies

$$(2.2) \quad \dot{P} = A^+ P - P A^+ + \tilde{Q},$$

the equations (2.1) are changed to

$$\dot{y} = (A^+ + \epsilon^2 Q_+(t))y,$$

where $Q_+ = (I + \epsilon P)^{-1} \tilde{Q} P$. If the above process can go on, then the perturbation term $\epsilon^2 Q_+(t)$ becomes smaller and smaller and the equations converge to constant coefficient equations.

The key to the iteration is to solve the equation (2.2) for P . Denote by $P = (p_{ij}(t))_{1 \leq i, j \leq n}, \tilde{Q} = (\tilde{q}_{ij}(t))_{1 \leq i, j \leq n}$. Expanding them in Fourier series and substituting them into the equations (2.2), by comparing the coefficients of both sides of the equations, we see formally that

$$p_{ij}^k = \frac{\tilde{q}_{ij}^k}{\langle k, \omega \rangle \sqrt{-1} + \lambda_i^+ - \lambda_j^+}, \forall k \in Z^r, |i - j| + |k| \neq 0.$$

If

$$|\langle \omega, k \rangle \sqrt{-1} + \lambda_i^+ - \lambda_j^+| \geq \frac{\alpha_+}{|k|^{\tau'}}, \forall 0 \neq k \in Z^r, \forall i, j = 1, 2, \dots, n,$$

where $\tau' = 3\tau$, then $|p_{ij}^k| \leq (|k|^{\tau'} / \alpha_+) |\tilde{q}_{ij}^k|$. Since \tilde{Q} is analytic on D_ρ , we have $\|\tilde{Q}_k\| \leq \|\tilde{Q}\|_{D_\rho} e^{-|k|\rho}$. So

$$(2.3) \quad \|P\|_{D_{\rho-s}} \leq \sum_{k \in Z^r} \|P_k\| e^{|k|(\rho-s)} \leq \left(\frac{1}{\delta} + \sum_{0 \neq k \in Z^r} \frac{|k|^{\tau'} e^{-s|k|}}{\alpha_+} \right) \|\tilde{Q}\|_{D_\rho} \leq \frac{c}{\alpha_+ s^v} \|Q\|_{D_\rho},$$

where $v = \tau' + r - 1$, $0 < s \leq \frac{1}{2}\rho$, $\delta = \min_{i \neq j} |\lambda_i^+ - \lambda_j^+|$, and c depends on τ', r, n . Thus, If ϵ is sufficiently small that $\|(I + \epsilon P)^{-1}\|_{D_{\rho-s}} \leq 2$, then

$$(2.4) \quad \|Q_+\|_{D_{\rho-s}} \leq \frac{2c}{\alpha_+ s^v} \|Q\|_{D_\rho}^2.$$

B. Iteration step. Consider the following equations:

$$(2.5) \quad \dot{x}_m = (A_m + \epsilon^{2^m} Q_m(t))x_m, \quad m \geq 0$$

where $A_m = \text{diag}(\lambda_1^m, \lambda_2^m, \dots, \lambda_n^m)$, and $Q_m(t)$ is an analytic quasiperiodic matrix on D_{ρ_m} with the frequencies $\omega_1, \omega_2, \dots, \omega_r$. Let

$$A_{m+1} = \text{diag}(\lambda_1^{m+1}, \lambda_2^{m+1}, \dots, \lambda_n^{m+1}),$$

where $\lambda_i^{m+1} = \lambda_i^m + \epsilon^{2^m} \bar{q}_{ii}^m, i = 1, 2, \dots, n$, with \bar{q}_{ii}^m being the average of $q_{ii}^m(t)$. Let $\bar{Q}_m(t) = Q_m(t) - \epsilon^{2^m} \text{diag}(\bar{q}_{11}^m, \bar{q}_{22}^m, \dots, \bar{q}_{nn}^m)$.

If

$$(2.6) \quad |\langle \omega, k \rangle \sqrt{-1} + \lambda_i^{m+1} - \lambda_j^{m+1}| \geq \frac{\alpha_m}{|k|^{3\tau}}, \quad |k| + |i - j| \neq 0,$$

then by the above discussions we have an analytic quasiperiodic matrix P_m on $D_{\rho_m - s_m}$ with the frequencies $\omega_1, \omega_2, \dots, \omega_r$, such that under the change of variables $x_m = (I + \epsilon^{2^m} P_m)x_{m+1}$, the equations (2.5) are changed to

$$\dot{x}_{m+1} = (A_{m+1} + \epsilon^{2^{m+1}} Q_{m+1}(t))x_{m+1}.$$

Moreover, we have

$$(2.7) \quad \|P_m\|_{D_{\rho_m - s_m}} \leq \frac{c}{\alpha_m s_m^v} \|Q_m\|_{D_{\rho_m}}^2.$$

If ϵ is sufficiently small that $\|(I + \epsilon^{2^m} P_m)^{-1}\|_{D_{\rho_m - s_m}} \leq 2$, we have

$$(2.8) \quad \|Q_{m+1}\|_{D_{\rho_m - s_m}} \leq \frac{2c}{\alpha_m s_m^v} \|Q_m\|_{D_{\rho_m}}^2.$$

Now we prove this iteration is convergent. At the first step, let $A_0 = A, Q_0(t) = Q(t), \alpha_0 = \alpha/2, \rho_0 = \rho, s_0 = \rho/4, D_0 = D_{\rho_0}, F_0 = (\epsilon \|Q_0\|_{D_0}) / (\alpha_0 s_0^v)$. At the m th step we choose $\alpha_m = \alpha_0 / (1 + m)^2, s_m = s_{m-1} / 2, \rho_{m+1} = \rho_m - s_m, D_m = D_{\rho_m}, F_m = (\epsilon^{2^m} \|Q_m\|_{D_m}) / (\alpha_m s_m^v)$. If

$$(2.9) \quad \|(I + \epsilon^{2^m} P_m)^{-1}\|_{D_{m+1}} \leq 2, \forall m \geq 0,$$

then by (2.8) we have $F_{m+1} \leq \bar{c} F_m^2$, where $\bar{c} = 2^{v+1}c$. So $\bar{c} F_{m+1} \leq (\bar{c} F_m)^2$. If $|\bar{c} F_0| < \frac{1}{2}$, then $\bar{c} F_m \leq (\frac{1}{2})^{2^m}$. By (2.7) it follows that $\|\epsilon^{2^m} P_m\|_{D_{m+1}} \leq \bar{c} F_m \leq (\frac{1}{2})^{2^m}$. So, we have

$$\|(I + \epsilon^{2^m} P_m)^{-1}\|_{D_{m+1}} \leq 1 + \sum_{l=1}^{\infty} \|\epsilon^{2^l} P_l\|_{D_{m+1}}^l \leq 2.$$

Let $\epsilon_1 > 0$ be so small that $\bar{c} F_0 \leq \frac{1}{2}$ for $\epsilon \in (0, \epsilon_1)$. Thus, for $\epsilon \in (0, \epsilon_1)$ satisfying all the small divisor conditions (2.6), all the above estimates hold.

Let $D_* = \bigcap_{m=0}^{\infty} D_m, P^m = (I + \epsilon^{2^m} P_m)(I + \epsilon^{2^{m-1}} P_{m-1}) \cdots (I + \epsilon P_0)$. Obviously, $D_{\frac{1}{2}\rho} \subset D_*$ and P^m is convergent as $m \rightarrow \infty$ under the norm $\|\cdot\|_{D_*}$. Let $\Phi = \lim_{m \rightarrow \infty} P^m$.

From the above discussion, it follows that $\|A_m - A_{m-1}\| \leq \epsilon^{2^m} \|Q_m\|_{D_m}$. So, A_m is also convergent. Let $\lim_{m \rightarrow \infty} A_m = B$. Obviously, $\lim_{m \rightarrow \infty} \|\epsilon^{2^m} Q_m\|_{D_{\frac{1}{2}\rho}} = 0$.

Thus, if $\epsilon \in (0, \epsilon_1)$ satisfy the small divisor conditions (2.6) for all $m \geq 0$, then, under the change of variables $x = \Phi y$, the equations (1.1) become $\dot{y} = By$. Moreover, Φ is an analytic quasiperiodic matrix on $D_{\frac{1}{2}\rho}$ with the frequencies $\omega_1, \omega_2, \dots, \omega_r$.

To finish the proof of the theorem, it remains to prove that there exist N_1, N_2, \dots, N_p such that if $|\mu_i| \geq N_i$ ($i = 1, 2, \dots, p$), then there exist $0 < \epsilon_0 < \epsilon_1$ and a nonempty Cantor subset $E \subset (0, \epsilon_0)$ such that for $\epsilon \in E$, the small divisor conditions (2.6) hold for all $m \geq 0$. Now we first prove that there exist N_1, N_2, \dots, N_p such that if $|\mu_i| \geq N_i, i = 1, 2, \dots, p$, then for all $m \geq 0$ and $i \neq j$, there exist $l \in \{l_1, l_2, \dots, l_p\}$ such that the l th derivative of $\lambda_i^{m+1} - \lambda_j^{m+1}$ with respect to ϵ at $\epsilon = 0$ does not vanish.

Suppose that $\lambda_i^1 - \lambda_j^1 = \mu_1 \epsilon^{l_1} + o(\epsilon^{l_1})$. Let $N_1 \geq 0$ be an integer such that $2^{\bar{N}_1} \leq l_1 \leq 2^{\bar{N}_1+1}$. Let

$$\bar{M}_i^1 = \left\| \frac{d^{l_1-2^i} Q_i}{d\epsilon^{l_1-2^i}} \Big|_{\epsilon=0} \right\|_{D_{\rho_i}}, \quad i = 1, 2, \dots, \bar{N}_1.$$

Then \bar{M}_i^1 ($i = 1, 2, \dots, \bar{N}_1$) only depends on τ, n, α, ρ , and $M_0, M_1, \dots, M_{l_1-2^i}$. By the construction of the transformation, \bar{M}_i^1 should depend on all derivatives up to the $l_1 - 2^i$ -th of $\lambda_{j_1}^i(\epsilon) - \lambda_{j_2}^i(\epsilon)$ with respect to ϵ at $\epsilon = 0$. But these derivatives only depend on $M_0, M_1, \dots, M_{l_1-2^i}$ and are independent of $\mu_1, \mu_2, \dots, \mu_p$. Let

$$N_1 = \frac{\bar{M}_1^1 + \bar{M}_2^1 + \dots + \bar{M}_{\bar{N}_1}^1}{l_1!}.$$

If $|\mu_1| > N_1$, we have at $\epsilon = 0$

$$\left| \frac{d^{l_1} (\lambda_i^{m+1} - \lambda_j^{m+1})}{d\epsilon^{l_1}} \right| \geq l_1! |\mu_1| - (\bar{M}_1 + \bar{M}_2 + \dots + \bar{M}_{\bar{N}_1}) > 0.$$

Similarly, let $\bar{N}_2 \geq \bar{N}_1$ be an integer such that $2^{\bar{N}_2} \leq l_2 < 2^{\bar{N}_2+1}$. Let

$$\bar{M}_i^2 = \left\| \frac{d^{l_2-2^i} Q_i}{d\epsilon^{l_2-2^i}} \Big|_{\epsilon=0} \right\|_{D_{\rho_i}}, \quad i = 1, 2, \dots, \bar{N}_2.$$

Then \bar{M}_i^2 ($i = 1, 2, \dots, \bar{N}_2$) only depends on $\tau, n, \alpha, \rho, M_0, M_1, \dots, M_{l_2-2^i}$ and the l -th order derivatives of $\mu_1 \epsilon^{l_1} + o(\epsilon^{l_1})$ with respect to ϵ , where $l \leq l_2 - 2^i$.

Let

$$N_2 = \frac{\bar{M}_1^2 + \bar{M}_2^2 + \dots + \bar{M}_{\bar{N}_2}^2}{l_2!}.$$

If $|\mu_2| > N_2$, then, when $\lambda_i^1 - \lambda_j^1 = \mu_2 \epsilon^{l_2} + o(\epsilon^{l_2})$, for all $m \geq 0$, the l_2 -th derivative of $\lambda_i^{m+1} - \lambda_j^{m+1}$ with respect to ϵ at $\epsilon = 0$ does not vanish. From the above we see that \bar{N}_2 may depend on \bar{N}_1 , but \bar{N}_1 is independent of \bar{N}_2 . In the same way we can obtain \bar{N}_p , which depends on $\tau, n, \alpha, \rho, M_0, M_1, \dots, M_{l_p-2}$ and the l -th derivatives of $\mu_i \epsilon^{l_i} + o(\epsilon^{l_i})$ with respect to $\epsilon, i = 1, 2, \dots, p - 1$, where $l \leq l_p - 2$. If $|\mu_p| > N_p$, then, when $\lambda_i^1 - \lambda_j^1$ consists of $\mu_p \epsilon^{l_p}$, for all $m \geq 0$, the l_p -th derivative of $\lambda_i^{m+1} - \lambda_j^{m+1}$ with respect to ϵ at $\epsilon = 0$ is not zero. It is easy to see that N_p

may depend on N_1, N_2, \dots, N_{p-1} . Thus, if $\lambda_i^1 - \lambda_j^1 = \mu_i \epsilon^{l_i} + o(\epsilon^{l_i})$, then

$$\frac{d^{l_i}(\lambda_i^{m+1} - \lambda_j^{m+1})}{d\epsilon^{l_i}}|_{\epsilon=0} \neq 0, \forall m \geq 0.$$

From the above iteration we see that the first step can only be done for all $\epsilon \in E_0 \subset (0, \epsilon_0)$, where E_0 is the set on which the small divisor conditions of the first step hold. Let E_{m-1} be the set on which the small divisor conditions of the m -th step hold. Then the m -th step can only be done for $\epsilon \in E_0 \cap E_1 \cap \dots \cap E_{m-1}$. Thus, the above iteration can only be convergent on the set $E = \bigcap_{m=0}^\infty E_m$. Since the small divisor conditions hold on a Cantor subset, in all the iteration steps the differentiations with respect to ϵ are understood in Whitney's sense [5]. By the Whitney extension theorem in [5], all differentiable functions on a closed set in Whitney's sense can be extended to the usual differentiable function on $(-\infty, +\infty)$ and the differentiation in Whitney's sense can be treated as the usual differentiation. Thus all the functions in the iteration step can be regarded as regular differentiable functions on $(-\infty, +\infty)$, and so all the estimates can be obtained without any difficulty. However, the iteration makes sense only for $\epsilon \in E$.

Now we prove E is a nonempty set. For this we prove that for most sufficiently small ϵ , the small divisor conditions (2.6) hold for all $m \geq 0$. Let $\lambda_i^{m+1} - \lambda_j^{m+1} = \mu_l \epsilon^l + o(\epsilon^l)$, where $l \leq l_p$. It is easy to see that the term $\mu_l \epsilon^l$ is unvariant when $m \geq N_p$. So there exists a sufficiently small $\epsilon_0 > 0$ such that if $|\epsilon| \leq \epsilon_0$, then $|\lambda_i^{m+1} - \lambda_j^{m+1}| \leq 2|\mu_l| \epsilon^l$ and $|\frac{d(\lambda_i^{m+1} - \lambda_j^{m+1})}{d\epsilon}| \geq \frac{1}{2}|\mu_l| \epsilon^{l-1}$ for all $m \geq 0$.

Let $f(\epsilon) = \langle \omega, k \rangle + \lambda_i^{m+1} - \lambda_j^{m+1}$, $i \neq j$, and

$$O_{ijm}^k = \{ \epsilon \in (0, \epsilon_0) \mid |f(\epsilon)| < \frac{\alpha_m}{|k|^{3\tau}} \}.$$

Since $\lambda_i^0 - \lambda_j^0 \neq 0, \forall i \neq j$, we choose ϵ_0 so small that if $|\epsilon| \leq \epsilon_0$, $|\lambda_i^{m+1} - \lambda_j^{m+1}| \geq \bar{\alpha} > 0$ holds for all $i \neq j$ and $m \geq 1$. So we only consider $k \neq 0$. Since

$$|\langle \omega, k \rangle + \lambda_i^{m+1} - \lambda_j^{m+1}| \geq |\langle \omega, k \rangle \sqrt{-1} + \lambda_i^0 - \lambda_j^0| - 2\mu_l \epsilon^l,$$

by the nonresonance conditions of the theorem, if $1/|k|^\tau > 4\mu_l \epsilon^l / \alpha_0$, then $|f(\epsilon)| \geq \alpha_0 / 2|k|^\tau > \alpha_m / |k|^{3\tau}$.

Suppose $\alpha_0 / 4\mu_l |k|^\tau < \epsilon^l \leq \epsilon_0$. Since

$$\left| \frac{df(\epsilon)}{d\epsilon} \right| \geq \frac{1}{2} |\mu_l| \epsilon^{l-1},$$

by the differentiation mean value theorem, we have

$$\text{meas}(O_{ijm}^k) \leq \frac{2\alpha_m}{|k|^{3\tau}} \frac{2}{|\mu_l| \epsilon^{l-1}} \leq \frac{\epsilon_0^{l+1}}{|k|^\tau (m+1)^2} \frac{4|\mu_l|}{\alpha_0} \leq \frac{8|\mu_l| \epsilon_0^{l+1}}{\alpha (m+1)^2 |k|^\tau}.$$

So

$$\text{meas}\left(\bigcup_{i \neq j} \bigcup_{0 \neq k \in Z^r} O_{ijm}^k\right) \leq \frac{8n^2 \max_l |\mu_l| \epsilon_0^2}{\alpha} \sum_{m=0}^\infty \frac{1}{(m+1)^2} \sum_{0 \neq k \in Z^r} \frac{1}{|k|^\tau} = c\epsilon_0^2,$$

where c depends on n, τ, α and μ_l . Let E be the subset of $(0, \epsilon_0)$ where the small divisor conditions (2.6) hold for all $m \geq 0$. Then $E = (0, \epsilon_0) - \bigcup_{m,i,j,k} O_{ijm}^k$. Thus $\text{meas}(E) \geq \epsilon_0 - c\epsilon_0^2 = \epsilon_0(1 - c\epsilon_0)$. If ϵ_0 is so small that $1 - c\epsilon_0 > 0$. then E is a nonempty set with positive Lebesgue measure. Noticing that $\{\frac{k}{|k|} \mid 0 \neq k \in Z^r\}$ is dense on the unit ball of R^r , we conclude that E is a Cantor set.

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