ON THE CHARACTERIZATION OF CERTAIN
SIMILARLY ORDERED SUPER-ADDITIVE FUNCTIONALS

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(Communicated by J. Marshall Ash)

ABSTRACT. Functionals which behave (sub-, super-) additively on similarly ordered functions occur quite naturally in many contexts. In the present paper we characterize (super-) additive functionals which are defined on a family of functions with the Stone-property in terms of their naturally adjoined dyadic martingales. As corollaries we obtain essential generalizations of integral representations as derived by Schmeidler (1986) and discussed in a recent monograph of Denneberg (1994).

1. INTRODUCTION AND MOTIVATION

In his systematic study of capacities Choquet [1] associated to increasing set functions φ vanishing on the empty set a functional Φ (often called the Choquet integral with respect to φ) defined by

$$\Phi(f) := \int_X f \, d\varphi := R \int_0^\infty \varphi(\{x \in X : f(x) \geq \alpha\}) \, d\alpha$$

$$=: R \int_0^\infty \varphi(f \geq \alpha) \, d\alpha, \qquad f : X \to \mathbb{R}^+,$$

where $R \int_0^\infty$ denotes the improper Riemann integral.

Dellacherie [2] observed that Φ as defined above behaves additively on similarly ordered functions.

Definition 1. Two functions $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are said to be similarly ordered, shortly f s.o. g, if

$$(f(x) - f(y))(g(x) - g(y)) \geq 0 \quad \text{for every } x, y \in X.$$

Let $\mathbb{L}$ denote a family of functions. A functional $\Phi : \mathbb{L} \to \mathbb{R}$ is said to be s.o.-additive if $\Phi(f + g) = \Phi(f) + \Phi(g)$ whenever f s.o. g and $f + g \in \mathbb{L}$.

Schmeidler [8] called such functionals comonotonic. For historical reasons—the notion similarly ordered has already been used in Hardy, Littlewood, and Pólya ([6], p. 168)—and in order not to confound comonotonic with monotonic in the same direction, we prefer the older more intuitive notion.

It should be noted that s.o.-additive functionals have played a prominent role in statistics since the publications of Dempster, Huber, Shafer and Strassen. They

naturally occur in other fields like mathematical economics, game theory and arti-
ficial intelligence.

Therefore it seems interesting to characterize them in a uniform way, thus making
it possible to apply the sophisticated and well understood methods of measure
theory instead of ad hoc considerations. For more background material and a
systematic approach to non-additive measures and integrals we refer the reader
to Denneberg [3]. The present paper was mainly motivated by the design of a
decision theory which is able to handle in a mathematically sound way paradoxical
situations like the ones considered by Allais (Nobel Prize 1988) and Ellsberg. Such
theories have recently been proposed by Gilboa [4] and Schmeidler [9]. Both theories
essentially depend on Schmeidler's [8] representation theorem. The latter result
is very special and only applicable if the functional already defines "distribution
functions" of the functions on which it is defined. The representation theorem then
is simply obtained by integrating the distribution functions. In the following we
give characterizations of some classes of s.o.-(super-) additive functionals in terms
of the naturally adjoined adapted stochastic process. For this purpose we only need
some basic results from the theory of martingales. Our standard reference will be
Kopp [7]. As immediate corollaries we obtain essential generalizations of integral
representations as discussed by Schmeidler [8]. Note that his ideas are restricted
to the case where $\Phi(1_{(f \geq \alpha)})$ is defined for all $\alpha \in [\inf f, \sup f]$, e.g. if $\mathbb{L}$ consists of
measurable functions or if $\mathbb{L}$ is a Stone lattice of functions which is conditionally
σ-completely embedded in $\overline{\mathbb{R}}^{\times}$.

2. Martingale characterizations
of certain s.o.-(super-) additive functionals

In the following we consider a family of functions $\mathbb{L}$ with the properties
  (i) $f(x) \geq 0$ for all $x \in X$, $\sup_{x \in X} f(x) < +\infty$,
and
  (ii) $\alpha f, f \wedge \alpha, f - f \wedge \alpha \in \mathbb{L}$ if $f \in \mathbb{L}$, $\alpha \in \mathbb{R}_+$.
For a functional $\Phi \colon \mathbb{L} \to \mathbb{R}_+$ the following properties will be of interest:
  (iii) $\Phi(\alpha f) = \alpha \Phi(f)$, $f \in \mathbb{L}$, $\alpha \in \mathbb{R}_+$.
  (iv) $\Phi(f + g) \geq \Phi(f) + \Phi(g)$, if f s.o. g and $f + g \in \mathbb{L}$.
  (v) Let I_k be a subset of $\{1, 2, \ldots, 2^{n_k}\}$ and let $|I_k| \frac{1}{2^{n_k}} \to 0$ as $k \to \infty$. Then

$$\sum_{i \in I_k} \Phi \left(f \wedge \frac{i}{2^{n_k}} - f \wedge \frac{i-1}{2^{n_k}} \right) \to 0.$$

  (vi) If $\|f_n - f\|_\infty \to 0$ as $n \to \infty$, then $\Phi(f_n) \to \Phi(f)$.
  (vii) $\Phi(f \wedge \alpha) \to 0$ as $\alpha \to 0$.

Remarks. 1. For notational convenience we always assume from now on that
$\sup f = 1$.

2. We state our results for s.o.-superadditive functionals. Corresponding results
for s.o.-subadditive functionals are obtained in the same way.

3. Property (vi) has been assumed by Schmeidler [8] for deriving his representa-
tion theorem. Obviously (vi)→(v) and for increasing functionals (v) and (vii) are
equivalent.

We define an increasing sequence $(\mathbb{F}_n)_{n\in\mathbb{N}}$ contained in the Lebesgue probability space $([0,1],\mathbb{B},\mu)$ by

$$\mathbb{F}_0 := \{\phi, [0,1]\}$$

and

$$\mathbb{F}_n := \sigma\text{-field generated by the atoms } \left[0,\frac{1}{2^n}\right),\ldots,\left[\frac{2^n-1}{2^n},1\right].$$

Let $\mathbb{F} := \sigma(\bigcup_{n\in\mathbb{N}}\mathbb{F}_n)$. It is easy to see that in this case the conditional expectations of an integrable random variable φ are obtained by

$$E(\varphi|\mathbb{F}_0) = \left(\int_0^1 \varphi\,d\mu\right)\cdot 1_{[0,1]},$$

$$E(\varphi|\mathbb{F}_n) = \sum_{i=1}^{2^n} 2^n\left(\int_{(i-1)/2^n}^{i/2^n}\varphi\,d\mu\right)1_{[\frac{i-1}{2^n},\frac{i}{2^n})}.$$

Lemma 1. *Let $(\mathbb{L},\Phi)$ satisfy (i)–(iv). To every $f\in\mathbb{L}$ there is naturally adjoined a positive dyadic supermartingale $\varphi^f = (\varphi_n^f,\mathbb{F}_n)_{n\in\mathbb{N}}$ which, in general, decomposes uniquely into three non-trivial components such that $\varphi^f = \varphi^{f1} + \varphi^{f2} + \varphi^{f3}$ where φ^{f1} is a uniformly integrable martingale, φ^{f2} is a martingale converging pointwise a.s. to 0 and φ^{f3} is a potential.*

Proof. Define $\varphi_0^f(\alpha) := \Phi(f)1_{[0,1]}$ (remember $\sup f = 1$!),

$$\varphi_n^f(\alpha) := \sum_{i=1}^{2^n} 2^n\Phi\left(f\wedge\frac{i}{2^n} - f\wedge\frac{i-1}{2^n}\right)1_{[\frac{i-1}{2^n},\frac{i}{2^n})}.$$

We observe that for any $f\in\mathbb{L}$ the partial sums of functions $f\wedge\frac{i}{2^n} - f\wedge\frac{i-1}{2^n}\in\mathbb{L}$ are s.o. and

$$E(\varphi_{n+1}^f|\mathbb{F}_n) = \sum_{i=1}^{2^n} 2^n\left[\Phi\left(f\wedge\frac{2i}{2^{n+1}} - f\wedge\frac{2i-1}{2^{n+1}}\right)\right.$$
$$\left. + \Phi\left(f\wedge\frac{2i-1}{2^{n+1}} - f\wedge\frac{2i-2}{2^{n+1}}\right)\right]1_{[\frac{i-1}{2^n},\frac{i}{2^n})}.$$

By (iv)

$$\Phi\left(f\wedge\frac{i}{2^n} - f\wedge\frac{i-1}{2^n}\right) \geq \Phi\left(f\wedge\frac{2i}{2^{n+1}} - f\wedge\frac{2i-1}{2^{n+1}}\right)$$
$$+ \Phi\left(f\wedge\frac{2i-1}{2^{n+1}} - f\wedge\frac{2i-2}{2^{n+1}}\right),$$

whence $\varphi_n^f(\alpha) \geq E(\varphi_{n+1}^f|\mathbb{F}_n)$ a.s.

As φ is a positive supermartingale it converges a.s. to an integrable limit (cf. Kopp [7], p. 51). Denote the a.s. limit of $(\varphi_n^f)_{n\in\mathbb{N}}$ by φ_∞^f and define $\varphi_n^{f1} := E(f_\infty|\mathbb{F}_n)$ and $\overline{\varphi}_n^{f2} := \varphi_n^f - \varphi_n^{f1}$. Then φ_n^{f1} is a uniformly integrable martingale and the supermartingale $(\overline{\varphi}_n^{f2})_{n\in\mathbb{N}}$ converges a.s. to 0. We now apply the Riesz decomposition (cf. Kopp [7], p. 68) to $(\overline{\varphi}_n^{f2})_{n\in\mathbb{N}}$ to obtain φ^{f2} and φ^{f3}. □

We are now going to characterize a special class of s.o. superadditive functionals.

Theorem 2. *Let $(\mathbb{F},\Phi)$ satisfy (i)–(iv). If Φ also satisfies (v), then for every $f\in\mathbb{L}$ the naturally adjoined dyadic supermartingale is uniformly integrable.*

Proof. Assume the contrary. Then for some $f \in \mathbb{L}$ the adjoined supermartingale φ^f is not uniformly integrable. In the Riesz decomposition $\varphi^f = \varphi^{f1} + \varphi^{f2}$ the potential φ^{f2} is uniformly integrable, whence the martingale φ^{f1} is not. Equivalently, $(\varphi^{f1})_{n \in \mathbb{N}}$ is not relatively weakly compact in L^1 (cf. Kopp [7], p. 30). The isometric isomorphism provided by the Radon-Nikodym theorem implies that the sequence $(\gamma_n)_{n \in \mathbb{N}}$ defined by

$$\gamma_n(A) := \int_A \varphi_n^{f1} \, d\mu, \qquad A \in \mathbb{B},$$

cannot be a relatively weakly compact subset of the space of regular Borel measures on $[0, 1]$. A celebrated result of Grothendieck [5] tells us that this is equivalent to the existence of an $\varepsilon > 0$ and a sequence of disjoint open sets $(0_k)_{k \in \mathbb{N}}$ together with a subsequence $(\gamma_{n_k})_{k \in \mathbb{N}}$ such that $\gamma_{n_k}(0_k) > \varepsilon$ for all $k \in \mathbb{N}$. This in turn gives a subsequence $(\varphi_{n_k})_{k \in \mathbb{N}}$ such that

$$\int_{0_k} \varphi_{n_k}^{f1} d\mu > \varepsilon \quad \text{for all } k \in \mathbb{N}.$$

As the set of all B with the property that for every $\delta > 0$ there exists a $C \in \bigcup_{n \in \mathbb{N}} \mathbb{F}_n$ such that $\mu(B \Delta C) < \varepsilon$ forms a monotone class which contains $\bigcup_{n \in \mathbb{N}} \mathbb{F}_n$, a dense subset of the Borel field $\mathbb{B}$, for every $k \in \mathbb{N}$ we can find a $C_k \in \mathbb{F}_{n_k}$ such that $\varphi_{n_k}^{f1}$ is $\mathbb{F}_{n_k}$-measurable and

$$\left| \int_{C_k} \varphi_{n_k}^{f1} d\mu - \int_{0_k} \varphi_{n_k}^{f1} d\mu \right| \le \int_{C_k \Delta 0_k} \varphi_{n_k}^{f1} d\mu \le \mu(C_k \Delta 0_k) \sup \varphi_{n_k}^{f1} < \frac{1}{2^{n_k}} \varepsilon,$$

whence

$$\int_{C_k} \varphi_{n_k}^{f1} d\mu > \frac{\varepsilon}{2} \quad \text{for all } k \in \mathbb{N}.$$

Each C_k consists of finitely many intervals of length $1/2^{n_k}$ to which in turn there correspond elements of the form $f \wedge \frac{i}{2^{n_k}} - f \wedge \frac{i-1}{2^{n_k}} \in \mathbb{L}$.

Observing the s.o.-additivity of the partial sums of such elements we obtain

$$\frac{\varepsilon}{2} < \int_{C_k} \varphi_{n_k}^{f1} d\mu = \sum_{i \in I_k} \Phi \left(f \wedge \frac{i}{2^{n_k}} - f \wedge \frac{i-1}{2^{n_k}} \right)$$

$$= \Phi \left(\sum_{i \in I_k} \left(f \wedge \frac{i}{2^{n_k}} - f \wedge \frac{i-1}{2^{n_k}} \right) \right) \quad \text{for some } I_k \subset \{1, 2, \dots, 2^{n_k}\}.$$

As $(\sup \varphi_{n_k}^{f1})_{k \in \mathbb{N}}$ cannot be a null-sequence, $(\mu(C_k \Delta 0_k))_{k \in \mathbb{N}}$ must be one and therefore $\mu(C_k) \to 0$ as $n \to \infty$. This gives a contradiction to (v) by observing

$$|I_k| \frac{1}{2^{n_k}} = \mu(C_k). \quad \square$$

The following corollaries now easily follow:

Corollary 3. *Let $(\mathbb{F}, \Phi)$ satisfy* (i)–(iii) *and assume that Φ is s.o.-additive. Then $\Phi(f) \ge \int_0^{1=\sup f} \varphi_\infty^f \, d\mu$ where φ_∞^f denotes the closure of the uniformly integrable part of the adjoined dyadic martingale. A proper integral representation, i.e. $\Phi(f) = \int_0^{1=\sup f} \varphi_\infty^f \, d\mu$, obtains iff Φ satisfies* (v).

Corollary 4. *Let* $(\mathbb{F}, \Phi)$ *satisfy* (i)–(iii) *and assume that* Φ *is s.o.-additive and increasing. Then* $\Phi(f) = R \int_0^{1=\sup f} \varphi_\infty^f(\alpha) \, d\alpha$ *if there exists an auxiliary function* $a_f \in \mathbb{L}$ *such that* $f(x) \neq 0 \Rightarrow a_f(x) \geq 1$. (*Here* $R \int$ *denotes the proper Riemann integral.*)

Remarks. 1. Without the preceding results a proof of Corollary 4 is easily obtained by applying Helley's selection theorem and Arzela's dominated convergence theorem.

2. For increasing functionals Φ it makes sense to define set functions $\varphi^\uparrow$ and $\varphi^\downarrow$ as below:

$$\varphi^\uparrow(\{f \geq \alpha\}) := \sup\{\Phi(g) \colon g \in \mathbb{L}, g \leq 1_{\{f \geq \alpha\}}\},$$

$$\varphi^\downarrow(\{f \geq \alpha\}) := \inf\{\Phi(g) \colon g \in \mathbb{L}, g \geq 1_{\{f \geq \alpha\}}\}, \qquad \alpha > 0.$$

It is easy to see that $\varphi^\uparrow(\{f \geq \alpha\}) \leq \varphi_\infty^f(\alpha) \leq \varphi^\downarrow(\{f \geq \alpha\})$ a.s. and the representation theorems discussed by Denneberg ([3], p. 155 ff) may now be inferred from Corollary 4.

If Φ has further continuity properties (properties (iv) and (v) in Denneberg ([3], p. 156)) we obtain representation results in terms of an improper Riemann integral.

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