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SUBAVERAGING ESTIMATE FOR CR FUNCTIONS DEFINED ON A HYPERSURFACE M OF $\mathbb{C}^n$

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Abstract. Let M be a smooth real hypersurface of $\mathbb{C}^n$ and N a compact submanifold of M. We generalize a result of A. Boggess and R. Dwilewicz giving, under some geometric conditions on M and N, an estimate of the submeanvalue on N of any CR function f on a neighbourhood of N, by the L^1 norm of f on a neighbourhood of N in M.

I. Introduction

Let M be a smooth real hypersurface of $\mathbb{C}^n$ and N a real compact submanifold of M of dimension k. In 1988, in the general case of M a smooth CR generic submanifold of $\mathbb{C}^n$, under some conditions of nonvanishing for the Levi form at every point p of M and of transversality for N, A. Boggess and R. Dwilewicz [BD] estimated the submeanvalue of any CR function f on a neighbourhood of N by the $L^1(M, d\sigma_M)$ norm of f, where $d\sigma_M$ is the measure on M. Their proof was based on the Henkin formulas and the upper-estimation of integral kernels.

In this article, in the hypersurface case, we generalize this result to a pseudoconvex hypersurface of maximal finite type m_0 (in the sense of Kohn [BG]).

Theorem 1. *Let M be a smooth real hypersurface of $\mathbb{C}^n$. Let N be a compact submanifold of M, not tangent complex of real dimension $k \geq 1$, without boundary. Suppose that M is pseudoconvex, of finite type in a neighbourhood of N. Then, for any neighbourhood ω of N in M, there exists a constant $C = C(\omega)$ such that for all functions $f \in CR(\omega)$ we have:*

$$(1.1)\qquad \Big| \int_N f d\sigma_N \Big| \leq C(\omega) \int_\omega |f| d\sigma_M .$$

Let us denote by $B_M(p,\delta)$ the nonisotropic ball of center p, of radius δ in the complex tangent directions and of radius δ^m (where m is the type of the point p) in the real tangent direction. Our proof of theorem 1 is based on the following ponctual estimation result **(1.2)** obtained by Boggess, Dwilewicz and Nagel [BDN] and generalized in [P]:

There exists $\delta_0 > 0$ such that, for all $p \in N$, for all $0 < \delta_p \leq \delta_0$ and for every point Z on the normal line to M through any point $q = \Pi(Z)$ of the nonisotropic ball $B_M(p, \delta_p)$, at a distance less than $\tau_0 {\delta_Z}^m$ (where $\tau_0 > 0$ is a uniform constant

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and $\delta_Z \in]0,\delta_0]$ *denotes the radius of the biggest nonisotropic ball centered in* q *and contained in* $B_M(p,\delta)$*), we have the following estimate:*

$\exists\, A,\, C>0$ *such that if* $|Z'-Z|\le C\cdot\delta_Z^m$*, we have for every* CR *function* u *in a neighbourhood of* N*:*

$$|u(Z')| \le \frac{A}{|B_M(\Pi(Z'),\delta)|}\int_{\zeta\in B_M(\Pi(Z'),\delta)} |u(\zeta)|d\sigma_M(\zeta), \tag{1.2}$$

where $|B_M(\Pi(Z'),\delta)|$ *is the measure of the nonisotropic ball for the measure* σ_M *of* M*, which is roughly* $\delta^{2n-2}\delta^m$*, where* m *is the type of* $\Pi(Z')$*.*

This property remains true for Π denoting a projection on M (in the sense of the definition on p.213 of [BDN]) and is based on the fact that the points Z and Z' are centers of analytic discs with boundary in $B_M(p,\delta_p)$.

If ω is a neighbourhood of N in M, we denote by $\mathfrak{O}$ the open set which lies in the pseudoconvex side, whose points Z' all satisfy **(1.2)** and to which every $CR(\omega)$ function f extends holomorphically to F.

II. Proof of the theorem

As in the article of Boggess and Dwilewicz [BD], assuming N not tangent complex allows us to construct locally a manifold $\tilde{N}$ of dimension $k+1$ transverse to M whose boundary contains N. If G is a (particular) parametrisation of N and U_j the open sets of $\mathrm{R}\times\mathrm{R}^{k-1}$ associated to this parametrisation, $\tilde{N}$ is locally written as

$$\tilde{N}_{j,\epsilon'} = \{G(x+iy,u);\ 0\le y\le\epsilon' \text{ and } (x,u)\in U_j\},$$

where $G(z,u)=G(x+iy,u)$ is a $\overline{\partial}$ flat extension of $G(x,u)$ on $\mathrm{C}\times\mathrm{R}^{k-1}$ (which means that $\forall N\in\mathrm{N}$, there exists a constant $C(N)$ such that $\big|\frac{\partial G}{\partial\overline{z}}(x+iy,u)\big|\le C(N)|y|^N$).

Now, under the assumption of theorem 1, we choose $\epsilon'>0$ such that $\tilde{N}_{j,\epsilon'}\subset\mathfrak{O}$.

By compactness of N, we extract from the $N_j=G(U_j)$ a finite covering of N, $N=\bigcup_{j=1}^{l}N_j$, and if $(\psi_j)_{j=1\ldots l}$ is a partition of unity on N associated to this covering, we write

$$\int_N f d\sigma_N = \sum_{j=1}^{l}\int_{N_j} F_j\psi_j d\sigma_N.$$

As in [BD], we then consider the function ϕ_j with compact support in U_j such that $\phi_j dxdu$ is the pull-back by G of $\psi_j d\sigma_N$, denoted by $G^*(\psi_j d\sigma_N)$. We extend ϕ_j $\overline{\partial}$ flat and obtain $\phi_j(z,u)$, whose support satisfies $\phi_j(x+iy,u)=0$ for $y\ge\frac{\epsilon'}{2}$. Thanks to G^{-1}, we then transport the measure $\phi_j(z,u)dzdu$ on $\tilde{N}_{j,\epsilon'}$ and get

$$I=\int_{N_j} F_j\psi_j d\sigma_N = \int_{N_j} F_j\Phi_j, \text{ where } \Phi_j = {G^{-1}}^*(\phi_j(z,u)dzdu).$$

To simplify, we write F for F_j, ϕ for ϕ_j, etc. Taking into account the support of Φ, the Stokes formula gives us $I=I_1+I_2$, where

$$I_1=\int_{0<y<\epsilon' z=x+iy}\int_{(x,u)\in U} F(G(z,u))\big(\frac{\partial\phi}{\partial\overline{z}}(z,u)\big)d\overline{z}\wedge dz\wedge du$$

and

$$I_2 = \sum_{l=1}^{l=n} \int_{\substack{0<y<\epsilon' \\ z=x+iy}} \int_{(x,u)\in U} \frac{\partial F}{\partial w_l}(G(z,u)) \frac{\partial G_l}{\partial \overline{z}}(z,u)\phi(z,u) d\overline{z}\wedge dz \wedge du.$$

Now, let's denote by δ^* the distance between the support of ϕ and the complement of N_j, and consider $\delta \leq \inf\left(\frac{\delta_0}{2}, \frac{\delta^*}{2}\right)$. From now on, we take

$$\epsilon' \leq \frac{\inf_{p\in N} \tau_0 \delta^{m_p}}{4}$$

and we project each point $Z = G(z,u) \in \tilde{N}_{j,\epsilon'} \subset \mathfrak{O}$ according to Π on M along $\tilde{N}_{j,\epsilon'}$.

Then, using the estimate **(1.2)**, we find that $\exists\, C_3 > 0$ such that

$$I_1 \leq \int_{0<y<\epsilon'} \int_{(x,u)\in U} \frac{C_3}{\delta_Z^{2(n-1)} \delta_Z^{m_{\Pi(Z)}}} \int_{B_M(\Pi(Z),\delta)} |F(\zeta)| d\sigma_M(\zeta) \frac{\partial \phi}{\partial \overline{z}}(z,u) d\overline{z} dz du.$$

Now, for all compact K in M, there exist two constants C_1, $C_2 > 0$ such that $\forall\, p \in K$, $\forall\, \delta_p$, $0 \leq \delta_p \leq 1$ we have $C_1 \delta_p^{m_0} \leq \delta_p^{m_p} \leq C_2 \delta_p^2$.

Taking $K = \overline{\bigcup_{Z\in\tilde{N}_{j,\epsilon'}} B_M\left(\Pi(Z), \frac{\delta_0}{2}\right)}$, we obtain a upper estimate of the inverse of δ_Z^2:

$$I_1 \leq \int_{0<y<\epsilon'} \int_{(x,u)\in U} \frac{C_3}{\left(\delta_Z^{m_{\Pi(Z)}}\right)^n} \int_{B_M(\Pi(Z),\delta)} |F(\zeta)| d\sigma_M(\zeta) \frac{\partial \phi}{\partial \overline{z}}(z,u) d\overline{z} dz du.$$

Now, because $\tilde{N}$ is transverse, there exists a constant C which doesn't depend on x, y, u such that for all $0 \leq y < \epsilon'$ and for all $(x,u) \in U$ we have

$$|G(z,u) - \Pi(G(z,u))| \geq Cte \left|\frac{\partial G}{\partial y}(x,u)\right| |y| \geq C|y|.$$

We also know that: $\left|\frac{\partial \phi}{\partial \overline{z}}(z,u)\right| \leq Cte(n)|y|^n$ for all $n \in \mathrm{N}$. So, we have

$$I_1 \leq Cte \left[\int_{0<y<\epsilon'} \int_{(x,u)\in G^{-1}\left(\bigcup_{p\in N} B_M(p,\frac{\delta_0}{2})\right)} d\overline{z} dz du\right] \int_{\bigcup_{p\in N} B_M(p,\delta)} |F| d\sigma_M,$$

which gives

$$I_1 \leq Cte\, \epsilon' \delta_0^{k-1} \delta_0^2 \int_{\bigcup_{p\in N} B_M(p,\delta)} |F| d\sigma_M \leq Cte\ \epsilon'\, \delta_0^{k+1} \int_\omega |F| d\sigma_M.$$

Now, to estimate I_2, we use the harmonicity of $\frac{\partial F}{\partial w_l}(G(z,u))$ on the C^n Euclidian ball $D(G(z,u),R)$ of center $G(z,u)$ and of radius R. Let us take $R = R_{z,u}$ ranging over the distance of $G(z,u)$ to M, i.e. roughly $|G(z,u) - \Pi(G(z,u))|$.

The transversality of $\tilde{N}_j$ allows us to write

$$\left|\frac{\partial F}{\partial w_l}(G(z,u))\right| \leq \frac{Cte}{|y|^{2n+1}} \int_{D(G(z,u),R)} |F(\zeta)| d\zeta.$$

Now, we know that each point $\zeta \in D(G(z,u),R)$ is the center of an analytic disc with boundary in M, and we can again use the subestimate **(1.2)**. Using the reasoning in the estimate for I_1 allows us to estimate I_2 from above:

$$I_2 \leq Cte\, {\epsilon'}^{2n+1}\, \delta_0^{k+1} \int_{\bigcup_{p\in N} B_M(p,\delta)} |F| d\sigma_M.$$

We then obtain an estimate on a neighbourhood of N in M which is of the form $\bigcup_{j=1}^{j=k} B_M(p_j,\delta)$, and so the expected result follows. ☐

III. Some remarks

Remark 1. We can't have a comparison between the $L^1(\omega, d\sigma_M)$ norm and the $L^1(N, d\sigma_N)$ norm of f. Indeed, consider the unit sphere $M = \partial B$ of $\mathbb{C}^2$ and the transverse circle $N = M \cap \{(z_1,z_2) \in \mathbb{C}^2\ ;\ z_2 = 0\}$. If such a comparison holds, we would have, for each function $F(z_1)$, depending only on z_1 and holomorphic on a neighbourhood of the ball B, the following estimate:

$$\int_0^{2\pi} |F(e^{i\theta})| d\theta \leq C \int_D |F(re^{i\theta})| r dr d\theta,$$

where $C > 0$ is a constant and D denotes the unit disc of $\mathbb{C}$.

This would mean that the Bergman space $L^1(D) \cap H(D)$ is included in the Hardy space $H^1(D)$, which is not the case.

Remark 2. Under an assumption of strict pseudoconvexity, we obtain a more precise version of **(1.1)**:

Proposition 1. *Let M be a smooth real hypersurface of $\mathbb{C}^n$ and N a compact submanifold of M, not tangent complex of real dimension k, without boundary and such that M is strictly pseudoconvex and of finite type 2 in a neighbourhood of N.*

Then, there exists a constant $C > 0$ such that for all δ, $0 < \delta \leq 1$, and for all CR functions on $\bigcup_{p\in N} B_M(p,\delta)$, we have

$$\Big| \int_N f d\sigma_N \Big| \leq \frac{C}{\delta^{2n-k-1}} \int_{\bigcup_{p\in N} B_M(p,\delta)} |f| d\sigma_M.$$

Proof. It is sufficient to prove it for δ small.

Now the support of ϕ is fixed, and we make δ smaller than $\inf(\frac{\delta_0}{2}, \frac{\delta^*}{2})$, so ϕ is not zero on $\{G(x+iy,u);\quad y = \epsilon' \text{ and } (x,u) \in U_j\}$, and then the Stokes formula involves an other integral, which is

$$J = \int_{\left\{\substack{y=\epsilon' \\ (x,u)\in U}\right.} F(G(x+i\epsilon',u))\phi(x+i\epsilon',u) dx du.$$

By the assumption of strict pseudoconvexity, we know that $\epsilon' = C\delta^2$, where C is a uniform constant. Using **(1.2)** again, we get the following estimate:

$$J \leq Cte \Big[\int_{(x,u)\in G^{-1}(\bigcup_{p\in N} B_M(p,\delta))} \frac{dx du}{\delta^{2n}_{G(x+i\epsilon',u)}} \Big] \int_{\bigcup_{p\in N} B_M(p,\delta)} |F| d\sigma_M.$$

All the points $G(x+i\epsilon', u)$ lie on $\tilde{N}_{j,\epsilon'}$ and so satisfy $\delta^2 = \delta^2_{G(x+i\epsilon',u)}$ for all $(x,u) \in U$. As the upper estimates of I_1 and I_2 are still available, we get that the constant $C(\omega)$ of **(1.1)** is of the form $\frac{Cte}{\delta^{2n-k-1}}$ for $\omega = \bigcup_{p\in N} B_M(p,\delta)$. □

Remark 3. Boggess and Dwilewicz gave a counterexample to estimate **(1.1)** if the assumption of transversality of N was not satisfied. Their example deals with the case where M is the Heisenberg domain and N a complex tangent curve to M. But M is then strictly pseudoconvex, and so every complex tangent curve in M is a peak set [DO]. So if we consider the associated n-powers of the peak function, we immediately get a contradiction of **(1.1)**. So, it could be better to consider the following counterexample:

Let M be the Heisenberg domain and consider the mixed curve (complex tangent at 0, and transverse to the complex tangent plane at a point different from zero) defined, for λ and μ given positive reals, by $z(1) = z(-1)$ and by

$$z(t) = (z_1(t), z_2(t)) = (te^{it}, t^2) \text{ for } |t| \le \lambda,$$

$$Re(z_2(t)) \ge \mu \text{ for } \lambda \le |t| \le 1.$$

Now consider the function $f_\epsilon(z) = \frac{1}{z_2+\epsilon}$; we can obtain, by computations analogous to those of [BD], constants C_1, C_2, C_3 and C_4 such that

$$\int_M \big|f_\epsilon\big| d\sigma_M \le -C_1C_2 ln(\epsilon) \quad \text{and} \quad \Big|\int_N f_\epsilon d\sigma_N\Big| \ge \frac{C_4}{\sqrt{\epsilon}} - C_3.$$

These estimates for the function f_ϵ $CR(M)$ contradict then the estimate **(1.1)**.

Remark 4. If N is a submanifold with boundary, we can't have an estimate of the form **(1.1)**. Indeed, take for instance for M the unit sphere S^3 of C^2 and for N a circle arc with extremities 0 and π contained in the circle $M \cap \{(z_1,z_2) \in \mathrm{C}^2 \,|\, z_2 = 0\}$. N is obviously not tangent complex to M. Then we consider the functions f holomorphic in a neighbourhood of the unit ball and depending only on the variable z_1. These are actually $CR(S^3)$ functions.

Let's denote by Δ the unit disc of C and write the inequality **(1.1)** for the neighbourhood $\omega = S^3$ and the functions f. So there exists $C(N, S^3)$ such that

$$\Big|\int_0^\pi f(e^{i\theta})d\theta\Big| \le C\,\Big|\int_\Delta r dr d\theta \frac{1}{2\pi}\int_{-\pi}^{+\pi} |f(re^{i\theta}|d\phi\Big| \le C\,||f||_{L^1(\Delta)}.$$

Then F defined on the Bergman space of the disc $B^2(\Delta)$ by

$$\begin{array}{lll} F: B^2(\Delta) & \to & \mathrm{C}, \\ f & \mapsto & \int_0^\pi f(e^{i\theta})d\theta, \end{array}$$

is a linear continuous bounded operator on $B^2(\Delta)$, which involves the existence of a unique function $g \in B^2(\Delta)$ such that

$$\int_0^\pi f(e^{i\theta})d\theta - \int_\Delta f(z)\overline{g(z)}d\lambda(z) = 0 \quad \forall f \in L^2(\Delta), \tag{1.3}$$

where $d\lambda$ denotes the Lebesgue measure on C.

In particular,

$$\int_0^\pi e^{in\theta}d\theta = \int_0^1\int_0^{2\pi} \overline{g(re^{i\theta})}r^{n+1}e^{in\theta}drd\theta.$$

Now, expressing g by the mean of its Fourier series $\sum_{p\in\mathbf{N}} a_p r^p e^{ip\theta}$, we obtain

$$\frac{2}{n} = \sum_{p\in\mathbf{N}} \overline{a_p} \int_0^1 r^{n+p+1} dr \int_0^{2\pi} e^{i(n-p)\theta} d\theta = \frac{\pi}{n+1} \overline{a_n}.$$

Now $g \in B^2$ if and only if $\displaystyle\sum_n \frac{|a_n|^2}{n+1}$ converges. But $\displaystyle\sum_n (n+1)\Big(\frac{2}{n}\Big)^2$ diverges, which gives a contradiction. □

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