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# ENVELOPING ALGEBRAS OF LIE COLOR ALGEBRAS: PRIMENESS VERSUS GRADED-PRIMENESS

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ABSTRACT. Let $G$ be a finite abelian group and let $L$ be a, possibly restricted, $G$-graded Lie color algebra. Then the enveloping algebra $U(L)$ is also $G$-graded, and we consider the question of whether $U(L)$ being graded-prime implies that it is prime. The first section of this paper is devoted to the special case of Lie superalgebras over a field $K$ of characteristic $\neq 2$. Specifically, we show that if $i = \sqrt{-1} \in K$ and if $U(L)$ has a unique minimal graded-prime ideal, then this ideal is necessarily prime. As will be apparent, the latter result follows quickly from the existence of an anti-automorphism of $U(L)$ whose square is the automorphism of the enveloping algebra associated with its $\mathbb{Z}_2$-grading. The second section, which is independent of the first, studies more general Lie color algebras and shows that if $U(L)$ is graded-prime and if most homogeneous components $L_g$ of $L$ are infinite dimensional over $K$, then $U(L)$ is prime. Here we use $\Delta$-methods to study the grading on the extended centroid $C$ of $U(L)$. In particular, if $G$ is generated by the infinite support of $L$, then we prove that $C = C_1$ is homogeneous.

## §1. SUPERALGEBRAS

Let $L = L_0 \oplus L_1$ be an ordinary (or restricted) Lie superalgebra over any field $K$ of characteristic not 2, and let $U(L)$ denote its ordinary (or restricted) enveloping algebra. Then $U(L) = U_0 \oplus U_1$ is $\mathbb{Z}_2$-graded, and we let $\sigma$ denote the corresponding automorphism of order 2 defined by $\sigma(u) = u$ if $u \in U_0$ and $\sigma(u) = -u$ if $u \in U_1$. Then it is clear that any subspace of $U(L)$ is graded if and only if it is $\sigma$-stable. In particular, an ideal $I \triangleleft U(L)$ is graded-prime if and only if it is $\sigma$-prime. The goal of this section is to show that if $i = \sqrt{-1} \in K$, then $U(L)$ is a graded-prime ring if and only if it is a prime ring. Of course, prime always implies graded-prime, so we are essentially concerned with the reverse implication.

**Lemma 1.1.** *Let $U(L)$ denote the (restricted) enveloping algebra of the (restricted) $K$-superalgebra $L$ and assume that $i = \sqrt{-1} \in K$. If $\tau\colon U(L) \to U(L)$ is given by $\tau(x) = -x$ if $x \in L_0$ and $\tau(x) = ix$ if $x \in L_1$, then $\tau$ is a well-defined algebra anti-automorphism of $U(L)$ and $\tau^2 = \sigma$.*

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*Proof.* Consider the map $\theta : L \to U(L)^{\mathrm{op}}$ given by $\theta(x) = -x^{\mathrm{op}}$ if $x \in L_0$ and $\theta(x) = ix^{\mathrm{op}}$ if $x \in L_1$. We show that $\theta$ is a Lie superalgebra homomorphism into the superalgebra of the $\mathbb{Z}_2$-graded ring $U(L)^{\mathrm{op}}$. To avoid confusion, let us use $[[\,,]]$ to denote the bracket defined by the latter algebra. Let $x \in L_0$ and let $y \in L_a$ with $a \in \{0, 1\}$. Then $[x, y] \in L_a$, so $\theta(y) = \lambda y^{\mathrm{op}}$ and $\theta([x, y]) = \lambda[x, y]^{\mathrm{op}}$ for the same scalar $\lambda$. Thus

$$\begin{aligned}[[\theta(x), \theta(y)]] &= [[-x^{\mathrm{op}}, \lambda y^{\mathrm{op}}]] = -\lambda(x^{\mathrm{op}}y^{\mathrm{op}} - y^{\mathrm{op}}x^{\mathrm{op}}) = -\lambda(yx - xy)^{\mathrm{op}}\\ &= \lambda(xy - yx)^{\mathrm{op}} = \lambda[x, y]^{\mathrm{op}} = \theta([x, y]).\end{aligned}$$

Similarly, if $x, y \in L_1$, then $[x, y] \in L_0$, so

$$\begin{aligned}[[\theta(x), \theta(y)]] &= [[ix^{\mathrm{op}}, iy^{\mathrm{op}}]] = i^2(x^{\mathrm{op}}y^{\mathrm{op}} + y^{\mathrm{op}}x^{\mathrm{op}}) = -(yx + xy)^{\mathrm{op}}\\ &= -(xy + yx)^{\mathrm{op}} = -[x, y]^{\mathrm{op}} = \theta([x, y]).\end{aligned}$$

In particular, if $L$ is an ordinary Lie superalgebra, then the universal property of $U(L)$ implies that $\theta$ extends to an algebra homomorphism $\theta : U(L) \to U(L)^{\mathrm{op}}$. By composing this with the natural algebra anti-isomorphism $U(L)^{\mathrm{op}} \to U(L)$, we conclude that $\tau$ exists and is an algebra anti-homomorphism. Since $\tau^2$ agrees with $\sigma$ on $L$, we conclude that $\tau^2 = \sigma$ and hence that $\tau$ is an algebra anti-automorphism of order dividing 4.

Finally, if $L$ is a restricted Lie superalgebra in characteristic $p > 2$, then the above argument will apply provided we show that $\theta$ also respects the $p$th power map ${}^{[p]} : L_0 \to L_0$. To this end, let $x \in L_0$ and note that $x^p = x^{[p]}$ in the restricted enveloping algebra $U(L)$. Thus

$$\theta(x^{[p]}) = -(x^{[p]})^{\mathrm{op}} = -(x^p)^{\mathrm{op}} = (-x^{\mathrm{op}})^p = \theta(x)^p,$$

and $\theta$ extends to an algebra homomorphism $\theta : U(L) \to U(L)^{\mathrm{op}}$, as required. □

We remark that any anti-automorphism $\tau$ of a ring $R$ permutes its ideals, preserving inclusion, and maps primes to primes. Furthermore, if $\tau$ commutes with an automorphism $\sigma$, then $\tau$ permutes the $\sigma$-stable ideals of $R$ and hence it maps $\sigma$-primes to $\sigma$-primes. With this observation, we have

**Theorem 1.2.** *Let $L$ be a (restricted) Lie superalgebra over the field $K$ of characteristic $\neq 2$, let $U(L)$ denote its (restricted) enveloping algebra, and assume that $i = \sqrt{-1} \in K$. If $U(L)$ has a unique minimal graded-prime ideal $P$, then $P$ is the unique minimal prime of $U(L)$.*

*Proof.* Let $\sigma$ be the automorphism of $U(L)$ of order 2 associated with the grading and, since $i = \sqrt{-1} \in K$, let $\tau$ be defined as in the preceding lemma. Then $\tau$ commutes with $\tau^2 = \sigma$, so $\tau$ permutes the graded-prime ideals of $U(L)$ and hence $\tau$ stabilizes $P$, the unique minimal graded-prime. In particular, $\tau$ gives rise to an anti-automorphism of the $\sigma$-prime ring $R = U(L)/P$. Since $\sigma$ is an automorphism of $R$ of order $\leq 2$, [P, page 133] implies that $R$ has at most two minimal primes, say $Q$ and $Q^\sigma$, and that $Q \cap Q^\sigma = 0$. But then the cyclic group $\{1, \tau, \tau^2, \tau^3\}$ permutes these $\leq 2$ minimal primes, and hence $\tau^2 = \sigma$ must act trivially as a permutation. Thus $Q = Q^\sigma$ and $0 = Q \cap Q^\sigma = Q$. This means that $R$ is a prime ring and hence that $P$ is a prime ideal of $U(L)$. Finally, if $T$ is any prime ideal of $U(L)$, then $T \cap T^\sigma$ is $\sigma$-prime, so $T \cap T^\sigma \supseteq P$ and consequently $T \supseteq P$. Thus $P$ is the unique minimal prime of $U(L)$. □

Note that [Bl] asks whether the enveloping algebra $U(L)$ of a finite-dimensional Lie superalgebra $L$ must necessarily have a unique minimal prime. In view of the above, we can essentially obtain an affirmative answer here by showing that $U(L)$ has a unique minimal graded-prime ideal. For example, in the following result, we are able to slightly sharpen the conclusion of [L, Theorem I(i)] concerning minimal graded-prime ideals when $L$ is solvable.

**Corollary 1.3.** *Let $L$ be a finite-dimensional solvable Lie superalgebra over a field $K$ of characteristic $0$. Then $U(L)$ has a unique minimal prime ideal.*

*Proof.* If $F$ is the algebraic closure of $K$, then [L, Theorem I(i)] and the preceding result imply that $F \otimes U(L) \cong U(F \otimes L)$ has a unique minimal prime, say $Q$. In particular, since $F \otimes U(L)$ is Noetherian, it follows that $Q$ is nilpotent. Now $U(L)$ is also Noetherian, so there exist minimal primes $P_1, P_2, \dots, P_n$ of $U(L)$, not necessarily distinct, with $P_1 P_2 \cdots P_n = 0$. Lifting these to ideals of $F \otimes U(L)$, we have

$$(F \otimes P_1)(F \otimes P_2) \cdots (F \otimes P_n) = 0 \subseteq Q$$

and therefore $Q \supseteq F \otimes P_k \supseteq P_k$ for some subscript $k$. But then $P_k$ is also nilpotent, and this easily implies that $P_k$ is the unique minimal prime of $U(L)$. □

This, of course, extends [KK, Theorem 3.8] and indicates why the following result may be of interest.

**Lemma 1.4.** *Let $L = L_0 \oplus L_1$ be a finite-dimensional ordinary Lie superalgebra and assume that $U(L)$ has a unique minimal prime ideal. Then every non-nilpotent ideal of $U(L)$ has a nonzero intersection with $U(L_0)$.*

*Proof.* If $T$ denotes the set of nonzero elements of $U(L_0)$, then clearly $T$ is a multiplicatively closed set of regular elements of $U(L)$. Furthermore, by [Bh, Section 4], $T$ is a right Ore set of $U(L)$ and $U(L)T^{-1}$ is the Artinian right classical quotient ring of the enveloping algebra. Now let $N$ denote the unique minimal prime ideal of $U(L)$, so that $N$ is also the unique largest nilpotent ideal of this Noetherian ring. If $I$ is a non-nilpotent ideal of $U(L)$, then $(I+N)/N$ is a nonzero ideal of the prime Noetherian ring $U(L)/N$, and hence $I$ contains an element $c$ which is regular modulo $N$. Indeed, [Sm, Theorem 2.12] implies that $c$ is regular in $U(L)$ and hence invertible in $U(L)T^{-1}$. Since the right ideal $IT^{-1}$ contains an invertible element, it follows that $1 \in IT^{-1}$, so $I \cap T \neq \emptyset$ and $I \cap U(L_0) \neq 0$. □

Another consequence of Theorem 1.2 is a slight sharpening of [BP, Corollary 1.3]. If $L$ is a Lie superalgebra, recall that $\Delta_L$ is defined by

$$\Delta_L = \{\, \ell \in L \mid \dim_K \ell \cdot \operatorname{ad} U(L) < \infty \,\}.$$

Then [BP, Lemma 2.1] implies that $\Delta_L$ is a characteristic (restricted) Lie superideal of $L$ and that it is the join of all finite-dimensional superideals of $L$. The following is an immediate consequence of Theorem 1.2 and [BP, Corollary 1.3(ii)].

**Corollary 1.5.** *Let $L$ be a Lie superalgebra over a field $K$ of characteristic $\neq 2$ and suppose that $i = \sqrt{-1} \in K$. If $\operatorname{char} K = p > 2$, assume further that $L$ is restricted. Then $U(L)$ is prime if and only if $U(\Delta_L)$ is graded $L$-prime.*

We use this to prove

**Corollary 1.6.** *Let $L$ be a solvable Lie superalgebra over a field $K$ of characteristic $0$. Then $U(L)$ is prime if and only if it is semiprime.*

*Proof.* Since prime always implies semiprime, we are concerned here with the reverse implication. Thus we suppose that $U(L)$ is semiprime, and we first consider the case where $i = \sqrt{-1} \in K$.

Suppose $A$ and $B$ are nonzero graded $L$-stable ideals of $U(\Delta_L)$ with $AB = 0$. Since $\Delta_L$ is generated by the finite-dimensional superideals of $L$, there exists such an ideal $I$ of $L$ with $\tilde{A} = A \cap U(I) \neq 0$ and $\tilde{B} = B \cap U(I) \neq 0$. Now $I$ is finite dimensional and solvable, so Corollary 1.3 implies that the Noetherian ring $U(I)$ has a unique minimal prime which is nilpotent. In particular, since $\tilde{A}\tilde{B} = 0$, we see that one of $\tilde{A}$ or $\tilde{B}$, say $\tilde{A}$, is nilpotent. But then $\tilde{A}$ is a nonzero graded $L$-stable nilpotent ideal of $U(I)$, so it is clear that $\tilde{A}U(L)$ is a nonzero nilpotent ideal of $U(L)$, a contradiction. It follows that $U(\Delta_L)$ is graded $L$-prime, and we conclude from the preceding corollary that $U(L)$ is prime.

Finally, let $K$ be an arbitrary field of characteristic 0 and set $F = K[i]$. Then $F/K$ is a finite Galois extension, so it follows that $F \otimes U(L) = U(F \otimes L)$ is also semiprime. Consequently, by the above, $F \otimes U(L)$ is prime, and hence so is $U(L)$. □

Note that the preceding argument really only requires the fact that each finite-dimensional superideal $I$ of $L$ is solvable. Indeed, we need only assume that, for each such $I$, $U(I)$ has a unique minimal prime ideal. We remark that if $H \triangleleft L$ and if $U(L)$ is semiprime, then it is not necessarily true that $U(H)$ is also semiprime. This occurs because superderivations need not stabilize minimal primes even if $\operatorname{char} K = 0$. For instance, we have

**Example 1.7.** If $K$ is any field of characteristic $\neq 2$, then there exists a metabelian (restricted) Lie superalgebra $L$ over $K$ such that $U(L)$ is prime while $U(\Delta_L)$ is not even semiprime.

*Proof.* Define $L = L_0 \oplus L_1$ so that $L_0$ has $K$-basis $\{a, b\}$ and $L_1$ has $K$-basis $\{c, d_0, d_1, d_2, \dots\}$. Furthermore, assume that $[c, d_0] = a$, $[b, d_j] = d_{j+1}$ for all $j \geq 0$, and that all other Lie brackets of basis elements are zero. Notice that $A = Ka + Kc$ and $B = Kb + Kd_0 + Kd_1 + Kd_2 + \cdots$ are super-subalgebras of $L$ and that $L = A \rtimes B$. Indeed, $B' = Kb + Kd_1 + Kd_2 + \cdots$ is a superideal of $B$ and $B \to B/B' \cong Kd_0 \to \operatorname{Der}(A)$ describes the action of $B$ on $A$. With this observation, it is clear that $L$ exists, and then $L$ is metabelian since $[L, L] = Ka + Kd_1 + Kd_2 + \cdots$ is abelian.

Now $A \subseteq \Delta_L$ and $B$ has no nonzero finite-dimensional super ideals, so $A = \Delta_L$ and $U(\Delta_L) = K[a, c \mid c^2 = 0]$ is not semiprime. On the other hand, the $\mathbb{Z}_2$-grading of this algebra is given by $U(\Delta_L) = K[a] \oplus K[a]c$ and, since $[c, d_0] = a$, it follows that any nonzero graded $L$-stable ideal of this ring meets $K[a]$ nontrivially. In particular, $U(\Delta_L)$ is graded $L$-prime, and if we now suppose that $i = \sqrt{-1} \in K$, then Corollary 1.5 implies that $U(L)$ is prime when $\operatorname{char} K = 0$.

If $\operatorname{char} K = p > 2$, let $\tilde{L}$ be the $K$-subspace of $U(L)$ with $K$-basis given by $\{a^{p^j}, b^{p^j}, c, d_j \mid j \geq 0\}$. Then we know that $\tilde{L}$ is a restricted Lie superalgebra and that its restricted enveloping algebra $U(\tilde{L})$ is precisely equal to $U(L)$. Furthermore,

$\Delta_{\tilde{L}}$ has $K$-basis $\{ a, a^p, a^{p^2}, \dots , c \}$, so $U(\Delta_{\tilde{L}}) = U(\Delta_L) = K[a, c \mid c^2 = 0]$ is graded $\tilde{L}$-prime. Thus, Corollary 1.5 again implies that $U(\tilde{L}) = U(L)$ is prime.

Finally, if $K$ is arbitrary, set $F = K[i]$. Then we conclude from the preceding argument applied to $F \otimes L$ that $F \otimes U(L) = U(F \otimes L)$ is prime, and consequently $U(L)$ is also prime. □

## §2. Color algebras

Let $G$ be a finite abelian group and let $\epsilon : G \times G \to K^\bullet$ be a bicharacter. If $L$ is a Lie color algebra associated with $G$ and $\epsilon$, then $L = \oplus \sum_{g \in G} L_g$ is a $G$-graded $K$-vector space and there is a $K$-bilinear map $[\, ,] : L \times L \to L$ which satisfies $[L_x, L_y] \subseteq L_{xy}$ for all $x, y \in G$. As usual, the support of $L$ is defined by

$$\operatorname{supp} L = \{ g \in G \mid L_g \neq 0 \}$$

and we can certainly assume that $G = \langle \operatorname{supp} L \rangle$ since the elements of $G$ outside of this supporting subgroup have no effect on the structure of $L$ or its enveloping algebra. Now each $g \in G$ gives rise to a linear character $\lambda_g \in \widehat{G} = \operatorname{Hom}(G, K^\bullet)$ defined by $\lambda_g(x) = \epsilon(g, x)$, and it is clear that the map $\Lambda : G \to \widehat{G}$ given by $g \mapsto \lambda_g$ is a group homomorphism. Again, we can assume that $\Lambda$ is one-to-one, since if $N = \ker \Lambda$ and $\bar{G} = G/N$, then $\epsilon$ determines a bicharacter $\bar{\epsilon} : \bar{G} \times \bar{G} \to K^\bullet$ and $L$ is naturally a Lie color algebra associated with $\bar{G}$ and $\bar{\epsilon}$. For convenience, we say that $G$ is minimal for $L$ if $G = \langle \operatorname{supp} L \rangle$ and $\Lambda : G \to \widehat{G}$ is one-to-one. In this case, it is clear that $G \cong \widehat{G}$.

Recall that $G$ is the disjoint union of the subsets $G_+ = \{ g \in G \mid \epsilon(g, g) = 1 \}$ and $G_- = \{ g \in G \mid \epsilon(g, g) = -1 \}$. Here $G_+$ is a subgroup of $G$ of index $\leq 2$ and $G_-$ is the remaining coset when the index is 2. If $L$ is an ordinary Lie superalgebra we let $U(L)$ denote its universal enveloping algebra, and we define the infinite support of $L$ by

$$\operatorname{supp}_\infty L = \{ g \in G_+ \mid L_g \neq 0 \} \cup \{ g \in G_- \mid \dim_K L_g = \infty \}.$$

Of course, $L$ is always an ordinary algebra when $\operatorname{char} K = 0$.

On the other hand, if $\operatorname{char} K = p > 0$, then $L$ may have the additional structure of a restricted color algebra. This means that there exists a $p$th power map

$$^{[p]} : L_g \to L_{g^p} \qquad \text{for all } g \in G_+,$$

and here we let $U(L)$ denote the restricted enveloping algebra of $L$. Furthermore, in this context the infinite support of $L$ is defined by

$$\operatorname{supp}_\infty L = \{ g \in G \mid \dim_K L_g = \infty \}.$$

See [BMPZ] or [S] for a more complete description of Lie color algebras and their enveloping algebras.

Now let $R$ be any $K$-algebra and let $G$ be a finite abelian group with $G \cong \widehat{G} = \operatorname{Hom}(G, K^\bullet)$. If $R = \oplus \sum_{g \in G} R_g$ is $G$-graded, then the dual group $\widehat{G}$ acts as automorphisms on $R$ by defining

$$\Big( \sum_{g \in G} r_g \Big)^\lambda = \sum_{g \in G} \lambda(g) r_g$$

for any $\lambda \in \widehat{G}$. Conversely, if $\widehat{G}$ acts on $R$, then $R$ is $G$-graded by the eigenspaces

$$R_g = \{ r \in R \mid r^\lambda = \lambda(g) r \text{ for all } \lambda \in \widehat{G} \},$$

which are indexed by the elements $g \in G$. Indeed, these are equivalent structures on $R$, since $G \cong \widehat{G}$ implies that for any two distinct elements $x, y \in G$ there exists $\lambda \in \widehat{G}$ with $\lambda(x) \neq \lambda(y)$.

Suppose, in addition, that $R$ is a semiprime $K$-algebra, and let $C = \mathcal{C}(R)$ denote its extended centroid (see [A] or [M]). If $R$ is $G$-graded, then we know that the action of $\widehat{G}$ on $R$ extends naturally to an action on the central closure $RC$ of $R$. Hence the $G$-grading on $R$ extends to a grading on $RC$ with $C$ a graded subring. Note that the assumption $G \cong \widehat{G}$ holds when $R = U(L)$ and $G$ is minimal for $L$. Our goal in this section is to study the grading on the extended centroid of $U(L)$ when this enveloping algebra is semiprime, and following [W, Proposition 2.8] we define

$$\Delta_L = \{\, \ell \in L \mid \dim_K \ell \cdot \operatorname{ad} U(L) < \infty \,\}.$$

Then $\Delta_L$ is a characteristic (restricted) Lie color ideal of $L$ which "controls" the linear identities of $U(L)$. We use this fact to prove

**Lemma 2.1.** *Assume that $R = U(L)$ is semiprime and let $c$ be a nonzero element of the extended centroid $C = \mathcal{C}(R)$. Then there exists a finite-dimensional graded subspace $V$ of $R$, which is stable under the adjoint action of $U(L)$ on $U(L)$, such that $0 \neq vc \in R$ for some $v \in V$.*

*Proof.* Let us first assume that either $\operatorname{char} K = 0$, or $\operatorname{char} K = p > 0$ and that $L$ is restricted. If $I = \{\, r \in R \mid rc \in R \,\}$, then the definition of $C$ implies that $I$ is an ideal of $R$ with zero annihilator in $R$ and hence in $RC$. In particular, $Ic \neq 0$ and we fix $\alpha \in I$ with $0 \neq \beta = \alpha c \in R$. Now let $r$ be any element of $I$ and set $s = rc \in R$. If $x \in R$, then we know that $xc = cx$ and, upon multiplying this equation on the left by $\alpha$ and on the right by $r$, we obtain

$$\alpha x s = \beta x r \qquad \text{for all } x \in R,$$

a linear identity in $R$. In particular, since $\beta \neq 0$, this linear identity and [W, Theorem 4.8] imply that there exist elements $\alpha'$ and $\beta' \neq 0$ in $U(\Delta_L)$, depending only upon $\alpha$ and $\beta$, such that $\alpha' s = \beta' r$. But $s = rc = cr$, so this yields $(\alpha' c - \beta')r = 0$ and, since $r \in I$ is arbitrary, we have $(\alpha' c - \beta')I = 0$ and hence $\alpha' c = \beta'$.

Now, by definition of $\Delta_L$ and $U(\Delta_L)$, it follows that there exists a finite-dimensional $\operatorname{ad} U(L)$-stable graded subspace $H$ of $L$ such that $\alpha' \in (K + H)^n$ for some integer $n \geq 0$. Thus $V = (K + H)^n$ is a finite-dimensional $\operatorname{ad} U(L)$-stable graded subspace of $U(L)$, and $\alpha' = v \in V$ satisfies $vc = \alpha' c = \beta' \in R \setminus 0$.

Finally, suppose that $L$ is an ordinary Lie color algebra in characteristic $p > 0$ and let $\tilde{L}$ be the restricted Lie color subalgebra of $U(L)$ generated by $L$. Then it is known that $U(\tilde{L})$, the restricted enveloping algebra of $\tilde{L}$, is precisely equal to $U(L)$. Thus, the result of the preceding paragraph applied to $U(\tilde{L})$ yields the corresponding result for $U(L)$. ∎

We remark that if $c$ above is assumed to be homogeneous, then we can find an appropriate finite-dimensional graded subspace $V$ with $0 \neq Vc \subseteq R$. This is essentially contained in the proof of the following result.

**Theorem 2.2.** *Let $L$ be a (restricted) $G$-graded Lie color algebra over the field $K$ and assume that $G$ is minimal for $L$. Suppose that the (restricted) enveloping algebra $U(L)$ is semiprime and let $C$ denote its extended centroid. If $G$ is generated by the infinite support of $L$, then $C = C_1$ is homogeneous of grade $1 \in G$.*

*Proof.* Suppose, by way of contradiction, that $C_x \neq 0$ for some $1 \neq x \in G$ and choose $0 \neq c \in C_x$. By the previous lemma, let $V$ be a finite-dimensional graded subspace of $R = U(L)$, stable under the adjoint action of $U(L)$, such that $0 \neq vc \in R$ for some $v \in V$. If $W = \{\, w \in V \mid wc \in R \,\}$, then, since $R$ is a graded subring of $RC$ and since $c$ is homogeneous, it follows that $W$ is a graded subspace of $V$. Furthermore, since $v \in W$, we see that $Wc = \sum_{g\in G} W_g c$ is a nonzero finite-dimensional graded subspace of $R$. Consequently, we can choose $\alpha \in W$ to be a homogeneous element such that $\deg \alpha c \geq \deg wc$ for all $w \in W$.

Since $G$ is minimal for $L$ and $x \neq 1$, we know that $1 \neq \lambda_x \in \widehat{G}$. In particular, $\ker \lambda_x = \{\, g \in G \mid \lambda_x(g) = 1 \,\}$ is a proper subgroup of $G$ and hence it cannot contain all members of $\operatorname{supp}_\infty L$, which, by hypothesis, generates $G$. Thus there exists $y \in \operatorname{supp}_\infty L$ with $1 \neq \lambda_x(y) = \epsilon(x,y)$. Let $\ell$ be any element of $L_y$ and, for convenience, use ${}^{\ell}$ to denote the adjoint action of $\ell$.

Since $\alpha$ is homogeneous, $c \in C_x$ and $\ell \in L_y$, we have $(\alpha c)^\ell = \alpha^\ell c + k'\alpha c^\ell$ for some nonzero $k' \in K$ depending upon the grades of $\alpha$ and $\ell$. Furthermore, since $c$ and $\ell$ commute, we have

$$c^\ell = \ell c - \epsilon(x,y)c\ell = (1-\epsilon(x,y))c\ell = k''c\ell,$$

where $k'' = 1 - \epsilon(x,y)$ is a nonzero element of $K$ since $\epsilon(x,y) = \lambda_x(y) \neq 1$. Thus, setting $k = -k'k'' \in K \setminus 0$, we have

$$\alpha^\ell c = (\alpha c)^\ell - k'\alpha c^\ell = (\alpha c)^\ell + k(\alpha c)\ell \in R.$$

In particular, since $\alpha^\ell \in V^\ell \subseteq V$, we have $\alpha^\ell \in W$. Now $\deg(\alpha c)^\ell \leq \deg \alpha c$ is always true and $\deg \alpha^\ell c \leq \deg \alpha c$ follows from the maximality of $\deg \alpha c$. Thus the equation $k(\alpha c)\ell = \alpha^\ell c - (\alpha c)^\ell$ implies that $\deg(\alpha c)\ell \leq \deg \alpha c$ for all $\ell \in L_y$.

Since $\alpha c$ is a nonzero element of $U(L)$, there exists a finite-dimensional graded subspace $H$ of $L$ with $\alpha c \in (K+H)^n$ for some $n \geq 0$. In particular, if $\dim_K L_y = \infty$, we can choose $\ell \in L_y \setminus H$, and consequently the PBW theorem implies that $\deg(\alpha c)\ell = 1 + \deg \alpha c > \deg \alpha c$, a contradiction. On the other hand, if $\dim_K L_y$ is finite, then the definition of $\operatorname{supp}_\infty L$ implies that $L$ is an ordinary Lie superalgebra and that $y \in G_+$. In this case, we know that $L_y \neq 0$ and, if $0 \neq \ell \in L_y$, then the PBW theorem again yields $\deg(\alpha c)\ell = 1 + \deg \alpha c > \deg \alpha c$, a contradiction. Thus the assumption that $C_x \neq 0$ for some $1 \neq x \in G$ is false, and the theorem is proved. □

The preceding argument actually yields information on the allowable support of $C$ when $G$ is not necessarily generated by $\operatorname{supp}_\infty L$. Specifically, it shows that if $x \in G$ satisfies $1 \neq \lambda_x(\operatorname{supp}_\infty L) = \epsilon(x, \operatorname{supp}_\infty L)$, then the homogeneous component $C_x$ must be zero.

Again, let $R$ be a $G$-graded $K$-algebra with $G \cong \widehat{G}$. Then any subspace $V$ of $R$ is $\widehat{G}$-stable if and only if it is $G$-graded. In particular, $R$ is $\widehat{G}$-prime if and only if it is graded-prime. Furthermore, when this occurs, then $R$ is $\widehat{G}$-semiprime and hence also semiprime since $\widehat{G}$ is finite. The following observation is well-known.

**Lemma 2.3.** *Let $R$ be a $G$-graded $K$-algebra with $G \cong \widehat{G}$, and assume that $R$ is graded-prime. If $I$ is a nonzero ideal of $R$, then there exist a homogeneous element $a \in R$ and an element $c \in C = \mathcal{C}(R)$ with $0 \neq ac \in I$.*

*Proof.* By assumption, $R = \oplus\sum_{g\in G} R_g$ is $G$-graded, and we can choose $0 \neq \alpha = \sum_{g\in S} \alpha_g \in I$ with support $S \subseteq G$ and with $|S|$ minimal. Fix $h \in S$ and let $x \in R_w$

for some $w \in G$. Then

$$\beta = \alpha_h x\alpha - \alpha x\alpha_h = \sum_{g\in S}(\alpha_h x\alpha_g - \alpha_g x\alpha_h) \in I,$$

and note that $\alpha_h x\alpha_g - \alpha_g x\alpha_h \in R_{hwg}$ since $G$ is commutative. Furthermore, the $g = h$ summand vanishes here, and thus $\beta \in I$ has support contained in $hw(S\setminus\{ h \})$, a set of size smaller than $|S|$. Thus $\beta = 0$, and indeed

$$\alpha_h x\alpha_g - \alpha_g x\alpha_h = 0$$

for all $g \in S$ and all $x \in R_w$. Since $R = \sum_{w\in G} R_w$, the above identity clearly holds for all $x \in R$.

Finally, note that the ideals $R\alpha_h R$ and $R\alpha_g R$ are nonzero and graded, and hence they have trivial annihilators in $R$ since $R$ is graded-prime. With this observation, it follows from the linear identity that, for each $g \in S$, there exists $c(g) \in C$ with $\alpha_g = \alpha_h c(g)$. (See, for example, the proof of [M, Theorem 1].) Thus

$$\alpha = \sum_{g\in S} \alpha_g = \alpha_h \sum_{g\in S} c(g) = ac$$

is a nonzero element of $I$ of the required form. □

As a consequence, we have

**Theorem 2.4.** *Let $L$ be a (restricted) $G$-graded Lie color algebra over the field $K$ and assume that $G$ is minimal for $L$. If the (restricted) enveloping algebra $U(L)$ is graded-prime and $G = \langle \mathrm{supp}_\infty L\rangle$, then $U(L)$ is prime.*

*Proof.* As we observed, $R = U(L)$ is semiprime since it is graded-prime, and thus we can let $C = \mathcal{C}(R)$ denote its extended centroid. Furthermore, since $G = \langle \mathrm{supp}_\infty L\rangle$, Theorem 2.2 implies that $C = C_1$ is homogeneous. Now let $I$ be any nonzero ideal of $R$. Then the preceding lemma implies that there exist a homogeneous element $a \in R$ and an element $c \in C$ with $0 \neq ac \in I$. But $C = C_1$, so $c$ is also homogeneous and $ac$ is a nonzero homogeneous element of $I$. Thus $I$ contains the nonzero graded ideal $RacR$ which has zero annihilator in the graded-prime ring $R$. It follows that $I$ also has zero annihilator in $R$, and therefore $R = U(L)$ is a prime ring. □

We remark that some hypothesis is needed in the preceding theorem, since graded-prime enveloping algebras are not necessarily prime. Indeed, we mention

**Example 2.5** ([Pr])**.** For any field $K$ of characteristic $\neq 2$ there exists a Lie color algebra $L$, graded by the fours group, such that $U(L)$ is a commutative algebra which is graded-prime but not prime.

*Proof.* Let $G = \{ 1, x, y, xy \} \cong \{ 1, x \} \times \{ 1, y \}$ be the fours group and let $K$ be a field of characteristic $\neq 2$. If $\epsilon : G\times G \to K^\bullet$ is the product of the super-bicharacters defined on $\{ 1, x \}$ and $\{ 1, y \}$, then $\epsilon(x, x) = \epsilon(y, y) = -1$ and $\epsilon(x, y) = 1$. Now let $L$ be the $G$-graded Lie color algebra with $K$-basis $\{ a, b, c \}$ such that $L_x = Ka$, $L_y = Kb$, $L_1 = Kc$ and $L_{xy} = 0$. Furthermore, suppose that $[a, a] = [b, b] = 2c$ and that all other Lie products among the basis elements are 0. Then it is easy to check that $L$ is indeed a Lie color algebra. Moreover, $U(L)$ is generated by $a, b, c$ with $a^2 = b^2 = c$ and $ab = ba$, since $[a, b] = 0$ and $\epsilon(x, y) = 1$. Thus the PBW theorem implies that $R = U(L) = K[a, b, c \mid a^2 = b^2 = c]$. Clearly $R$ is a commutative ring, which is not prime since $a \neq \pm b$ but $0 = a^2 - b^2 = (a - b)(a + b)$. On the other hand, the grading of $R$ is given by $R_1 = K[c]$, $R_x = K[c]a$, $R_y = K[c]b$ and

$R_{xy} = K[c]ab$. Consequently, each nonzero homogeneous element of $R$ is regular in $R$, and hence $R$ is graded-prime.

Note that the above mentioned ring is an ordinary enveloping algebra. However, when $\operatorname{char} K = p > 2$, then the same ring $R$ is also the restricted enveloping algebra of the restricted Lie color algebra $\tilde{L}$ with $K$-basis $\{ a, b, c, c^p, c^{p^2}, \dots \}$. Thus there also exist restricted enveloping algebras which are graded-prime but not prime. □

Finally, Theorem 2.4 allows us to slightly sharpen [W, Theorem 5.4(ii)] when $G$ is generated by the infinite support of $L$. Specifically, we have

**Corollary 2.6.** *Let $L$ be a $G$-graded Lie color algebra over the field $K$, and assume that $G$ is minimal for $L$ and that $G = \langle \operatorname{supp}_\infty L \rangle$. Furthermore, let $U(L)$ denote the enveloping algebra of $L$ if* $\operatorname{char} K = 0$ *or the restricted enveloping algebra of the restricted color algebra $L$ if* $\operatorname{char} K = p > 0$. *Then $U(L)$ is prime if and only if $U(\Delta_L)$ is graded $L$-prime.*

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