PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 126, Number 6, June 1998, Pages 1791–1798
S 0002-9939(98)04193-8

THE γ_p PROPERTY AND THE REALS

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(Communicated by Andreas R. Blass)

ABSTRACT. The γ_p property may be generalized by using filters on ω in a very natural way. We analyze the necessary requirements for a space $\mathcal{X}$ to have property $\gamma_{\mathcal{F}}$ for a filter $\mathcal{F}$. We construct special filters for which $\mathbb{R}$ has the $\gamma_{\mathcal{F}}$ property, in particular a P-point and a Q-point.

1. INTRODUCTION

Our terminology is standard, but we review the main concepts and notation. The set of natural numbers will be denoted by ω; $\mathcal{P}(\omega)$ denotes the collection of all its subsets. Given a set X, we write $[X]^\omega$ and $[X]^{<\omega}$ to denote the collection of infinite subsets and finite subsets of X respectively; if we wish to be more specific, we write $[X]^n$ and $[X]^{\leq n}$ to denote subsets of size n or at most n respectively. We use the well-known "almost inclusion" ordering between members of $[\omega]^\omega$, i.e. $X \subseteq^* Y$ if $X \setminus Y$ is finite. We identify $\mathcal{P}(\omega)$ with ${}^\omega 2$ via characteristic functions. The space ${}^\omega 2$ is further equipped with the product topology of the discrete space $\{0,1\}$; a basic neighbourhood is then a set of the form

$$\mathcal{O}_s = \{f \in {}^\omega 2 : s \subseteq f\},$$

where $s \in {}^{<\omega}2$, the collection of finite binary sequences. The terms "nowhere dense", "meager", "Baire property" all refer to this topology.

A filter is a collection of subsets of ω closed under finite intersections, supersets and containing all cofinite sets. It is called proper if it does not contain the empty set; thus the collection of cofinite sets is the smallest proper filter, it is called the *Fréchet* filter and is denoted by $\mathfrak{Fr}$. To avoid trivialities, we shall assume that all filters under discussion are proper. Given a collection of sets $\mathcal{A} \subseteq [\omega]^\omega$, we denote by $\langle \mathcal{A} \rangle$ the filter generated by $\mathcal{A}$, that is, the smallest filter containing each member of $\mathcal{A}$. For a filter $\mathcal{F}$, $\mathcal{F}^+$ denotes the set of all sets X such that $\langle \mathcal{F} \cup \{X\} \rangle$ is a proper filter, and we call such an X compatible with $\mathcal{F}$. Another filter $\mathcal{G}$ is called compatible with $\mathcal{F}$ if every element of $\mathcal{G}$ is compatible with $\mathcal{F}$, that is, $\mathcal{F} \cup \mathcal{G}$ generates a proper filter.

The Rudin-Keisler ordering on filters is defined by

$$\mathcal{F} \leq_{RK} \mathcal{G} \text{ if } (\exists f \in {}^\omega\omega)\, \mathcal{G} \supseteq f^{-1}(\mathcal{F}) := \{f^{-1}[X] : X \in \mathcal{F}\}.$$

The following Lemma from [6] combinatorially describes F_σ filters.

Received by the editors June 5, 1996 and, in revised form, November 21, 1996.
1991 *Mathematics Subject Classification.* Primary 04A20; Secondary 03E05, 03E15, 03E35.
The research of both authors was partially supported by the Instituto de Matemáticas, México. The research of the second author was also partially supported by the NSERC of Canada.

Lemma 1.1. *Let $\mathcal{F}$ be an F_σ filter. Then there is an increasing sequence of natural numbers $\langle n_k : k \in \omega\rangle$ and sets $a_i^k \subseteq [n_k, n_{k+1})$, $i < m_k$, such that*

1. $(\forall x \in [m_k]^{\leq k})\ \bigcap_{i\in x} a_i^k \neq \emptyset$,
2. $(\forall X \in \mathcal{F})(\forall^\infty k)(\exists i < m_k)\ a_i^k \subseteq X$.

Proof. Let $\mathcal{F} = \bigcup_n \mathcal{C}_n$, where each $\mathcal{C}_n$ is closed, and put

$$\mathcal{C} = \{X \cup n : n \in \omega \text{ and } X \in \mathcal{C}_n\}.$$

Then again $\mathcal{C}$ is a closed set and every member of $\mathcal{F}$ is almost equal to a member of $\mathcal{C}$.

Let $n_0 = 0$ and, having defined n_j for $j \leq k$, choose an $n_{k+1} > n_k$ such that

$$(\forall X_0, X_1, \cdots, X_{k-1} \in \mathcal{C})\ \bigcap_{i<k} X_i \cap [n_k, n_{k+1}) \neq \emptyset.$$

The existence of such an n_{k+1} follows from the fact that $\mathcal{C}$ is closed and $\mathcal{F}$ only contains infinite sets. Finally enumerate $\{X \cap [n_k, n_{k+1}) : X \in \mathcal{C}\}$ as $\{a_i^k : i < m_k\}$. This completes the proof. □

It is worth noticing that conversely, given a family $\langle\langle a_i^k : i < m_k\rangle; k \in \omega\rangle$ with the above properties, the collection

$$\{X : (\forall k)(\exists i < m_k)\ a_i^k \subseteq X\}$$

is a closed set generating an F_σ filter.

To define the γ property, we recall first the notion of an ω-cover.

Definition 1.2. A collection $\mathcal{O}$ of open sets is an ω-cover of a topological space $\mathcal{X}$ if $(\forall x \in [\mathcal{X}]^{<\omega})(\exists O \in \mathcal{O})\ \ x \subseteq O$.

In this paper, we will assume to avoid trivialities that if $\mathcal{O}$ is an ω-cover of a space $\mathcal{X}$, then $\mathcal{X} \notin \mathcal{O}$.

Traditionally, a space $\mathcal{X}$ is said to have the γ property if every ω-cover $\mathcal{O}$ of $\mathcal{X}$ contains a sequence $\langle O_n : n \in \omega\rangle$ such that

$$\mathcal{X} = \lim_n O_n,$$

where $\lim_n O_n = \bigcup_n \bigcap_{k\geq n} O_k$.

Following [1] and [4], we can extend the limit notion $\lim_n O_n$ using an arbitrary filter $\mathcal{F}$ by

$$\lim_{\mathcal{F}} O_n := \bigcup_{A\in\mathcal{F}} \bigcap_{k\in A} O_k.$$

Definition 1.3. Given a filter $\mathcal{F}$ on ω, a space $\mathcal{X}$ is said to have the:

1. $\gamma_{\mathcal{F}}$ property if every ω-cover $\mathcal{O}$ of $\mathcal{X}$ contains a sequence $\langle O_n : n \in \omega\rangle$ such that $\mathcal{X} = \lim_{\mathcal{F}} O_n$,
2. $\gamma'_{\mathcal{F}}$ property if for every sequence $\langle \mathcal{G}_n : n \in \omega\rangle$ of ω-covers, there is $O_n \in \mathcal{G}_n$ such that $\mathcal{X} = \lim_{\mathcal{F}} O_n$,
3. $\gamma''_{\mathcal{F}}$ property if for every sequence $\langle \mathcal{G}_n : n \in \omega\rangle$ of ω-covers, there is $\mathcal{H}_n \in [\mathcal{G}_n]^{<\omega}$ and a sequence $\langle O_n : n \in \omega\rangle \subseteq \bigcup_n \mathcal{H}_n$ such that $\mathcal{X} = \lim_{\mathcal{F}} O_n$.

The classical γ property can therefore be rephrased as the $\gamma_{\mathfrak{Fr}}$ property, and the properties $\gamma_{\mathfrak{Fr}}$, $\gamma'_{\mathfrak{Fr}}$ and $\gamma''_{\mathfrak{Fr}}$ are all equivalent ([3]); but notice that the larger the filter $\mathcal{F}$, the weaker these properties become. Indeed, a subset of $\mathbb{R}$ with the

γ property necessarily has strong measure zero, but we will see later that $\mathbb{R}$ itself can have the $\gamma_{\mathcal{F}}$ property for certain (large) filters $\mathcal{F}$; we will also show that the properties $\gamma_{\mathcal{F}}$ and $\gamma'_{\mathcal{F}}$ are not equivalent in general. Also observe that if a space $\mathcal{X}$ has the $\gamma_{\mathcal{F}}$ property, then every open ω-cover has a countable ω-subcover; this last property is denoted by ϵ in [3], and we shall make use of the fact that every σ-compact space has this property.

The authors thank the referee for a thorough report and helpful comments. The second author wishes moreover to thank the Instituto de Matemáticas of the Universidad Nacional Autónoma de México, and S. Garcia-Ferreira for making his visit to the Institute possible. Most of the results presented here were obtained there.

2. Basic properties

We are most interested in the $\gamma_{\mathcal{F}}$ property relative to subsets of the reals $\mathbb{R}$ (see [3] and [4]), but many basic properties will hold for any σ-compact space.

Definition 2.1. Let $\mathcal{O} = \langle O_n : n \in \omega \rangle$ be a countable ω-cover of a space $\mathcal{X}$. For $x \in [\mathcal{X}]^{<\omega}$, let $A_x = \{n : x \subseteq O_n\}$ and put

$$\mathcal{H}(\mathcal{O}) = \langle \{A_x : x \in [\mathcal{X}]^{<\omega}\} \rangle.$$

If $x \in \mathcal{X}$, we often abuse notation and use A_x instead of $A_{\{x\}}$. Thus $\mathcal{H}(\mathcal{O})$ is a filter on ω which will play an important role in the study of the $\gamma_{\mathcal{F}}$ property; it shouldn't come as a surprise that if a space $\mathcal{X}$ has the $\gamma_{\mathcal{F}}$ property then the filter $\mathcal{F}$ should be somewhat related to $\mathcal{H}(\mathcal{O})$ for any ω-cover $\mathcal{O}$ of $\mathcal{X}$. In this section, we develop various tools regarding this filter.

Proposition 2.2. *A space $\mathcal{X}$ has the $\gamma_{\mathcal{F}}$ property if and only if $\mathcal{X}$ has the ϵ property and $\mathcal{F} \geq_{RK} \mathcal{H}(\mathcal{O})$ for every countable ω-cover $\mathcal{O}$ of $\mathcal{X}$.*

Proof. We already noted that a space with the $\gamma_{\mathcal{F}}$ property must have the ϵ property. Further, if $\mathcal{O} = \langle O_n : n \in \omega \rangle$ is a countable ω-cover of $\mathcal{X}$, then by assumption there is a subsequence $\langle n_j : j \in \omega \rangle$ such that

$$\mathcal{X} = \bigcup_{A \in \mathcal{F}} \bigcap_{j \in A} O_{n_j}.$$

Define $f \in {}^{\omega}\omega$ by $f(j) = n_j$; we claim that $f^{-1}(\mathcal{H}(\mathcal{O})) \subseteq \mathcal{F}$. Indeed, fix $x \in \mathcal{X}$ and choose $B \in \mathcal{F}$ such that $x \in \bigcap_{j \in B} O_{n_j}$; but this means that $n_j \in A_x$ whenever $j \in B$, i.e. $f[B] \subseteq A_x$ and hence $B \subseteq f^{-1}[A_x]$, and thus $\mathcal{F} \geq_{RK} \mathcal{H}(\mathcal{O})$.

Conversely, to prove that $\mathcal{X}$ has the $\gamma_{\mathcal{F}}$ property, let $\mathcal{O}$ be an ω-cover of $\mathcal{X}$. By the ϵ property, we can assume that $\mathcal{O} = \langle O_n : n \in \omega \rangle$. Now if $f^{-1}(\mathcal{H}(\mathcal{O})) \subseteq \mathcal{F}$, put $n_j = f(j)$, and then for $x \in \mathcal{X}$

$$\begin{aligned} \{j : x \in O_{n_j}\} &= \{j : x \in O_{f(j)}\} \\ &= \{j : f(j) \in A_x\} \\ &= f^{-1}[A_x] \in \mathcal{F}. \end{aligned}$$

Thus $\mathcal{X} = \lim_{\mathcal{F}} O_{n_j}$, and this completes the proof. $\square$

Corollary 2.3. *If $\mathcal{F} \leq_{RK} \mathcal{G}$, then any space with $\gamma_{\mathcal{F}}$ has $\gamma_{\mathcal{G}}$.*

The following lemma is straightforward.

Lemma 2.4. *If $\mathcal{O} = \langle O_n : n \in \omega\rangle$ is a countable ω-cover of a space $\mathcal{X}$, then for any $A \in [\omega]^\omega$,*

$$A \text{ is compatible with } \mathcal{H}(\mathcal{O}), \text{ iff } \langle O_n : n \in A\rangle \text{ is an } \omega\text{-cover of } \mathcal{X}.$$

Corollary 2.5. *If $A \in [\omega]^\omega$ is compatible with $\mathcal{H}(\mathcal{O})$, where $\mathcal{O}$ is a countable ω-cover of $\mathcal{X}$, then*

$$\langle \{A\} \cup \mathcal{H}(\mathcal{O})\rangle \subseteq \mathcal{H}(\tilde{\mathcal{O}}),$$

where $\tilde{\mathcal{O}} = \langle \tilde{O}_n : n \in \omega\rangle$ is the ω-cover of $\mathcal{X}$ with $\tilde{O}_n = O_n$ if $n \in A$, and $\tilde{O}_n = \emptyset$ otherwise.

Lemma 2.6. *If $\mathcal{H} = \bigcup_{n\in\omega} \mathcal{H}(\mathcal{O}_n)$ generates a proper filter, where each $\mathcal{O}_n$ is a countable ω-cover of a σ-compact space $\mathcal{X}$, then $\mathcal{H} \subseteq \mathcal{H}(\mathcal{O})$ for some ω-cover $\mathcal{O}$ of $\mathcal{X}$.*

Proof. Write $\mathcal{X} = \bigcup_n \mathcal{C}_n$, where $\langle \mathcal{C}_n : n \in \omega\rangle$ is an increasing sequence of compact sets, and let $\mathcal{O}_n = \langle O_k^n : k \in \omega\rangle$. We build an ω-cover $\mathcal{O} = \langle O_n : n \in \omega\rangle$ by induction as follows.

Let $n_0 = 0$ and, given n_k, choose n_{k+1} large enough so that

$$(\forall x \in [\mathcal{C}_k]^{\leq k})(\exists i \in [n_k, n_{k+1}))\ x \subseteq \bigcap_{j\leq k} O_i^j.$$

This is possible by the compactness of each $\mathcal{C}_i$ and the fact that $\mathcal{H}$ is assumed to generate a proper filter. Now for $n_k \leq i < n_{k+1}$, define

$$O_i = \bigcap_{j\leq k} O_i^j.$$

So $\mathcal{O}$ is indeed an ω-cover of $\mathcal{X}$. Finally, if $\{A^0_{x_0}, \dots, A^m_{x_m}\} \subseteq \mathcal{H}$, where $x_i \in [\mathcal{X}]^{<\omega}$ and $A^i_{x_i} \in \mathcal{H}(\mathcal{O}_i)$, then let $x = \bigcup_{i\leq m} x_i$ and choose k large enough so that $k \geq \max\{|x|, m\}$ and $x \subseteq \mathcal{C}_k$. If we let $A_x = \{n : x \subseteq O_n\}$, then $A_x \in \mathcal{H}(\mathcal{O})$ and $A_x \setminus n_k \subseteq A^0_{x_0} \cap \dots \cap A^m_{x_m}$, as desired. □

We can also pull back ω-covers via a surjective map:

Lemma 2.7. *Let $f \in {}^\omega\omega$ be onto and $\mathcal{O} = \langle O_n : n \in \omega\rangle$ a countable ω-cover of $\mathcal{X}$. Then $f^{-1}(\mathcal{O}) := \langle \tilde{O}_n : n \in \omega\rangle$ is an ω-cover of $\mathcal{X}$, where $\tilde{O}_n = O_{f(n)}$ for each n, and*

$$f^{-1}(\mathcal{H}(\mathcal{O})) = \mathcal{H}(f^{-1}(\mathcal{O}))$$

Lemma 2.8. *If $\mathcal{X}$ is a σ-compact space and $\mathcal{O}$ is a countable ω-cover of $\mathcal{X}$, then $\mathcal{H}(\mathcal{O})$ is contained in some F_σ filter. In particular, $\mathcal{H}(\mathcal{O})$ is a meager filter.*

Proof. Write $\mathcal{X} = \bigcup_n \mathcal{C}_n$, where $\langle \mathcal{C}_n : n \in \omega\rangle$ is an increasing sequence of compact sets, and let $\mathcal{O} = \langle O_n : n \in \omega\rangle$. As above, let $n_0 = 0$ and, given n_k, choose n_{k+1} large enough so that

$$(\forall x \in [\mathcal{C}_k]^{\leq k^2})(\exists i \in [n_k, n_{k+1}))\ x \subseteq O_i.$$

Now for $k \in \omega$ and $x \in [\mathcal{C}_k]^{\leq k^2}$, let $a^k_x = \{i \in [n_k, n_{k+1}) : x \subseteq O_i\}$. Notice now that, given $x_0, \dots, x_{k-1} \in [\mathcal{C}_k]^{\leq k}$, we have $\bigcap_{i<k} a^k_{x_i} = a^k_{\cup_{i<k} x_i} \neq \emptyset$, and therefore $\mathcal{H}(\mathcal{O})$ is included in the F_σ filter defined by the family $\langle\langle a^k_x : x \in [\mathcal{C}_k]^{\leq k}\rangle; k \in \omega\rangle$. □

It follows in particular from Lemma 2.8 that if $\mathcal{X}$ is σ-compact and $\mathcal{O}$ is a countable ω-cover of $\mathcal{X}$, then $\mathcal{H}(\mathcal{O})$ is a meager filter.

Corollary 2.9. *Given two countable ω-covers $\mathcal{O}_1$ and $\mathcal{O}_2$ of a σ-compact space $\mathcal{X}$, there is an $f \in {}^\omega\omega$ such that $f^{-1}(\mathcal{H}(\mathcal{O}_2))$ is compatible with $\mathcal{H}(\mathcal{O}_1)$.*

Proof. As $\mathcal{H}(\mathcal{O}_1))$ is contained in some F_σ filter, it must be meager; indeed the proof of Lemma 2.8 provides a sequence of integers $\langle n_k : k \in \omega\rangle$ such that

$$(\forall A \in \mathcal{H}(\mathcal{O}_1))(\forall^\infty k) \;\; A \cap [n_k, n_{k+1}) \neq \emptyset.$$

Now we define $f \in {}^\omega\omega$ by $f^{-1}(k) = [n_k, n_{k+1})$. Clearly$f^{-1}(\mathcal{H}(\mathcal{O}_2))$ is compatible with $\mathcal{H}(\mathcal{O}_1)$. ☐

Finally, here is a well-known result ([2]) about the Rudin-Keisler ordering that we quote without proof.

Proposition 2.10. *The Rudin-Keisler ordering is $2^{\aleph_0}$ upward directed.*

3. Special filters

In this section we investigate various filters $\mathcal{F}$ for which $\mathbb{R}$ has the $\gamma_\mathcal{F}$ property; clearly Proposition 2.2, Lemma 2.8 and Proposition 2.10 combined imply the following:

Proposition 3.1. *Every σ-compact space $\mathcal{X}$ has the $\gamma_\mathcal{F}$ property for some filter $\mathcal{F}$.*

and therefore the following result of [4]:

Corollary 3.2. *There is a filter $\mathcal{F}$ for which $\mathbb{R}$ has the $\gamma_\mathcal{F}$ property.*

Now for any σ-compact space $\mathcal{X}$ and filter $\mathcal{F}$, the $\gamma_\mathcal{F}$ property is equivalent to the $\gamma''_\mathcal{F}$ property; indeed in general, the $\gamma''_\mathcal{F}$ property is equivalent to the $\gamma_\mathcal{F}$ property plus the property that given any sequence $\langle \mathcal{G}_n : n \in \omega\rangle$ of ω-covers, there is $\mathcal{H}_n \in [\mathcal{G}_n]^{<\omega}$ such that $\bigcup_n \mathcal{H}_n$ is again an ω-cover (more on this below), a property clearly satisfied by σ-compact spaces. However a subset of $\mathbb{R}$ with the $\gamma'_\mathcal{F}$ property must have strong measure zero, and therefore $\mathbb{R}$ itself cannot have this stronger property for any $\mathcal{F}$.

If one wishes to investigate what possible filters $\mathcal{F}$ could be built for which $\mathbb{R}$ has the $\gamma_\mathcal{F}$ property, it is essential that we understand the combinatorics satisfied by filters of the form $\mathcal{H}(\mathcal{O})$. They should be quite close to F_σ filters and turn out to be what is called a P^+-filter in [7], where it is proved that F_σ filters themselves are P^+-filters, and this is all we will need.

Definition 3.3. A filter $\mathcal{F}$ is called a P^+-filter if, given any decreasing sequence $\langle X_0 \,{}^*\!\supseteq X_1 \,{}^*\!\supseteq X_2 \,{}^*\!\supseteq \ldots\rangle$ from $\mathcal{F}^+$, there is an $X \in \mathcal{F}^+$ such that $X \subseteq^* X_n$ for each n.

Proposition 3.4. *Given a σ-compact space $\mathcal{X}$ and a countable ω-cover $\mathcal{O}$ of $\mathcal{X}$, then the filter $\mathcal{H}(\mathcal{O})$ is a P^+-filter.*

Proof. $\mathcal{H}(\mathcal{O})$ is in fact a *strong* P^+-filter as defined in [7], but we shall not need this stronger property. Write $\mathcal{X} = \bigcup_n \mathcal{C}_n$, where $\langle \mathcal{C}_n : n \in \omega\rangle$ is an increasing sequence of compact sets, and let $\mathcal{O} = \langle O_n : n \in \omega\rangle$. Fix a sequence $\langle X_0 \,{}^*\!\supseteq X_1 \,{}^*\!\supseteq X_2 \,{}^*\!\supseteq \ldots\rangle$ from $\mathcal{H}(\mathcal{O})^+$.

By compactness and by Lemma 2.4, we can choose for each k a finite set $s_k \in [X_k]^{<\omega}$ such that

$$(\forall x \in [\mathcal{C}_k]^{\leq k})(\exists i \in s_k)\; x \subseteq O_i.$$

Then $X = \bigcup_k s_k \subseteq^* X_n$ for each n, and by Lemma 2.4 again it is compatible with $\mathcal{H}(\mathcal{O})$, as $\langle O_n : n \in X\rangle$ is an ω-cover of $\mathcal{X}$. ☐

Now we are ready to show that certain special filters $\mathcal{F}$ will provide $\mathbb{R}$ with the $\gamma_{\mathcal{F}}$ property.

Proposition 3.5. *Assuming the Continuum Hypothesis (or Martin's Axiom, $P(\mathfrak{c})$, ...), there is a P-point p such that $\mathbb{R}$ has the γ_p property.*

Proof. Assuming the Continuum Hypothesis, list all possible countable ω-covers of $\mathbb{R}$ as $\langle \mathcal{O}_\alpha : \alpha < \aleph_1 \rangle$ and all possible sequences $\langle X_0^{\alpha\ *} \supseteq X_1^{\alpha\ *} \supseteq \ldots \rangle$ of infinite subsets of ω for $\alpha < \aleph_1$.

We build an increasing sequence of filters $\langle \mathcal{H}_\alpha : \alpha < \aleph_1 \rangle$ as follows. Having constructed compatible filters $\mathcal{H}_\beta$ each of the form $\mathcal{H}(\mathcal{O})$ for $\beta < \alpha$, choose by Lemmas 2.6 and 2.9 a countable ω-cover $\mathcal{O}$ such that $\bigcup_{\beta<\alpha} \mathcal{H}_\beta \subseteq \mathcal{H}(\mathcal{O})$ and $\mathcal{H}(\mathcal{O}) \geq_{RK} \mathcal{H}(\mathcal{O}_\alpha)$. Further, by Proposition 3.4 and Corollary 2.5, we can assume that some X_n^α either is not compatible with $\mathcal{H}(\mathcal{O})$ or else contains some infinite set $X \subseteq^* X_n^\alpha$ for each n.

Clearly $\mathcal{H} = \bigcup_{\alpha<\aleph_1} \mathcal{H}_\alpha$ is a P-point for which $\mathbb{R}$ has the $\gamma_{\mathcal{H}}$ property. □

It is shown in [4] that $\mathbb{R}^\omega$ cannot have the γ_p property whenever p is a P-point; thus we have, assuming the Continuum Hypothesis, found a filter p for which $\mathbb{R}$ has the γ_p property but $\mathbb{R}^\omega$ doesn't; and this answers a question of [4] and [5].

We can also easily get a Q-point q for which $\mathbb{R}$ has the γ_q property.

Proposition 3.6. *Assuming the Continuum Hypothesis (or Martin's Axiom, $P(\mathfrak{c})$, ...), then for each filter p there is a Q-point q such that $q \geq_{RK} p$.*

Proof. Fix p and choose a sequence $\langle r_n : n \in \omega \rangle$ of pairwise non-isomorphic selective ultrafilters; then

$$q = p \otimes \langle r_n : n \in \omega \rangle = \{A \subseteq \omega \times \omega : \{n : \{m : \langle n, m \rangle \in r_n\} \in p\}$$

is a Q-point above p in the RK ordering. □

Corollary 3.7. *Assuming the Continuum Hypothesis (or Martin's Axiom, $P(\mathfrak{c})$, ...), there is a Q-point q for which $\mathbb{R}$ has the γ_q property.*

We remind the reader that an ultrafilter that is both a P-point and a Q-point is called selective.

The next proposition slightly improves a result from [5], where selective is used instead of semiselective.

Proposition 3.8. *If s is a semiselective ultrafilter and $\mathcal{X}$ is a space with the γ_s property, then for every sequence $\langle \mathcal{G}_n : n \in \omega \rangle$ of ω-covers of $\mathcal{X}$, there are an ultrafilter $\tilde{s}$ isomorphic to s and for each $n \in \omega$, $W_n \in \mathcal{G}_n$ such that $\mathcal{X} = \lim_{\tilde{s}} W_n$.*

Proof. Consider a sequence $\langle \mathcal{G}_n : n \in \omega \rangle$ of ω-covers of a space $\mathcal{X}$. We may assume that $\mathcal{X}$ is infinite, and we therefore fix an infinite subset $\{x_n : n \in \omega\}$ of $\mathcal{X}$.

Now choose an increasing sequence $\langle \pi_n : n \in \omega \rangle$ of natural numbers such that $\lim_n(\pi_{n+1} - \pi_n) = \infty$ and for each n, redefine

$$\mathcal{G}'_n = \{ \bigcap_{j=\pi_n}^{\pi_{n+1}} V_j \setminus \{x_n\} : V_j \in \mathcal{G}_j \}$$

and let $\mathcal{G} = \bigcup_n \mathcal{G}'_n$.

Then $\mathcal{G}$ is an ω-cover of $\mathcal{X}$, and by the γ_s property there must be a sequence $\langle U_n : n \in \omega \rangle$ from $\mathcal{G}$ such that $\mathcal{X} = \lim_s U_n$.

Now let $B_k = \{n : U_n \in \mathcal{G}'_k\}$; and notice that $B_k \notin s$. Therefore there must be an $X \in s$ such that $|X \cap B_k| \leq k$ for each k. But this means that we can define a one-to-one function $\sigma : X \to \omega$ such that $\pi_k \leq \sigma(i) < \pi_{k+1}$ whenever $i \in B_k$.

Finally for $\pi_k \leq j < \pi_{k+1}$, $j = \sigma(i)$, choose $W_j \in \mathcal{G}_j$ such that $U_i \subseteq W_j$; choose W_j arbitrary from $\mathcal{G}_j$ if j is not in the range of σ.

Then $\mathcal{X} = \lim_{\tilde{s}} W_n$, where $\tilde{s} = \sigma(s)$. □

Although properties $\gamma_{\mathcal{F}}$ and $\gamma''_{\mathcal{F}}$ are invariant under isomorphic transformations of the filter $\mathcal{F}$, this does not appear to be so with the $\gamma'_{\mathcal{F}}$ property which is very unnatural while working with filters. The property $\gamma'_{\tilde{s}}$ for some isomorphic ultrafilter $\tilde{s}$ is called *strictly* $W\gamma_{T(s)}$ in [5]; there is however a more natural formulation of this property.

Definition 3.9. A space $\mathcal{X}$ is said to have the:

1. $C''(\omega)$ property if for every sequence $\langle \mathcal{G}_n : n \in \omega \rangle$ of ω-covers of $\mathcal{X}$, there is a sequence $\langle O_n \in \mathcal{G}_n : n \in \omega \rangle$ forming an ω-cover of $\mathcal{X}$.
2. Hurewicz(ω) property if for every sequence $\langle \mathcal{G}_n : n \in \omega \rangle$ of ω-covers of $\mathcal{X}$, there are $\mathcal{H}_n \in [\mathcal{G}_n]^{<\omega}$ such that $\bigcup_n \mathcal{H}_n$ is an ω-cover of $\mathcal{X}$.

These are generalizations of and actually imply the classical properties C'' and *Hurewicz* respectively; further, it is proved in [8] that a space $\mathcal{X}$ has property $C''(\omega)$ if and only if each finite power $\mathcal{X}^n$ has property C''. Note that the $\gamma''_{\mathcal{F}}$ property defined above is clearly equivalent to the $\gamma_{\mathcal{F}}$ property together with the Hurewicz(ω) property. It is not hard to show that the *strict* $W\gamma_{T(s)}$ property is nothing else than the γ_s property together with the $C''(\omega)$ property; this is a much more natural property than $\gamma'_{\mathcal{F}}$ as far as filters are concerned.

We have shown that (assuming CH) there are P-points p and Q-points q for which $\mathbb{R}$ has the γ_p and γ_q property; it is proved in [4] that if r is selective, then any space $\mathcal{X}$ with the γ_r property has property C'' and therefore every subset of $\mathbb{R}$ with the γ_r property has strong measure zero. A slight modification of this result shows that if s is a semiselective ultrafilter that is both rapid and a P-point, then any space $\mathcal{X}$ with the γ_s property has property C''.

Proposition 3.10. *If s is a semiselective ultrafilter, then every space with the γ_s property has property $C''(\omega)$.*

Proof. Let $\mathcal{X}$ be a space with the γ_s property. By Theorem 2.1 (c) of [4], every finite power of $\mathcal{X}$ has the γ_s property. According to the remark quoted above, every finite power of $\mathcal{X}$ has property C''. It then follows from Sakai's result [8] that $\mathcal{X}$ has property $C''(\omega)$. □

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