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RECURSIVE CONDITION FOR POSITIVITY OF THE ANGLE FOR MULTIVARIATE STATIONARY SEQUENCES

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Abstract. In this note a recursive type condition for positivity of the angle between past and future for q-variate stationary sequences is provided. In the case $q = 2$ it gives a simple different proof of a result due to Solev and Tserkhtsvadze on basicity of bivariate stationary sequences.

Let Z denote the set of all integers, C be the set of complex numbers and C^q be the Cartesian product of q copies of C. The elements of C^q will be identified with column vectors. By $L^2(C^q)$ we will denote the Hilbert space of all C^q valued functions on $(-\pi, \pi]$ that are square integrable w.r.t. the Lebesgue measure dt. Let $X = \{X^k(n) \in H : \quad n \in Z, \quad k = 1, \dots, q\}$ be a q-variate ($q < \infty$) stationary sequence in a Hilbert space H; i.e. for every $k, j = 1, \dots, q$, the inner product $(X^k(n), X^j(m))$ depends only on $n-m$. Let $M(X) = \overline{sp}\{X^k(n) : \ k = 1, \dots, q, \ n \in Z\}$, $M_+(X) = \overline{sp}\{X^k(n) : \ k = 1, \dots, q, \ n \geq 0\}$ and $M_-(X) = \overline{sp}\{X^k(n) : \ k = 1, \dots, q, \ n < 0\}$.

Definition 1. A stationary sequence X is said to be of *positive angle* iff

$$\sup\{|(x, y)| : \ x \in M_+(X), \ y \in M_-(X), \ ||x|| = ||y|| = 1\} < 1. \tag{1}$$

If a stationary sequence X has a spectral density F' w.r.t. the Lebesgue measure (cf. [4]), then any $q \times q$ matrix G satisfying $G(t)^*G(t) = F'(t)$, dt-a.e., will be called a *square root* of F'. Note that for every $x \in C^q$, $G(\cdot)x \in L^2(C^q)$. From [7], Theorem 4.5, it follows that X is of positive angle iff there exists a constant C such that for every C^q-valued trigonometric polynomial f

$$\int_{-\pi}^{\pi} |G(t)P_+ f(t)|^2 dt \leq C \int_{-\pi}^{\pi} |G(t)f(t)|^2 dt \tag{2}$$

where P_+ is the orthogonal projection in $L^2(C^q)$ onto $L^2_+(C^q) = \overline{sp}\{e^{in\cdot} e_k : \ n \geq 0, \ k = 1, \dots, q\}$.

Definition 2. The class of all $q \times q$ matrix valued functions G with $G(\cdot)x \in L^2(C^q)$ for all $x \in C^q$, such that (2) holds true for some C and all C^q-valued trigonometric polynomials f will be denoted $\mathcal{P}A(q)$.

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It is well known that if a q-variate stationary sequence X is of positive angle then the spectral measure F of X is absolutely continuous w.r.t. the Lebesgue measure. Therefore X is of positive angle iff there is a square root G of the spectral density of X such that $G \in \mathcal{P}A(q)$. If $q = 1$ then in [2] it was proved that X is of positive angle iff $F' = |G|^2$ satisfies the so-called A_2 condition, i.e. there is a constant C such that for each intrerval or its complement $I \subset (-\pi, \pi]$

$$\left(\int_I |G(t)|^2 dt\right)\left(\int_I |G(t)|^{-2} dt\right) \leq C|I|^2 \tag{3}$$

where $|I|$ is the length of I.

In 1986 Solev and Tserkhtsvadze [8] obtained necessary and sufficient conditions for positivity of the angle for full rank bivariate stationary sequences. The conditions were in terms of the coefficients of a triangular square root of F'. The main idea of [8] can be summarized as follows.

Theorem 1 ([8]). *Let* $G = \begin{bmatrix} \sigma & 0 \\ \tau & r \end{bmatrix}$. *The following conditions are equivalent:*

1. *G belongs to $\mathcal{P}A(2)$.*
2. *$|\sigma|^2$ and $|r|^2$ satisfy the A_2 condition and the mapping*

$$\mathcal{A} : f \longrightarrow (\tau P_+ \frac{1}{\sigma} - r P_+ \frac{\tau}{\sigma r}) f$$

   extends to a continuous operator in $L^2(C)$.
3. *$|\sigma|^2$ and $|r|^2$ satisfy the A_2 condition and the two mappings*

$$\mathcal{A}_- : f \longrightarrow r P_- \frac{\tau}{r} P_+ \frac{1}{\sigma} f,$$

$$\mathcal{A}_+ : f \longrightarrow r P_+ \frac{\tau}{r} P_- \frac{1}{\sigma} f$$

   extend to continuous operators in $L^2(C)$.

Having the above lemma the analytic conditions for the positivity of the angle provided by Solev and Tserkhtsvadze follow readily from Fefferman's theorem on the conjugate space of H^1 (see [1]).

Below we prove a version of the theorem above for the q-variate case. In particular this leads to a simpler proof of Theorem 1.

Recall that if X is of positive angle then X is so-called J_0 regular ([6]) and hence the range of the spectral density $F' = \frac{dF}{dt}$ is constant dt-a.e. and for each x in the range the function $|G^{\#}(\cdot)x|^2$ is integrable (see e.g. [3], Section 5), where $G^{\#}$ stands for the generalized inverse matrix. Therefore, without loss of generality we can assume that the sequence X has a density which is of full rank.

Lemma 1. *Suppose that G^{-1} exists dt-a.e. Then $G \in \mathcal{P}A(q)$ if and only if the operator GP_+G^{-1} sending $f(\cdot) \longrightarrow G(\cdot)P_+G^{-1}(\cdot)f(\cdot)$ is bounded in $L^2(C^q)$.*

Proof. If $G \in \mathcal{P}A(q)$ then from the remark preceding the lemma it follows that $G^{-1}(\cdot)x \in L^2(C^q)$ for all $x \in C^q$. Moreover the inequality (2) extends to all f such that $G(\cdot)f(\cdot) \in L^2(C^q)$. Letting in (2) $f = G^{-1}g$, where g is a trigonometric polynomial, we obtain

$$\int_{-\pi}^{\pi} |G(t)(P_+G^{-1}(t)g(t))|^2 dt \leq C \int_{-\pi}^{\pi} |g(t)|^2 dt. \tag{4}$$

So GP_+G^{-1} is bounded. Conversely, if (4) extends to $L^2(C^q)$, then letting $g = Gf$, where f is a trigonometric polynomial we obtain (2). ☐

Theorem 2. *Suppose that* $G(t) = [g_{ij}(t)]_{i,j=1,\dots,q}$ *is an invertible lower triangular matrix for every* $t \in (-\pi, \pi]$. *Denote* $G^{-1}(t) = H(t) = [h_{ij}(t)]_{i,j=1,\dots,q}$. *Let* $G_u(t)$ *and* $G_l(t)$ *be the* $(q-1)\times(q-1)$ *matrices obtained from* $G(t)$ *by removing the q-th row and the q-th column, and the 1-st row and the 1-st column, respectively. Then* $G \in \mathcal{P}A(q)$ *if and only if*

i) $G_u \in \mathcal{P}A(q-1)$,
ii) $G_l \in \mathcal{P}A(q-1)$,
iii) *the mapping* $f \longrightarrow \sum_{j=1}^{q} g_{qj}P_+h_{j1}f$ *extends to a continuous operator in* $L^2(C)$.

Proof. Note that since we are working with lower triangular matrices $H_u = (G_u)^{-1}$ and $H_l = (G_l)^{-1}$. Write G as a sum of two $q \times q$ matrices

$$\begin{aligned} G &= \begin{bmatrix} \begin{bmatrix} & G_u & \end{bmatrix} & \begin{matrix} 0 \\ \vdots \end{matrix} \\ 0 \quad \dots & 0 \end{bmatrix} + \begin{bmatrix} \begin{bmatrix} & 0 & \end{bmatrix} \\ g_{q1} \quad \dots \quad g_{qq} \end{bmatrix} \\ &= \begin{bmatrix} 0 & \dots \quad 0 \\ \begin{matrix} \vdots \\ 0 \end{matrix} & \begin{bmatrix} & G_l & \end{bmatrix} \end{bmatrix} + \begin{bmatrix} \begin{matrix} g_{11} \\ \vdots \\ g_{q1} \end{matrix} & \begin{bmatrix} & 0 & \end{bmatrix} \end{bmatrix}. \end{aligned}$$

If we write G^{-1} as

$$G^{-1} = \begin{bmatrix} \begin{matrix} h_{11} \\ \vdots \\ h_{q1} \end{matrix} & \begin{bmatrix} & 0 & \end{bmatrix} \end{bmatrix} + \begin{bmatrix} 0 & \dots \quad 0 \\ \begin{matrix} \vdots \\ 0 \end{matrix} & \begin{bmatrix} & G_l^{-1} & \end{bmatrix} \end{bmatrix},$$

then we obtain that

$$\begin{aligned} GP_+G^{-1} \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_q \end{bmatrix} &= GP_+G^{-1} \begin{bmatrix} f_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + GP_+G^{-1} \begin{bmatrix} 0 \\ f_2 \\ \vdots \\ f_q \end{bmatrix} \\ &= \begin{bmatrix} G_uP_+ \begin{bmatrix} h_{11}f_1 \\ \vdots \\ h_{q-1,1}f_1 \end{bmatrix} \\ \sum_{j=1}^{q} g_{qj}P_+h_{j1}f_1 \end{bmatrix} + \begin{bmatrix} 0 \\ G_lP_+G_l^{-1} \begin{bmatrix} f_2 \\ \vdots \\ f_q \end{bmatrix} \end{bmatrix}. \end{aligned}$$

Therefore GP_+G^{-1} is bounded in $L^2(C^q)$ iff

A) $G_lP_+G_l^{-1}$ is bounded in $L^2(C^{q-1})$,
B) $f \to G_uP_+ \begin{bmatrix} h_{11}f \\ \vdots \\ h_{q-1,1}f \end{bmatrix}$ is bounded from $L^2(C)$ to $L^2(C^{q-1})$ and
C) $f \to \sum_{j=1}^{q} g_{qj}P_+h_{j1}f$ is bounded in $L^2(C)$.

If now we write

$$G^{-1}=\begin{bmatrix}\begin{bmatrix} & & \\ & G_u^{-1} & \\ & & \end{bmatrix} & \begin{matrix}0\\ \vdots\end{matrix}\\ 0\ \ \ldots & 0\end{bmatrix}+\begin{bmatrix}\begin{bmatrix} & & \\ & 0 & \\ & & \end{bmatrix}\\ h_{q1}\ \ \ \ldots\ \ \ h_{qq}\end{bmatrix},$$

then

$$GP_+G^{-1}\begin{bmatrix}f_1\\ \vdots\\ f_q\end{bmatrix}=\begin{bmatrix}G_uP_+G_u^{-1}\begin{bmatrix}f_1\\ \vdots\\ f_{q-1}\end{bmatrix}\\ 0\end{bmatrix}$$

$$+\begin{bmatrix}0\\ \vdots\\ 0\\ \sum_{j=1}^{q} g_{qj}P_+\phi_j\end{bmatrix}+\begin{bmatrix}0\\ \vdots\\ 0\\ g_{qq}P_+\sum_{j=1}^{q} h_{qj}f_j\end{bmatrix}$$

where ϕ_j is the j-th coordinate of $G_u^{-1}\begin{bmatrix}f_1\\ \vdots\\ f_{q-1}\end{bmatrix}$. This shows that if GP_+G^{-1} is bounded in $L^2(C^q)$ then $G_uP_+G_u^{-1}$ is bounded in $L^2(C^{q-1})$, which combined with A) and C) and Lemma 1 proves the necessity of conditions i) - iii).

Conversely, suppose that the conditions i) - iii) hold. Since $G_uP_+G_u^{-1}$ is bounded, its restriction

$$G_uP_+G_u^{-1}\begin{bmatrix}f\\ 0\\ \vdots\\ 0\end{bmatrix}=G_uP_+\begin{bmatrix}h_{11}f\\ \vdots\\ h_{q-1,1}f\end{bmatrix}$$

is also bounded and we conclude that the conditions A), B) and C) are satisfied. Thus GP_+G^{-1} is bounded, which in view of Lemma 1 completes the proof. □

Solev and Tserkhtsvadze's Theorem 1 is a simple consequence of the theorem above. It is enough to note that under the assumptions of Theorem 1

$$G^{-1}=\begin{bmatrix}\frac{1}{\sigma} & 0\\ -\frac{\tau}{r\sigma} & \frac{1}{r}\end{bmatrix}.$$

For the equivalence of the second and third conditions observe first that if $\sigma, r\in\mathcal{P}A(1)$ then $rP_-\frac{1}{r}=1-rP_+\frac{1}{r}$ and similarly $\sigma P_-\frac{1}{\sigma}$ is bounded and so $\mathcal{A}_-=rP_-\frac{1}{r}\mathcal{A}$ and $\mathcal{A}_+=-\sigma P_-\frac{1}{\sigma}\mathcal{A}$ are bounded. Conversely, boundedness of $\mathcal{A}_+$ and $\mathcal{A}_-$ clearly implies the boundedness of $\mathcal{A}$ for $\mathcal{A}=\mathcal{A}_- - \mathcal{A}_+$.

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