AN ANNULUS DIFFEOMORPHISM
WITH NON-DENJOY MINIMAL SETS

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ABSTRACT. We construct an annulus diffeomorphism with the property that a countably dense set of irrational rotation numbers are represented only by pseudocircles on which the diffeomorphism acts minimally but is not semiconjugate to rigid rotation on the circle. This answers a question of Boyland about whether such behavior is possible only at the maximum or minimum of the rotation set.

INTRODUCTION

We begin with two notions, rotation number and rotation set, central to the study of annulus homeomorphisms. Given an annulus homeomorphism, the rotation number, ρ, of a point is the limiting average angular displacement of the point under iteration, if the limit exists. For an introduction to rotation numbers see, among others, [D].

The rotation set, $\rho(f)$, of a homeomorphism f is the set of numbers realized as rotation numbers. For each member of the rotation set we can ask about the smallest closed invariant sets (minimal sets) with this rotation number. In the case of area preserving monotone twist maps the Aubry-Mather theorem provides an answer to this question. Monotone twist maps are those where points closer to the inner circle of the annulus have a smaller (or larger) angular displacement than points closer to the outer circle of the annulus. (For a complete definition see [Bd].) We state here only the part of the theorem concerning irrational rotation numbers.

Theorem (Aubry-Mather). *Let f be an area preserving monotone twist map of the annulus. Then for all $\alpha \in \rho(f)$ with $\alpha \notin \mathbb{Q}$, f has a monotone, minimal set X with $\rho(f|X) = \alpha$ and $f|X$ is semiconjugate to rigid rotation on the circle.*

An example of Handel [H] shows that the Aubry-Mather theorem does not extend when the monotone twist condition is removed. Handel constructs a C^∞ area preserving diffeomorphism of the annulus with an irrational in the rotation set whose only minimal set (for this irrational) is the pseudocircle [B1], [B2], [F]. This minimal set is not monotone, and the diffeomorphism restricted to the pseudocircle is *not* semiconjugate to rigid rotation on the circle.

Boyland notes that in Handel's construction the irrational rotation number is the maximum of the rotation set and poses Question 3.3 in [Bd].

Question (Boyland). Let f be an area preserving homeomorphism of the annulus. For each irrational, α, which lies between the rotation numbers of f restricted to the boundary circles, does f have a minimal set X so that $f|X$ is semiconjugate to rigid rotation by α on the circle?

We show that Handel's example can be modified so that the irrational α, for which the pseudocircle is the only minimal set, is between the boundary rotation numbers. This provides a negative answer to Boyland's question.

Further, we show that this behavior need not be isolated to a single rotation number.

Theorem. *There exists a C^∞, area preserving diffeomorphism of the annulus with the property that a countably infinite dense set of irrational rotation numbers are represented only by pseudocircles on which the dynamics are minimal but not semiconjugate to rigid rotation on the circle.*

The construction

We remain consistent with the notation used in [H] and omit here some details of the construction found there. Also, we will work with the universal cover of the annulus $[-1, 1] \times \mathbb{R}/\mathbb{Z}$, as reflected in the figures.

Handel's example involves a sequence of nested annuli embedded in $\mathbb{R}^2$, and an associated sequence of functions. The first annulus A_1 is divided into p_1 pieces, and f_1 is defined as the rotation which permutes these pieces: $f_1(x, y) = (x + \frac{1}{p_1}, y)$. (See Figure 1.)
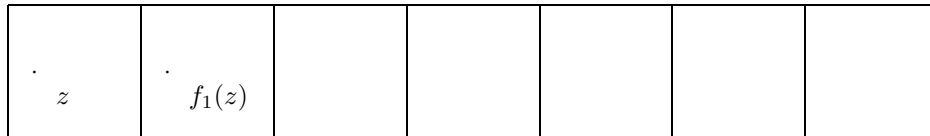

FIGURE 1. $p_1 = 7$

Next we embed a circle C_2 in A_1 in a homotopically nontrivial way. (See Figure 2.) Each embedding is done "crookedly" as defined in [B1]. This crookedness, at this and future stages of the construction, is responsible for the "pathological" nature of the intersection of the annuli, the pseudocircle. We also require that C_2 be invariant under f_1. The annulus A_2 is created by thickening the circle C_2.
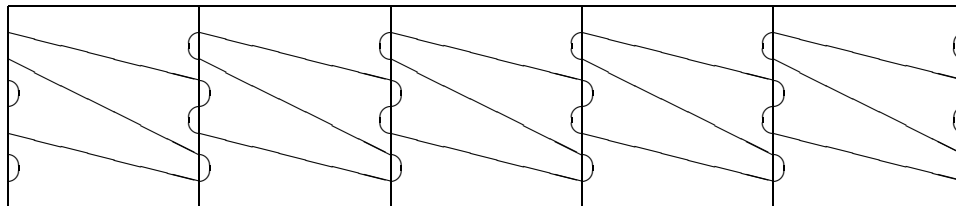

FIGURE 2. A crooked embedding of C_n in A_{n-1}

Then A_2 is divided into $p_2 \in \mathbb{Z}$ pieces and f_2 is defined (on A_2) as the composition of f_1 with the rotation that permutes the p_2 pieces of A_2. We modify Handel's

construction on the definition of f_2 on the *lower* part of A_1 and A_2. (See Figure 3. Note that in the figures here and to follow, the scale is misrepresented and the crookedness has been suppressed.)

For any circle C we define $U(C)$ as the unbounded component of the complement and $L(A)$ as the bounded component of the complement, or in the universal cover that part of $\mathbb{R}/\mathbb{Z} \times [-1, 1]$ which is above or below C, respectively. Also, we define $R_\alpha(x, y) = (x + \alpha, y)$ as rigid rotation by α.

$$f_2 = \begin{cases} f_1 & \text{on} \quad U(C_2 \times \{1\}) \\ R_{\phi_1(t)} & \text{on} \quad C_2 \times [\frac{1}{2}, 1] \quad \text{where} \quad \phi_1 : [\frac{1}{2}, 1] \to [\alpha_2, \alpha_1] \\ R_{\alpha_2} & \text{on} \quad C_2 \times [-\frac{1}{2}, \frac{1}{2}] \\ R_{\phi_2(t)} & \text{on} \quad C_2 \times [-1, \frac{-1}{2}] \quad \text{where} \quad \phi_2 : [-1, \frac{-1}{2}] \to [\alpha_2 + \frac{1}{p_2}, \alpha_2] \\ R_{\alpha_2 + 1/p_2} & \text{on} \quad L(C_2 \times \{-1\}) \end{cases}$$

Here we set $\alpha_2 = \alpha_1 + \frac{1}{p_2}$, and the rotation R_{α_2} is a rotation in the annular structure of A_2.
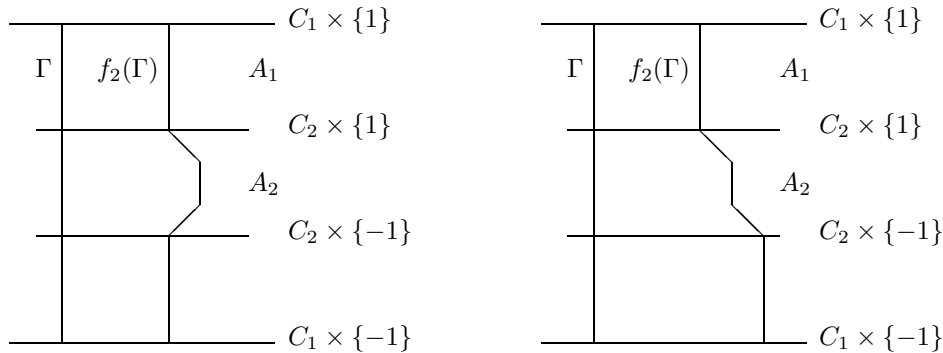

FIGURE 3. Handel's example(left), Handel's example modified(right)

The construction continues in a like manner. A circle C_3 is crookedly embedded in A_2 in a way that leaves it invariant under f_2. This circle is thickened to form the annulus A_3 which is then divided into p_3 pieces. The modified f_3 is defined (on A_3) as the composition of f_2 with the rotation in A_3 that permutes the p_3 pieces. Below A_3 is where the map is modified and differs from Handel's construction.

Continuing, each embedding is accomplished by embedding a circle C_n in the annulus A_{n-1} of the previous stage and then thickening C_n to form the next annulus A_n. Each annulus is divided into p_n pieces. Then the next annulus is crookedly embedded with the condition that it be invariant under the rotation in the previous annulus. The resulting diffeomorphism is the infinite composition of the f_n.

The map f_n is defined as follows. (See Figure 4.) Again, $\alpha_n = \alpha_{n-1} + \frac{1}{p_n}$, each rotation R is in the appropriate annular structure, and the ϕ_i are diffeomorphisms.

$$
f_n = \begin{cases}
f_{n-1} & \text{on} \quad U(\{C_n \times \{1\}\}) \\
R_{\phi_1(t)} & \text{on} \quad C_n \times [\frac{1}{2}, 1] \quad \text{where} \quad \phi_1 : [\frac{1}{2}, 1] \to [\alpha_n, \alpha_{n-1}] \\
R_{\alpha_n} & \text{on} \quad C_n \times [\frac{-1}{2}, \frac{1}{2}] \\
R_{\phi_2(t)} & \text{on} \quad C_n \times [-1, \frac{-1}{2}] \quad \text{where} \quad \phi_2 : [-1, \frac{-1}{2}] \to [\alpha_n + \frac{1}{p_n}, \alpha_n] \\
R_{\alpha_n + 1/p_n} & \text{on} \quad U(C_{n-1} \times \{\frac{-1}{2}\}) \cap L(C_n \times \{-1\}) \\
R_{\phi_3(t)} & \text{on} \quad C_{n-1} \times [-1, \frac{-1}{2}] \quad \text{where} \quad \phi_3 : [-1, -1/2] \\
& \qquad\qquad\qquad\qquad\qquad \to [\alpha_{n-1} + \frac{1}{p_{n-1}}, \alpha_n + \frac{1}{p_n}] \\
f_{n-1} & \text{on} \quad L(C_{n-1} \times \{-1\})
\end{cases}
$$

We show later that each f_n is area preserving and C^∞. Given ϵ_n we can choose p_n sufficiently large so that f_n and f_{n-1} are ϵ_n close in the C^n topology. By sending ϵ_n rapidly to 0 we guarantee that the limiting map f restricted to P is C^∞.
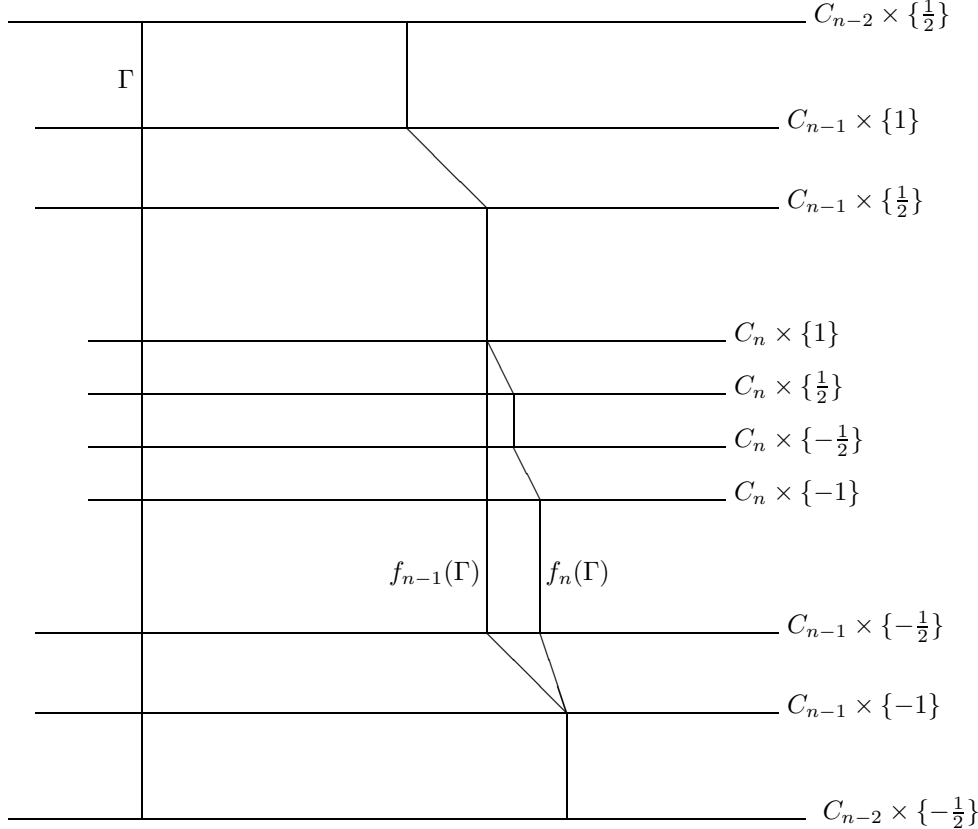

FIGURE 4. Induction step, $f_{n-1} \longrightarrow f_n$

We point out that the invariant set P (the intersection of the annuli) and the limiting map f restricted to this invariant set P is exactly as defined in Handel's example. So the topological properties ascribed to P, the pseudocircle, remain intact. Also, P remains a minimal set for f, and $f|P$ has a well defined irrational rotation number even though $f|P$ is not semi-conjugate to a rotation of S^1. What

remains for us to establish is the greater rotation number below the pseudocircle and the area preservation and differentiability of the limiting map on the entire annulus.

Off the pseudocircle the annulus A_1 is divided into a sequence of smooth topological annuli and on each of these f is either a finite composition of rotations (on $L(C_{n-1} \times \{\frac{1}{2}\}) \cap U(C_n \times \{1\})$ and $U(C_{n-1} \times \{-\frac{1}{2}\}) \cap L(C_n \times \{-1\})$) or an interpolating map (on $C_{n-1} \times [\frac{1}{2}, 1]$ and $C_{n-1} \times [-1, -\frac{1}{2}]$). The rotation number of f on these regions is easy to compute.

On $L(C_{n-1} \times \{\frac{1}{2}\}) \cap U(C_n \times \{1\})$ the rotation number of f is α_{n-1} and on $C_{n-1} \times [\frac{1}{2}, 1]$ interpolates between α_{n-1} and α_{n-2}. This is as in Handel's example. On $U(C_{n-1} \times \{-\frac{1}{2}\}) \cap L(C_n \times \{-1\})$ the rotation number of f is $\alpha_n + \frac{1}{p_n} = \alpha_{n-1} + \frac{2}{p_n}$ and on $C_{n-1} \times [-1, -\frac{1}{2}]$ the rotation number interpolates between $\alpha_{n-1} + \frac{1}{p_{n-1}} = \alpha_n + \frac{2}{p_{n-1}}$ and $\alpha_n + \frac{1}{p_n}$. This differs from Handel's example, and the rapid increase of the sequence $\{p_n\}$ guarantees that $\alpha_n + \frac{1}{p_n} > \alpha_{n+1} + \frac{1}{p_{n+1}} > \lim_{n \to \infty} \alpha_n = \alpha$ where α is the rotation number of $f|P$.

Next we show that the "rotations" R used in the definition of the f_n exist as area preserving, C^∞ maps conjugate to rigid rotation on the standard annulus. We let A be a smooth topological annulus and μ be a smooth measure on A. We let $\bar{A} = [0,1] \times \mathbb{R}/\mathbb{Z}$ be the standard annulus and m be Lebesgue measure on $\bar{A}$.

From Moser's result [M] on the equivalence of smooth measures, we have a diffeomorphism $h : A \to \bar{A}$ with the property that for any $B \subset A$ we have $\mu(B) = m(h(B))$. If $\bar{R}_\alpha$ is rigid rotation on $\bar{A}$, then we define $R_\alpha = h^{-1} \circ \bar{R}_\alpha \circ h$ and this map preserves μ, is smooth, and is conjugate to rigid rotation by α. $\square$

NUMBER THEORETIC RESTRICTIONS

The rotation number α of the diffeomorphism restricted to the pseudocircle is the sum of a very rapidly decreasing sequence:

$$\alpha = \sum_{n=1}^{\infty} \frac{1}{p_n}.$$

In [T] this rotation number is shown to be very well approximable in the following sense:

$$\exists\, C,\ 1 < C < 2, \exists\, \left\{\frac{a_n}{b_n}\right\},\ \frac{a_n}{b_n} \to \alpha,\ \text{and } |\alpha - \frac{a_n}{b_n}| < \frac{3}{C^n}.$$

Question 1. Let f be an area preserving homeomorphism of the annulus. For each sufficiently poorly approximable irrational, α, in the rotation set, does f have a minimal set X so that $f|X$ is semiconjugate to rigid rotation by α on the circle?

EXTENSION TO A COUNTABLY DENSE SUBSET OF THE ROTATION SET

We note that in this example there is only one number, the irrational α, in the rotation set which is not realized by a "nice" minimal set. However, every other rotation number between the boundary rotation numbers is represented (at least) by a smooth topological circle on which the diffeomorphism is conjugate to rigid rotation. The region between any two such circles with different rotation numbers is an annulus on which the previous construction can be carried out.

Choosing any rational number $\{a/b\}$, and any sufficiently small $\epsilon > 0$, we take the region that represents the interval of rotation numbers $[a/b, a/b + \epsilon]$. This region

is a smooth topological annulus which we divide into b pieces and on which we define f_1 to be rotation by a/b. At this point we continue as before, embedding a second annulus in the first and dividing this second annulus into p_2 pieces. By choosing the subsequent p_n large we preserve the smoothness of the diffeomorphism.

Continuing with a dense set of rationals and a sequence of ϵ's tending (rapidly) to zero, we generate a countably dense set of numbers in the rotation set for which the only corresponding minimal set would be the pseudocircle. This leads us to another question with which we conclude.

Question 2. Let f be an area preserving homeomorphism of the annulus. For each irrational, α, in the rotation set, with the exception of some countable subset, does f have a minimal set X so that $f|X$ is semiconjugate to rigid rotation by α on the circle?

Acknowledgements

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