

ON SOME FINITELY BASED REPRESENTATIONS OF SEMIGROUPS

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ABSTRACT. In this paper we present a method of obtaining finitely based linear representations of possibly infinitely based semigroups.

Let $R\{x_1, x_2, \dots\}$ be a free associative algebra over a commutative ring R with the countable set of free generators $\{x_1, x_2, \dots\}$. An endomorphism α of $R\{x_1, x_2, \dots\}$ is called a *semigroup endomorphism* if $x_1\alpha, x_2\alpha, \dots$ are monomials (i.e. finite products of x_i 's). An ideal I of $R\{x_1, x_2, \dots\}$ is called an *S-ideal* if I is closed under all semigroup endomorphisms of $R\{x_1, x_2, \dots\}$. Let S be a semigroup, M a faithful module over R . A multiplicative homomorphism $\psi: S \rightarrow \text{End}_R M$ is called a (linear) *representation of S on M*. An element $p = p(x_1, x_2, \dots, x_n)$ of $R\{x_1, x_2, \dots\}$ is called an *identity* of the representation ψ if

$$p(\psi(s_1), \dots, \psi(s_n)) = 0 \text{ for all } s_1, \dots, s_n \in S.$$

It is not hard to show that the set of all identities of any representation is an *S-ideal* of $R\{x_1, x_2, \dots\}$. An *S-ideal* I of $R\{x_1, x_2, \dots\}$ is called *finitely S-generated* if there exist $p_1, \dots, p_k \in I$ such that I is the least *S-ideal* of $R\{x_1, x_2, \dots\}$ containing p_1, \dots, p_k . A representation ψ is called *finitely based* if the *S-ideal* I of its identities is finitely *S-generated*. Any set $\{p_1, \dots, p_k\} \subseteq I$ such that the elements p_1, \dots, p_k *S-generate* I is called a *finite basis of identities* of ψ .

S.M.Vovsi and N.H.Shon in [VSh] proved that every representation of a finite group over a field is finitely based. A semigroup version of this problem is open: it is not known whether every representation of a finite semigroup is finitely based.

Let A be an associative algebra over R , $S = \langle A, \cdot \rangle$ its multiplicative semigroup, and M be an R -module. Let $\phi: A \rightarrow \text{End}_R M$ be a homomorphism (R -linear, additive, and multiplicative). The kernel of ϕ , $\text{Ker}\phi$, is a two-sided ideal in A . Define a representation ψ of S on M by $m\psi(s) = m\phi(s)$ for all $m \in M$, $s \in S$. The representation ψ is a linear representation of S on M . Call the representation ψ *associated* with ϕ .

From now on we assume that the ring R and the module M have at least one of the following properties: (i) for any integer m there exist $r_1, \dots, r_m \in R$ such that $\prod_{i < j} (r_i - r_j)m = 0$ implies $m = 0$ for each $m \in M$; (ii) the ring R is finitely generated. In this paper we prove the following theorem.

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Theorem 1. *Let A be an associative ring, $S = \langle A, \cdot \rangle$ its multiplicative semigroup, M an R -module, and R and M satisfy at least one of the conditions (i) and (ii) mentioned above. Suppose $\phi: A \rightarrow \text{End}_R M$ is a homomorphism (R -linear, additive, and multiplicative). Then the associated representation ψ of S on M is finitely based if and only if there is a finite basis of identities of the ring $A/\text{Ker } \phi$.*

L.Al'shanskii and A.Kushkuley proved in [AK] that the natural representation of the (infinitely based as a semigroup) 6-element Brandt monoid is finitely based. Theorem 1 above provides a number of similar examples. So, one can use finite basis results from ring theory to get the finite basis property for some semigroup representations. For instance, it follows from Theorem 1 that the regular representation (by right multiplications) of the multiplicative semigroup S of all square $n \times n$ matrices, $n > 1$, over a finite associative ring with 1 is finitely based, while it is known that this semigroup has no finite basis of its identities, see [S]. Note that the associative algebra of all square $n \times n$ matrices, $n > 1$, over a finite associative ring is finitely based, see [L, K].

Recall that a polynomial $p = p(x_1, \dots, x_n) \in R\{x_1, x_2, \dots\}$ is called an identity of an associative algebra A over a ring R if $p(a_1, a_2, \dots, a_n) = 0$ for all $a_1, \dots, a_n \in A$. The set of all identities of the algebra A forms a T -ideal of $R\{x_1, x_2, \dots\}$ (i.e. an ideal which is closed under all endomorphisms of $R\{x_1, x_2, \dots\}$). Note that each T -ideal is also an S -ideal. Recall also that the *first linearization* in x_i of a polynomial $p = p(x_1, \dots, x_n) \in R\{x_1, x_2, \dots\}$ is the polynomial

$$(1) \quad \begin{aligned} & q(x_1, \dots, x_n, x_{n+1}) \\ &= p(x_1, \dots, x_{i-1}, x_i + x_{n+1}, x_{i+1}, \dots, x_n) \\ & - p(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n) \\ & - p(x_1, \dots, x_{i-1}, x_{n+1}, x_{i+1}, \dots, x_n). \end{aligned}$$

Call a polynomial q a *linearization* of a polynomial p if there is a finite sequence of polynomials $p = p_1, p_2, \dots, p_k = q$ such that for each $i = 2, \dots, k$ the polynomial p_i is the first linearization of p_{i-1} in some indeterminate x_l . It follows from the definition of a T -ideal that any T -ideal is closed under taking linearizations. Note that the set of all possible linearizations of a polynomial is finite up to renumberings of the indeterminates. (For a thorough account of linearizations see, for example, [R1].)

Let $r_1, \dots, r_s \in R$. Bring in a set of transformations related to r_1, \dots, r_s . Let $p = p(x_1, \dots, x_n)$ be a polynomial and let d_i be the degree of p in x_i . Write

$$p = p_0 + p_1 + \dots + p_{d_i},$$

where the polynomials p_0, \dots, p_{d_i} are homogeneous in x_i with $\deg(p_j) = j$ for all j . So,

$$(2) \quad \begin{aligned} & p(x_1, \dots, x_{i-1}, r_j x_i, x_{i+1}, \dots, x_n) \\ &= p_0(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n) \\ &+ r_j p_1(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n) \\ &+ \dots + r_j^{d_i} p_{d_i}(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n). \end{aligned}$$

Then the polynomial

$$(3) \quad \begin{aligned} & q(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n) \\ &= r_j^{d_i} p(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n) \\ & - p(x_1, \dots, x_{i-1}, r_j x_i, x_{i+1}, \dots, x_n) \end{aligned}$$

is called the *homogenization* of p in x_i relative to r_j . Call a polynomial q a *homogenization* of p if q is a result of several consecutive homogenizations of p . It follows from the definition of a T -ideal that any T -ideal is closed under taking homogenizations. Note that the set of all possible homogenizations of a polynomial is finite up to renumberings of the indeterminates (since the set r_1, \dots, r_s is finite).

Lemma 1. *A T -ideal is finitely generated (as a T -ideal) if and only if it is finitely S -generated.*

Proof. Suppose that a T -ideal I is finitely generated as a T -ideal by the finite set of polynomials $P = \{p_1, \dots, p_n\}$. This means that the set of polynomials of the form $p(f_1, \dots, f_i)$, where $p \in P$, i is an integer, and f_1, \dots, f_i are arbitrary polynomials, generates I as a two-sided ideal. Our goal is to find a finite set of polynomials $P' = \{p'_1, \dots, p'_m\}$ which S -generates I . This means in turn that the set of polynomials of the form $p'(m_1, \dots, m_j)$, where $p' \in P'$, j is an integer, and m_1, \dots, m_j are arbitrary monomials (but not arbitrary polynomials!), generates I as a two-sided ideal.

Since I is a T -ideal, I is closed under renaming indeterminates. So, without loss of generality we may assume that the polynomials p_1, \dots, p_n involve only indeterminates x_1, \dots, x_k . Let P_1 be the set of all linearizations of polynomials $\{p_1, \dots, p_n\}$ together with the polynomials p_1, \dots, p_n themselves. Since a polynomial has only finitely many linearizations, the set P_1 is finite. The definition of the linearization (1) can be rewritten as

$$\begin{aligned}
 & p(x_1, \dots, x_{i-1}, x_i + x_{n+1}, x_{i+1}, \dots, x_n) \\
 (4) \quad & = q(x_1, \dots, x_n, x_{n+1}) \\
 & - p(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n) \\
 & - p(x_1, \dots, x_{i-1}, x_{n+1}, x_{i+1}, \dots, x_n).
 \end{aligned}$$

Let $p = p_i(x_1, \dots, x_k) \in \{p_1, \dots, p_n\}$. In particular, the formula (4) implies that

$$\begin{aligned}
 & p_i(\alpha_1 m_1, \dots, \alpha_{i-1} m_{i-1}, \beta_1 w_1 + \beta_2 w_2, \alpha_{i+1} m_{i+1}, \dots, \alpha_k m_k) \\
 (5) \quad & = q(\alpha_1 m_1, \dots, \alpha_{i-1} m_{i-1}, \beta_1 w_1, \alpha_{i+1} m_{i+1}, \dots, \alpha_k m_k, \beta_2 m_2) \\
 & - p_i(\alpha_1 m_1, \dots, \alpha_{i-1} m_{i-1}, \beta_1 w_1, \alpha_{i+1} m_{i+1}, \dots, \alpha_k m_k) \\
 & - p_i(\alpha_1 m_1, \dots, \alpha_{i-1} m_{i-1}, \beta_2 w_2, \alpha_{i+1} m_{i+1}, \dots, \alpha_k m_k)
 \end{aligned}$$

for any choice of $\alpha_1, \dots, \alpha_k, \beta_1, \beta_2 \in R$ and arbitrary monomials $m_1, \dots, m_k, w_1, w_2$. Now let f_1, \dots, f_k be arbitrary polynomials. Look at $p_i(f_1, \dots, f_k)$. By induction on the total number of '+' signs in polynomials f_1, \dots, f_k one shows that the polynomial $p_i(f_1, \dots, f_k)$ can be written as a linear combination of polynomials of the form $q(\alpha_1 m_1, \dots, \alpha_l m_l)$, where the polynomial q is a linearization of p_i , l is an integer, $\alpha_1, \dots, \alpha_l \in R$, m_1, \dots, m_l are monomials and, moreover, for each $i = 1, \dots, l$ the term $\alpha_i m_i$ is an additive term of some f_j , $j = 1, \dots, k$. The base for this induction is provided by formula (5). The induction step is also implicit in formula (4). (Suppose $f_i = g_1 + g_2$ is a nontrivial sum of two polynomials. Then in (4) one needs to set $x_1 = f_1, \dots, x_{i-1} = f_{i-1}, x_i = g_1, x_{i+1} = f_{i+1}, \dots, x_k = f_k, x_{k+1} = g_2$.) So, for arbitrary polynomials f_1, \dots, f_k the polynomial $p_i(f_1, \dots, f_k)$ can be written as a linear combination of polynomials of the form $q(\alpha_1 m_1, \dots, \alpha_l m_l)$, where $q \in P_1$, $\alpha_1, \dots, \alpha_l \in R$, and m_1, \dots, m_l are monomials. Therefore, the T -ideal I is generated as a two-sided ideal by the set of all polynomials of the form $p(\alpha_1 m_1, \dots, \alpha_l m_l)$, where $p \in P_1$, $\alpha_1, \dots, \alpha_l \in R$, l is an integer, and m_1, \dots, m_l are monomials.

Case 1. The ring R and the module M satisfy condition (i). In this case, the Vandermonde argument (see for example [R1, R2]) can be applied to polynomials from P_1 to show that the T -ideal I can be generated by a finite number of completely homogeneous polynomials. (A polynomial is completely homogeneous if all its monomials have the same degree in each indeterminate.) In fact, the Vandermonde argument shows that in this case if a polynomial belongs to the T -ideal I , then all completely homogeneous components of this polynomial (i.e. the sums of all monomials (together with their coefficients) having the same degree in each indeterminate) also belong to the T -ideal I . So in this case, let P_2 be the set of all completely homogeneous components of polynomials from P_1 . Note that the set P_2 is finite. Moreover, if $p = p(x_1, \dots, x_k) \in P_2$, and $\deg_i p = d_i$, then for arbitrary $\alpha_1, \dots, \alpha_k \in R$ we have $p(\alpha_1 x_1, \dots, \alpha_k x_k) = \alpha_1^{d_1} \cdots \alpha_k^{d_k} p(x_1, \dots, x_k)$, since p is completely homogeneous. Thus the T -ideal I can be generated as a two-sided ideal by the set of all polynomials of the form $p(m_1, \dots, m_l)$ where $p \in P_2$ and m_1, \dots, m_l are monomials. In other words, the T -ideal I is S -generated by the finite set $P' = P_2$.

Case 2. The ring R is finitely generated. Let r_1, \dots, r_s be a finite system of generators for R . Since the elements r_1, \dots, r_s generate the ring R , each element $\alpha \in R$ can be written as a finite sum $\alpha = \sum \delta_i v_i$, where $\delta_i = \pm 1$ and the v_i 's are (commutative) monomials in r_1, \dots, r_s (the monomials v_i need not be distinct). Now we refine the foregoing linearization argument. We have shown that the T -ideal I is generated as a two-sided ideal by the set of all polynomials of the form $p(\alpha_1 m_1, \dots, \alpha_l m_l)$, where $p \in P_1$, $\alpha_1, \dots, \alpha_l \in R$, l is an integer, and m_1, \dots, m_l are monomials. Write each of the elements $\alpha_1, \dots, \alpha_l$ as a finite sum of monomials in r_1, \dots, r_s with coefficients ± 1 . Since the set P_1 is closed under taking linearizations, applying the linearization argument once more we get that the T -ideal I is generated as a two-sided ideal by the set of all polynomials of the form $p(\pm v_1 m_1, \dots, \pm v_l m_l)$, where $p \in P_1$, l is an integer, v_1, \dots, v_l are monomials in r_1, \dots, r_s , and m_1, \dots, m_l are monomials in x_1, x_2, \dots . Let P_2 be the set of all polynomials of the form $p((-1)^{i_1} x_1, \dots, (-1)^{i_k} x_k)$, where $p \in P_1$, and $i_1, \dots, i_k \in \{0, 1\}$. Since the set P_1 is finite, the set P_2 is also finite. So, the T -ideal I is generated as a two-sided ideal by the set of all polynomials of the form $p(v_1 m_1, \dots, v_l m_l)$, where $p \in P_2$, l is an integer, v_1, \dots, v_l are monomials in r_1, \dots, r_s , and m_1, \dots, m_l are monomials in x_1, x_2, \dots . Now let P_3 be the set of all homogenizations of polynomials from P_2 relative to r_1, \dots, r_s together with polynomials from P_2 themselves. The set P_3 is finite, since a polynomial has only finitely many homogenizations relative to a finite subset of R . Rewrite the formula (3) as

$$\begin{aligned}
 & p(x_1, \dots, x_{i-1}, r_j x_i, x_{i+1}, \dots, x_l) \\
 (6) \quad & = r_j^{d_i} p(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_l) \\
 & - q(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_l).
 \end{aligned}$$

Note that if $p \in P_3$, then $q \in P_3$ and formula (6) shows that the polynomial $p(x_1, \dots, x_{i-1}, r_j x_i, x_{i+1}, \dots, x_l)$ belongs to the S -ideal generated by P_3 . By induction on the total degree of the monomials v_1, \dots, v_l one shows that a polynomial of the form $p(v_1 m_1, \dots, v_l m_l)$, where $p \in P_2$, v_1, \dots, v_l are monomials in r_1, \dots, r_s , and m_1, \dots, m_l are monomials in x_1, x_2, \dots , can be written as an R -linear combination of polynomials of the form $p(m_1, \dots, m_j)$, where $p \in P_3$, and m_1, \dots, m_j

are monomials in x_1, x_2, \dots (To make the induction step, suppose $v_j = r_j \cdot v'_j$ is a nontrivial factorization of v_j . Then set in (6) $x_1 = v_1 m_1, \dots, x_i = v'_j m_i, \dots, x_l = v_l m_l$.) Thus, in this case the T -ideal I is generated as a two-sided ideal by the set of all polynomials of the form $p(m_1, \dots, m_l)$, where $p \in P_3$, l is an integer, and m_1, \dots, m_l are monomials in x_1, x_2, \dots . In other words, the T -ideal I is S -generated by the finite set $P' = P_3$.

Conversely, if a finite set S -generates a T -ideal, then this set also certainly generates this T -ideal as a T -ideal.

Lemma 1 is proved.

Proof of Theorem 1. Let $p = p(x_1, \dots, x_n)$ be an identity of the representation ψ . Then

$$m \cdot p(\psi(a_1), \dots, \psi(a_n)) = 0 \text{ for all } a_1, \dots, a_n \in A, m \in M$$

if and only if

$$p(\phi(a_1), \dots, \phi(a_n)) = 0 \text{ for all } a_1, \dots, a_n \in A$$

if and only if

$$\phi(p(a_1, \dots, a_n)) = 0 \text{ for all } a_1, \dots, a_n \in A$$

if and only if

$$p(a_1, \dots, a_n) \in \text{Ker } \phi \text{ for all } a_1, \dots, a_n \in A$$

if and only if

$$p = p(x_1, \dots, x_n) \text{ is an identity of } A/\text{Ker } \phi.$$

This means that the S -ideal of identities of ϕ coincides with the T -ideal of identities of $A/\text{Ker } \phi$. Now Lemma 1 finishes the proof of the theorem.

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REFERENCES

- [VSh] Vovsi S.M., Shon N.H. *Identities of stable-by-finite representations of groups*, Mat. Sbornik, **132(174)**, 1987, 578–591; English transl., Math. USSR Sb. **60** (1988), 569–581. MR **88f**:20020
- [AK] Al'shanskii L., Kushkuley A. *Identities of the natural representation of the infinitely based semigroup*, Proc. Amer. Math. Soc. **118**, 1993, 931–938. MR **93i**:20071
- [S] Sapir M.V., *Problems of Burnside type and the finite basis property in varieties of semigroups*, Izv. AN SSSR, Ser. Mat., 51(2), 319–340, 1987; transl. in Math USSR. Izv., **30(2)**, 1988, 295–314. MR **88h**:20078
- [L] Lvov I.V., *On varieties of associative rings I*, Algebra and Logic, **12** (1973), 150–167. MR **52**:10802
- [K] Kruse R.L., *Identities satisfied by a finite ring*, J. Algebra, **26**, 298–318. MR **48**:4025

- [R1] Rowen L.H., *Polynomial identities in ring theory*, Academic Press, New York, 1980. MR **82a**:16021
- [R2] Rowen L.H., *Ring theory*, v.2, Academic Press, San Diego, 1988. MR **89h**:16001

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