

## ON ASYMMETRY OF TOPOLOGICAL CENTERS OF THE SECOND DUALS OF BANACH ALGEBRAS

F. GHAHRAMANI, J. P. MCCLURE, AND M. MENG

(Communicated by Theodore W. Gamelin)

ABSTRACT. Let  $\mathfrak{A}$  be a Banach algebra with a bounded approximate identity and let  $Z_1(\mathfrak{A}^{**})$  and  $Z_2(\mathfrak{A}^{**})$  be the left and right topological centers of  $\mathfrak{A}^{**}$ . It is shown that i)  $\mathfrak{A}^*\mathfrak{A} = \mathfrak{A}\mathfrak{A}^*$  is not sufficient for  $Z_1(\mathfrak{A}^{**}) = Z_2(\mathfrak{A}^{**})$ ; ii) the inclusion  $\widehat{\mathfrak{A}}Z_1(\mathfrak{A}^{**}) \subseteq \widehat{\mathfrak{A}}$  is not sufficient for  $Z_2(\mathfrak{A}^{**})\widehat{\mathfrak{A}} \subseteq \widehat{\mathfrak{A}}$ ; iii)  $Z_1(\mathfrak{A}^{**}) = Z_2(\mathfrak{A}^{**}) = \widehat{\mathfrak{A}}$  is not sufficient for  $\mathfrak{A}$  to be weakly sequentially complete. These results answer three questions of Anthony To-Ming Lau and Ali Ülger.

Suppose that  $\mathfrak{A}$  is a Banach algebra and let  $\mathfrak{A}^*$  be the dual space of  $\mathfrak{A}$ . Then  $\mathfrak{A}^*$  can be made into a Banach  $\mathfrak{A}$ -bimodule as follows: for  $f \in \mathfrak{A}^*$ ,  $a \in \mathfrak{A}$ ,  $fa$  and  $af$  are defined by

$$\begin{aligned}\langle fa, b \rangle &= \langle f, ab \rangle, \\ \langle af, b \rangle &= \langle f, ba \rangle \quad (b \in \mathfrak{A})\end{aligned}$$

where  $\langle \cdot, \cdot \rangle$  is used for the dual pairing between elements of  $\mathfrak{A}^*$  and  $\mathfrak{A}$ .

Let  $\mathfrak{A}^*\mathfrak{A} = \{fa : f \in \mathfrak{A}^*, a \in \mathfrak{A}\}$  and  $\mathfrak{A}\mathfrak{A}^* = \{af : a \in \mathfrak{A}, f \in \mathfrak{A}^*\}$ . The dual space  $\mathfrak{A}^*$  is said to factor on the left (resp. right) if  $\mathfrak{A}^* = \mathfrak{A}^*\mathfrak{A}$  (resp.  $\mathfrak{A}^* = \mathfrak{A}\mathfrak{A}^*$ ). In a recent paper [5] Anthony To-Ming Lau and Ali Ülger have obtained various necessary and sufficient conditions for factoring of  $\mathfrak{A}^*$ . Here we answer three questions left open in [5].

The second dual space  $\mathfrak{A}^{**}$  of a Banach algebra  $\mathfrak{A}$  admits two Banach algebra products known as first (left) and second (right) Arens products. Each of these products extends the product of  $\mathfrak{A}$  as canonically embedded in  $\mathfrak{A}^{**}$  (see [1] or [3]). We briefly recall the definition of these products. For  $m, n \in \mathfrak{A}^{**}$ , their first (left) Arens product indicated by  $m \square n$  is given by

$$\langle m \square n, f \rangle = \langle m, nf \rangle \quad (f \in \mathfrak{A}^*),$$

where  $nf \in \mathfrak{A}^*$  is defined by

$$\langle nf, a \rangle = \langle n, fa \rangle \quad (a \in \mathfrak{A}),$$

where  $fa$  is as defined earlier.

---

Received by the editors December 5, 1996.

1991 *Mathematics Subject Classification*. Primary 46H99.

*Key words and phrases*. Arens product, topological center, Banach module.

The first author was supported by NSERC grant OGP 003664 and the second author by NSERC grant A8069.

The second Arens product of elements  $m, n \in \mathfrak{A}^{**}$  indicated by  $m \diamond n$  is defined by

$$\langle m \diamond n, f \rangle = \langle n, fm \rangle \quad (f \in \mathfrak{A}^*),$$

where  $fm$  is an element of  $\mathfrak{A}^*$  defined by

$$\langle fm, a \rangle = \langle m, af \rangle \quad (a \in \mathfrak{A}).$$

Again we note that  $af$  is as defined earlier.

For each  $n \in \mathfrak{A}^{**}$ , the mapping  $m \mapsto m \square n$  (resp.  $m \mapsto n \diamond m$ ) is weak\*-weak\* continuous. However for certain  $n$ , the mapping  $m \mapsto n \square m$  (resp.  $m \mapsto n \diamond m$ ) may fail to be weak\* continuous. Due to this lack of symmetry the left (resp. right) topological center  $Z_1(\mathfrak{A}^{**})$  (resp.  $Z_2(\mathfrak{A}^{**})$ ) of  $\mathfrak{A}^{**}$  is defined by

$$Z_1(\mathfrak{A}^{**}) = \{m \in \mathfrak{A}^{**} : n \mapsto m \square n \text{ is weak*}-\text{weak* continuous}\},$$

$$Z_2(\mathfrak{A}^{**}) = \{m \in \mathfrak{A}^{**} : n \mapsto n \diamond m \text{ is weak*}-\text{weak* continuous}\}.$$

It follows easily from the definition of  $Z_1(\mathfrak{A}^{**})$ (resp.  $Z_2(\mathfrak{A}^{**})$ ) that  $m \in Z_1(\mathfrak{A}^{**})$  (resp.  $m \in Z_2(\mathfrak{A}^{**})$ ) if and only if  $m \square n = m \diamond n$  (resp.  $n \diamond m = n \square m$ ) for all  $n \in \mathfrak{A}^{**}$ . In general  $Z_1(\mathfrak{A}^{**})$  and  $Z_2(\mathfrak{A}^{**})$  need not be equal (see [5], Example 2.5 and Remark 6.j, p.1211), but they both contain  $\hat{\mathfrak{A}}$  (= the image of  $\mathfrak{A}$  in  $\mathfrak{A}^{**}$  under the canonical mapping).

Anthony To-Ming Lau and Ali Ülger have shown in [5] that if  $\mathfrak{A}^*$  factors on one side but not on the other, then  $Z_1(\mathfrak{A}^{**}) \neq Z_2(\mathfrak{A}^{**})$ . In connection to this result they have asked:

**Question.** Suppose that  $\mathfrak{A}$  has a bounded approximate identity. Does the equality  $\mathfrak{A}^*\mathfrak{A} = \mathfrak{A}\mathfrak{A}^*$  imply that  $Z_1(\mathfrak{A}^{**}) = Z_2(\mathfrak{A}^{**})$  ?

We answer this question in the negative. In fact our counter example is a modification of one of the examples in [5].

For a Banach algebra  $\mathfrak{A}$  we let  $\mathfrak{A}^\sharp$  be the unitization of  $\mathfrak{A}$ . Thus  $\mathfrak{A}^\sharp = \mathfrak{A} \oplus \mathbf{C}$ , where  $\mathbf{C}$  is the field of complex numbers. For  $a, b \in \mathfrak{A}, \alpha, \beta \in \mathbf{C}$  we have  $\| (a, \alpha) \| = \| a \| + | \alpha |$  and  $(a, \alpha)(b, \beta) = (ab + \alpha b + \beta a, \alpha\beta)$ .

**1. Lemma.** For any Banach algebra  $\mathfrak{A}, Z_1((\mathfrak{A}^\sharp)^{**}) = Z_1(\mathfrak{A}^{**}) \oplus \mathbf{C}$  and a similar result holds for the right topological centers.

*Proof.* First we note that the two Banach algebras  $(\mathfrak{A}^\sharp)^{**}$  and  $(\mathfrak{A}^{**})^\sharp$  are isomorphic, when they both have the first Arens product. Then the rest is straightforward from the definition of topological centers. □

The following answers question (6d) in [5].

**2. Theorem.** There exists a Banach algebra  $\mathfrak{A}$  possessing a bounded approximate identity and satisfying  $\mathfrak{A}^*\mathfrak{A} = \mathfrak{A}\mathfrak{A}^* = \mathfrak{A}^*$ , but  $Z_1(\mathfrak{A}^{**}) \neq Z_2(\mathfrak{A}^{**})$ .

*Proof.* It is known that there exists a Banach algebra  $\mathfrak{B}$  having a bounded approximate identity for which  $Z_1(\mathfrak{B}^{**}) \neq Z_2(\mathfrak{B}^{**})$  (see [5]). Then for  $\mathfrak{A} = \mathfrak{B}^\sharp$ , by Lemma 1 we have  $Z_1(\mathfrak{A}^{**}) \neq Z_2(\mathfrak{A}^{**})$ . However,  $\mathfrak{A}^*\mathfrak{A} = \mathfrak{A}\mathfrak{A}^* = \mathfrak{A}^*$ , since  $\mathfrak{A}$  is unital. □

For the origin of the following question see [5, question 6e].

**Question.** Does the inclusion  $\hat{\mathfrak{A}}Z_1(\mathfrak{A}^{**}) \subseteq \hat{\mathfrak{A}}$  imply  $Z_2(\mathfrak{A}^{**})\hat{\mathfrak{A}} \subseteq \hat{\mathfrak{A}}$  ?

To answer this question, we first recall that if  $\mathfrak{A}$  is a Banach algebra then  $\mathfrak{A}^{op}$  is the algebra obtained by reversing the order of multiplication in  $\mathfrak{A}$ ; i.e.,  $\mathfrak{A}^{op}$  has the product  $\circ$  given by  $a \circ b = ba$ , for every  $a, b \in \mathfrak{A}$ .

**3. Theorem.** *There exists a Banach algebra  $\mathfrak{A}$  with a bounded approximate identity for which  $\hat{\mathfrak{A}}Z_1(\mathfrak{A}^{**}) \subset \hat{\mathfrak{A}}$ , but  $Z_2(\mathfrak{A}^{**})\hat{\mathfrak{A}}$  is not contained in  $\hat{\mathfrak{A}}$ .*

*Proof.* There exists a Banach algebra  $\mathfrak{B}$ , with a bounded approximate identity, such that  $Z_2(\mathfrak{B}^{**}) = \hat{\mathfrak{B}}$  but  $Z_1(\mathfrak{B}^{**})$  is larger than  $\hat{\mathfrak{B}}$  (see [5]). It is well known that switching to the opposite multiplication in a Banach algebra results in interchanging the first and the second Arens products in its second dual space. From this and by Lemma 1, for  $\mathfrak{A} = (\mathfrak{B}^{op})^\#$  we have

$$Z_1(\mathfrak{A}^{**}) = \mathfrak{B}^{op} \oplus \mathbf{C} = \hat{\mathfrak{A}}$$

and

$$Z_2(\mathfrak{A}^{**}) = Z_1(\mathfrak{B}^{**}) \oplus \mathbf{C}.$$

Hence

$$\hat{\mathfrak{A}}Z_1(\mathfrak{A}^{**}) = (\mathfrak{B}^{op} \oplus \mathbf{C})(\mathfrak{B}^{op} \oplus \mathbf{C}) = \mathfrak{B}^{op} \oplus \mathbf{C} = \hat{\mathfrak{A}},$$

and

$$Z_2(\mathfrak{A}^{**})\hat{\mathfrak{A}} = (Z_1(\mathfrak{B}^{**}) \oplus \mathbf{C})(\mathfrak{B}^{op} \oplus \mathbf{C}) = Z_1(\mathfrak{B}^{**}) \oplus \mathbf{C}.$$

But  $Z_1(\mathfrak{B}^{**}) \oplus \mathbf{C}$  is larger than  $\hat{\mathfrak{A}}$ , since  $Z_1(\mathfrak{B}^{**})$  is larger than  $\hat{\mathfrak{B}}$ . □

In [5], Lau and Ülger ask (question 6j): if  $Z_1(\mathfrak{A}^{**}) = Z_2(\mathfrak{A}^{**}) = \hat{\mathfrak{A}}$ , must  $\mathfrak{A}$  be weakly sequentially complete?

We give an example of a Banach algebra  $\mathfrak{A}$  with identity, and such that  $Z_1(\mathfrak{A}^{**}) = Z_2(\mathfrak{A}^{**}) = \hat{\mathfrak{A}}$ , but with  $\mathfrak{A}$  not weakly sequentially complete. First we describe a general construction; this construction is known and has been used by workers in the area of Automatic Continuity (see [2], p. 647).

Let  $A$  be a Banach algebra, and let  $M$  be a Banach  $\mathfrak{A}$ -bimodule. Then  $\mathfrak{A} = M \oplus A$ , with norm  $\| (m, a) \| = \| m \| + \| a \|$ , and product  $(m, a)(n, b) = (mb + an, ab)$  is a Banach algebra. If  $A$  has a bounded approximate identity  $(e_j)$  and if  $M$  factors ( $M = MA = AM$ ), then  $(0, e_j)$  is a bounded approximate identity for  $\mathfrak{A}$ . In particular, if  $A$  has an identity  $e$ , and  $M$  is unital, then  $(0, e)$  is an identity for  $\mathfrak{A}$ . Note that  $\mathfrak{A}^* = M^* \times A^*$  (with norm  $\| (s, f) \| = \max \{ \| s \|, \| f \| \}$ ), and that  $\mathfrak{A}^{**} = M^{**} \oplus A^{**}$ . It is routine to check that  $\mathfrak{A}$  is weakly sequentially complete if and only if each of  $A$  and  $M$  is weakly sequentially complete. Also, it can be checked that the first Arens product on  $\mathfrak{A}^{**}$  satisfies

$$(1) (\mu, \alpha) \square (\nu, \beta) = (\mu \square \beta + \alpha \square \nu, \alpha \square \beta), \text{ for } \mu, \nu \in M^{**} \text{ and } \alpha, \beta \in A^{**}.$$

Here, in all cases,  $\square$  has been used to denote that extension of a bilinear operation to second duals, which is weak\*-weak\* continuous in the left-hand argument; for instance,  $(\mu, \beta) \mapsto \mu \square \beta$  denotes the extension to  $M^{**} \times A^{**}$  of the right module action  $(m, b) \mapsto mb$  of  $A$  on  $M$  which is weak\*-weak\* continuous in  $\mu$ .

Now we identify  $Z_1(\mathfrak{A}^{**})$ . Since a net  $(\nu_j, \beta_j)$  in  $\mathfrak{A}^{**}$  converges weak\* to  $(\nu, \beta)$  if and only if  $\nu_j \rightarrow \nu$  weak\* in  $M^{**}$  and  $\beta_j \rightarrow \beta$  weak\* in  $A^{**}$ , it is straightforward to check that  $(\mu, \alpha) \in Z_1(\mathfrak{A}^{**})$  if and only if

- (2)  $\alpha \in Z_1(A^{**})$ ;
- (3)  $\nu \mapsto \alpha \square \nu$  is weak\*-weak\* continuous on  $M^{**}$ ; and
- (4)  $\beta \mapsto \mu \square \beta$  is weak\*-weak\* continuous  $A^{**} \rightarrow M^{**}$ .

Now suppose  $Z_1(A^{**}) = \hat{A}$ . Then (2) implies  $\alpha \in \hat{A}$ , and it is then straightforward to verify that (3) holds. So in this case,  $(\mu, \alpha) \in Z_1(\mathfrak{A}^{**})$  if and only if  $\alpha \in \hat{A}$  and (4) holds.

Finally, we consider the particular case  $M = A^*$ . Recall that we are assuming  $A$  has a bounded approximate identity. To ensure that  $\mathfrak{A}^{**}$  has a bounded approximate identity, we suppose that  $A^*$  factors, in the sense of [5]. Note that the case that  $A$  has an identity is included.

Now we have  $M^* = A^{**}$  and  $M^{**} = A^{***}$ , and the meanings of the various bilinear operations have to be carefully distinguished; however, it can be shown that, for  $\mu \in M^{**}$ ,  $\beta \in A^{**}$ , and  $\alpha \in M^* = A^{**}$ ,

$$\langle \mu \square \beta, \alpha \rangle = \langle \mu, \beta \square \alpha \rangle;$$

that is, the bilinear map  $(\mu, \beta) \mapsto \mu \square \beta$  on  $M^{**} \times A^{**}$  to  $M^{**}$  is the right dual module action of  $A^{**}$  on  $A^{***} = M^{**}$  corresponding to the first Arens product on  $A^{**}$ . Suppose (4) holds, and take  $\alpha = E$ , a right identity for the first Arens product in  $A^{**} = M^*$ . If  $\beta_j \rightarrow \beta$  weak\* in  $A^{**}$ , then by (4),

$$\langle \mu, \beta_j \rangle = \langle \mu, \beta_j \square E \rangle = \langle \mu \square \beta_j, E \rangle \rightarrow \langle \mu \square \beta, E \rangle = \langle \mu, \beta \rangle.$$

That is, (4) implies that  $\mu$  is a weak\*-continuous linear functional on  $A^{**}$ , so that  $\mu \in (A^*)^\wedge = \hat{M}$ . The converse is easy, so we have proved the conclusion  $Z_1(\mathfrak{A}^{**}) = \hat{M} \oplus \hat{A} = \hat{\mathfrak{A}}$  in the following proposition; the arguments for  $Z_2(\mathfrak{A}^{**})$  are quite similar.

**6. Proposition.** *Let  $A$  be a Banach algebra with a bounded approximate identity.*

(1) *Assume  $A^*$  factors (in the sense of [5]). Then  $\mathfrak{A} = A^* \oplus A$  has a bounded approximate identity.*

(2) *Assume  $Z_1(A^{**}) = Z_2(A^{**}) = \hat{A}$ . Then  $Z_1(\mathfrak{A}^{**}) = Z_2(\mathfrak{A}^{**}) = \hat{\mathfrak{A}}$ .*

(3)  *$\mathfrak{A}$  is weakly sequentially complete if and only if each of  $A$  and  $A^*$  is weakly sequentially complete.*

If we take  $A = l^1(\mathbb{Z})$  (the group algebra of the discrete group of integers), then  $Z_1(A^{**}) = Z_2(A^{**}) = \hat{A}$  (see [4]), and all the assumptions above are satisfied; in particular,  $A$  has an identity. Also,  $A^* = l^\infty(\mathbb{Z})$  is not weakly sequentially complete. So  $\mathfrak{A} = A^* \oplus A$  has an identity, satisfies  $Z_1(\mathfrak{A}^{**}) = Z_2(\mathfrak{A}^{**}) = \hat{\mathfrak{A}}$ , and  $\mathfrak{A}$  is not weakly sequentially complete. That answers (negatively) question 6j of [5].

#### REFERENCES

- [1] R. Arens, 'The adjoint of a bilinear operation', Proc. Amer. Math. Soc. 2 (1951), 839 - 848. MR **13**:659f
- [2] H.G. Dales, 'The uniqueness of the functional calculus', Proc. London Math. Soc. (3) 27 (1973), 638-648. MR **48**:12062
- [3] J. Duncan and S.A.R. Hosseiniun, 'The second dual of a Banach algebra', Proc. Roy. Soc. Edinburgh Sect. A84 (1979), 309 - 325. MR **81f**:46057
- [4] A.T. Lau and V. Losert, On the second conjugate algebra of  $L^1(G)$  of a locally compact group, J. London Math Soc. 37 (1988), 464-470. MR **89e**:43007
- [5] A.T. Lau and A. Ülger, 'Topological centers of certain dual algebras', Trans. Amer. Math. Soc. vol. 348, no. 3 (1996), 1191 - 1212. MR **96h**:43003

DEPARTMENT OF MATHEMATICS AND ASTRONOMY, UNIVERSITY OF MANITOBA, WINNIPEG R3T 2N2, CANADA