

ON REPRESENTATION AND REGULARITY OF CONTINUOUS PARAMETER MULTIVALUED MARTINGALES

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ABSTRACT. In this paper we study multivalued martingales in continuous time. First we show that every multivalued martingale in continuous time can be represented as the closure of a sequence of martingale selections. Then we prove two results concerning the cadlag modifications of continuous time multivalued martingales, in Kuratowski-Mosco convergence and in convergence in the Hausdorff metric respectively.

§1. INTRODUCTION

Multivalued processes (set-valued processes) in discrete time have been studied by many authors. This notion extends those of real and vector random processes, and it is interesting that there are a lot of convergence and representation results analogous to those existing for single-valued processes. This line of recent research can be traced in the works of Bagchi [1, 2], Dam [9], Daures [10], Giné-Hahn-Zinn [15], Gut-Schmidt [16], Hess [17, 18], Hiai-Umegaki [20], Luu [22], Papageorgiou [25]-[29], Salinetti-Wets [31] and Wang-Xue [34]. The more recent works among this list obtained many profound convergence results and representation theorems concerning martingale or amart selections for multivalued martingale-like processes.

As for the study of the regularity of vector-valued processes which admit cadlag modifications, we can refer to the works of Choi-Sucheston [8] concerning uniform amarts and Frangos [13] concerning separable positive integrable submartingales on an ordered continuous Banach lattice.

The purpose of the present paper is to discuss some questions of continuous parameter multivalued martingales. The questions are their representations and their cadlag modifications. Some preliminaries are given in Section 2. In Section 3, a representation result for closed martingales is given; the main result of this section is the representation theorem for continuous parameter multivalued martingales which have weakly compact and convex values or whose dual space \mathbf{X}^* is strongly separable. In Section 4, a general limit theorem analogous to Theorems 2.3 and 3.2 of Choi-Sucheston [8] is given first. Then we prove two convergence results for closed martingales in the ascending and descending cases in Kuratowski-Mosco (K-M) convergence and in convergence in the Hausdorff metric. Finally, Section 5

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provides two other main results concerning the cadlag modifications for multivalued martingales in continuous time. Theorem 5.3 admits the right continuity and the left-limit in the cadlag modification with respect to K-M convergence, and every such process is uniformly bounded (for $t \geq 0$) by a multifunction which has weakly compact and convex values. Then Theorem 5.4 gives convergence in the Hausdorff metric under the restriction that these processes can be obtained as the limit of simple multifunctions in the complete metric space $(\mathcal{L}_{fc}^1(\mathbf{X}), \Delta)$.

§2. NOTATIONS AND PRELIMINARIES

Throughout this paper, $(\Omega, \mathcal{F}, \mathbf{P})$ is a *complete* probability space. We denote by \mathbf{Z} (resp. $\mathbf{J}, \bar{\mathbf{J}}$) the set of (resp. positive, nonnegative) integers and by \mathbf{R} (resp. \mathbf{R}_+) the set of real numbers (resp. nonnegative real numbers). Let $(\mathcal{F}_t, t \geq 0)$ be a sub σ -field filtration on $(\Omega, \mathcal{F}, \mathbf{P})$, which satisfies the *usual conditions* (complete and right continuity). τ is a stopping time of (\mathcal{F}_t) . An increasing sequence τ_1, τ_2, \dots of stopping times is said to *announce* τ if $\lim_{n \rightarrow \infty} \tau_n = \tau$ and $\tau_n < \tau$ (except on $\{\tau = 0\}$). Similarly a decreasing sequence of stopping times is said to *recall* τ if $\lim_{n \rightarrow \infty} \tau_n = \tau$ and $\tau_n > \tau$ (except on $\{\tau = \infty\}$). A *predictable time* is a stopping time that is announced by some sequence (τ_n) . A stopping time is called *simple* if it takes finitely many finite values. Let \mathbf{T} denote the set of all simple stopping times and $\mathbf{T}(\mathbf{S})$ the set of all simple stopping times with values in a countable dense subset \mathbf{S} of \mathbf{R}_+ containing 0.

\mathbf{X} denotes a separable Banach space with the dual space \mathbf{X}^* , \mathcal{C} the family of nonempty bounded close convex subsets of \mathbf{X} , \mathcal{K} the family of nonempty weakly compact convex subset of \mathbf{X} . We denote by \mathbf{D} a countable dense subset of \mathbf{X} and by \mathbf{D}^* (resp. $\mathbf{D}^*(M)$) a countable dense subset of \mathbf{X}^* in strong topology (resp. the Mackey topology $m(\mathbf{X}^*, \mathbf{X})$). Given subsets A, B and C of \mathbf{X} , the *distance function* $d(\cdot, C)$, the *support function* $s(\cdot, C)$ and the *Hausdorff distance* $h(A, B)$ are defined as in Hiai-Umegaki [20], Hess [17] and Wang-Xue [34]. Let \mathcal{G} be a sub σ -field of \mathcal{F} ; the concepts and notations such as \mathbf{X} -valued Bochner integrable random variables space $L^1[\Omega, \mathbf{X}]$, measurable multifunctions $\mathcal{M}[\mathcal{F}]$, Castaing representations, integrable \mathcal{C} -valued multifunctions space \mathcal{L}_{fc}^1 , integrable compact \mathcal{C} -valued multifunctions \mathcal{L}_{cc}^1 , integrable \mathcal{K} -valued multifunctions \mathcal{L}_{wkc}^1 , \mathcal{G} -measurable integral set $S_F^1(\mathcal{G})$ of multifunction F , conditional expectations, etc. are the same as in the above references too.

Let \mathbf{Q} be a directed index subset of \mathbf{R} containing infinity, $t \in \mathbf{Q}$, and let $(C_r)_{r \in \mathbf{Q}}$ be a class of subsets of \mathbf{X} . We put

$$(2.1) \quad s - \underline{\lim}_{s \rightarrow t} C_s := \{x \in \mathbf{X} \mid x = \lim_{s \rightarrow t} x_s, x_s \in C_s, \forall s \in \mathbf{Q}\},$$

$$(2.2) \quad w - \overline{\lim} C_s := \{x \in \mathbf{X} \mid (x_k) \text{ is weakly convergent to } x, \\ x_k \in C_{s(k)}, \forall k \in \mathbf{J}\}$$

where $(C_{s(k)})_{k \in \mathbf{J}}$ is a subsequence of $(C_s)_{s \in \mathbf{Q}}$ and $s(k) \rightarrow t$. A directed class (C_r) is said to *converge to C in the sense of Kuratowski-Mosco (K-M)*, if

$$C = s - \underline{\lim}_{s \rightarrow t} C_s = w - \overline{\lim}_{s \rightarrow t} C_s.$$

We write this limit as $(\text{K-M})\lim_{s \rightarrow t} C_s$ or $C_s \xrightarrow{\text{K-M}} C$. We say it *converges to C in the sense of Wijsman*, if for any $x \in \mathbf{X}$, $\lim_{s \rightarrow t} d(x, C_s) = d(x, C)$, that it *weakly converges to C*, if for any $x^* \in \mathbf{X}^*$, $\lim_{s \rightarrow t} s(x^*, C_s) = s(x^*, C)$, and that

the Hausdorff distance converges to C , if $\lim_{s \rightarrow t} h(C_s, C) = 0$. We denote the above limits by $C_s \xrightarrow{Wij_s} C$, $C_s \xrightarrow{w} C$ and $C_s \xrightarrow{h} C$ respectively. The relations among the above four kinds of convergence were mentioned in Wang-Xue [34, p.809].

For $F, G \in \mathcal{L}_{fc}^1$, define $\Delta(F, G) = \mathbf{E}h(F, G)$; then (\mathcal{C}, h) and $(\mathcal{L}_{fc}^1, \Delta)$ are complete metric spaces. Let \mathbf{L}_c^1 be the closure of the set of all simple functions in $(\mathcal{L}_{fc}^1, \Delta)$ (cf. Hiai-Umegaki [20, p.160]). When \mathbf{X} is a finite dimensional space, $\mathbf{L}_c^1 = \mathcal{L}_{fc}^1$.

Definition 2.1. An adapted \mathcal{L}_{fc}^1 -valued process $\{F_t, \mathcal{F}_t, t \in \mathbf{Q}\}$ (i.e., for every $t \in \mathbf{Q}$, the multifunction F_t is \mathcal{F}_t -measurable) is called a (multivalued) martingale, submartingale, or supermartingale, if

$$\mathbf{E}(F_t | \mathcal{F}_s) =, \supset, \text{ or } \subset F_s \quad a.s.$$

for all $s, t \in \mathbf{Q}, s < t$.

Definition 2.2. An adapted \mathbf{X} -valued process $\{f_t, \mathcal{F}_t, t \in \mathbf{Q}\}$ is called a martingale selection of an adapted multivalued process $\{F_t, \mathcal{F}_t, t \in \mathbf{Q}\}$ if it satisfies the following two conditions:

1. $\{f_t, \mathcal{F}_t, t \in \mathbf{Q}\}$ is a vector-valued martingale,
2. $\forall t \in \mathbf{Q}, f_t \in S_{F_t}^1(\mathcal{F}_t)$.

The set of martingale selections of a multivalued process $\{F_t, \mathcal{F}_t, t \in \mathbf{Q}\}$ will be denoted by $\text{MS}(F_t)$ (or $\text{MS}(F_n)$, if $\mathbf{Q} \subset \mathbf{Z}$).

We start with the following two simple lemmas.

Lemma 2.1. Suppose that $(C, C_n, n \in \mathbf{J}) \subset \mathcal{K}$ and $C \cup (\bigcup_{n \geq 1} C_n) \subset B \in \mathcal{K}$. Then

1. $C_n \xrightarrow{Wij_s} C$ if and only if

$$(2.3) \quad \lim_{n \rightarrow \infty} d(x, C_n) = d(x, C), \quad \forall x \in \mathbf{D}.$$

2. $C_n \xrightarrow{w} C$ if and only if

$$(2.4) \quad \lim_{n \rightarrow \infty} s(x^*, C_n) = s(x^*, C), \quad \forall x^* \in \mathbf{D}^*(M).$$

Thus we have the following lemma directly from Hess [17, Lemma 5.4]

Lemma 2.2. Under the conditions in Lemma 2.1, the following are equivalent:

1. $C_n \xrightarrow{\text{K-M}} C$,
2. $\begin{cases} \lim_{n \rightarrow \infty} d(x, C_n) = d(x, C), & \forall x \in \mathbf{D}, \\ \lim_{n \rightarrow \infty} s(x^*, C_n) = s(x^*, C), & \forall x^* \in \mathbf{D}^*(M). \end{cases}$

Now we define a useful distance on \mathcal{K} to make it a metric space. Let $\mathbf{D} = (x_k, k \in \mathbf{J})$ and $\mathbf{D}^*(M) = (x_k^*, k \in \mathbf{J})$. For any $A, B \in \mathcal{K}$

$$(2.5) \quad \rho(A, B) := \sum_{k=1}^{\infty} \frac{1}{2^k} \left(\frac{|d(x_k, A) - d(x_k, B)|}{1 + |d(x_k, A) - d(x_k, B)|} + \frac{|s(x_k^*, A) - s(x_k^*, B)|}{1 + |s(x_k^*, A) - s(x_k^*, B)|} \right)$$

(\mathcal{K}, ρ) is a metric space. Under the same hypothesis as in Lemma 2.1, we have the following equivalence by Lemma 2.2:

$$(2.6) \quad C_n \xrightarrow{\text{K-M}} C \iff \rho(C_n, C) \rightarrow 0.$$

§3. MARTINGALE SELECTIONS AND REPRESENTATION THEOREMS

In this section we establish the representation theorem for continuous parameter multivalued martingales: *Every multivalued martingale can be represented as the closure of a sequence of its martingale selections.* Luu [22] obtained it for \mathcal{L}^1_{fc} -valued martingales with discrete parameter. Hess [17] generalized the result to martingales with unbounded values.

Let $(\mathcal{F}_n, n \in \mathbf{Z})$ be an increasing σ -field filtration, $\mathcal{F}_\infty = \bigvee_{n \geq 1} \mathcal{F}_n$, $\mathcal{F}_{-\infty} = \bigcap_{n \leq -1} \mathcal{F}_n$. First we give a simple extension of Proposition 1.4 in Luu [22] to closed multivalued martingales. We begin with the following lemma.

Lemma 3.1. *Suppose that $F \in \mathcal{L}^1_{fc}$ and the σ -field $\mathcal{G} \subset \mathcal{F}$. Then there exists a sequence $\{f^i, i \geq 1\} \subset S^1_F$ such that $F(\omega) = cl\{f^i(\omega); i \geq 1\}, \forall \omega \in \Omega$, and $\mathbf{E}(F | \mathcal{G}) = cl\{\mathbf{E}(f^i | \mathcal{G}); i \geq 1\}, a.s.$*

Proof. There exist $(f^n, n \geq 1) \subset S^1_F$ and $(g^n, n \geq 1) \subset S^1_{\mathbf{E}(F|\mathcal{G})}(\mathcal{G}) = cl\{\mathbf{E}(f | \mathcal{G}); f \in S^1_F\}$ such that

$$(3.1) \quad \begin{aligned} F(\omega) &= cl\{f^n(\omega); n \geq 1\}, \quad a.s. \\ \mathbf{E}(F | \mathcal{G})(\omega) &= cl\{g^n(\omega); n \geq 1\}, \quad a.s. \end{aligned}$$

For any positive integer m , there exists a sequence $(f^{m,k}, k \geq 1) \subset S^1_F$ such that $\|g^m - \mathbf{E}(f^{m,k} | \mathcal{G})\|_1 \rightarrow 0 (k \rightarrow \infty)$. Thus there exists a subsequence $(k(j), j \geq 1)$ such that $\|g^m - \mathbf{E}(f^{m,k(j)} | \mathcal{G})\| \rightarrow 0$ almost surely, from which we have

$$(3.2) \quad \mathbf{E}(F | \mathcal{G}) = cl\{\mathbf{E}(f^{m,k(j)} | \mathcal{G}); m \geq 1, j \geq 1\}, \quad a.s.$$

Moreover, there exists a null set $N \in \mathcal{F}$ such that for all $\omega \in N^c, f^{m,k(j)}(\omega) \in F(\omega), \forall m \geq 1, j \geq 1$. Let

$$(3.3) \quad f^{m,k(j)}(\omega) = \begin{cases} f^{m,k(j)}(\omega) & \omega \in N^c, \\ f^1(\omega) & \omega \in N. \end{cases}$$

for $m \geq 1$ and $j \geq 1$. Then we have

$$(3.4) \quad \begin{aligned} F &= cl\{f^n, f^{m,k(j)}; n \geq 1, m \geq 1, j \geq 1\}, \\ \mathbf{E}(F | \mathcal{G}) &= cl\{\mathbf{E}(f^n | \mathcal{G}), \mathbf{E}(f^{m,k(j)} | \mathcal{G}); n \geq 1, m \geq 1, j \geq 1\} \quad a.s. \end{aligned}$$

This completes the proof of Lemma 3.1. □

From Lemma 3.1 we can easily prove the following theorem.

Theorem 3.1. *Suppose that $F \in \mathcal{L}^1_{fc}$. Let $F_n = \mathbf{E}(F | \mathcal{F}_n), -\infty \leq n \leq +\infty$. Then there exists a sequence $(g^i, i \geq 1) \subset S^1_F$ such that $F = cl\{g^i; i \geq 1\}$ and*

$$(3.5) \quad F_n = cl\{\mathbf{E}(g^i | \mathcal{F}_n); i \geq 1\} \quad a.s., -\infty \leq n \leq +\infty.$$

In the sequel, we prove the representation theorem for continuous parameter multivalued martingales.

Lemma 3.2. *If $F, G \in \mathcal{L}^1_{wkc}$ satisfy $F \subset G$ a.s. and $\mathbf{E}(F) = \mathbf{E}(G)$, then $F = G$, a.s.*

Proof. For all $x^* \in \mathbf{X}^*, s(x^*, F) \leq s(x^*, G)$ a.s. is obvious. But from Theorem 2.2 in Hiai-Umegaki [20], we have

$$(3.6) \quad \begin{aligned} \mathbf{E}(s(x^*, F)) &= s(x^*, \mathbf{E}(F)) = s(x^*, \mathbf{E}(G)) \\ &= \mathbf{E}(s(x^*, G)). \end{aligned}$$

So $s(x^*, F(\omega)) = s(x^*, G(\omega))$ for all ω except on a null set $N(x^*)$ related to the point x^* . Let $N = \bigcup_{x^* \in \mathbf{D}^*(M)} N(x^*)$; then $\mathbf{P}(N) = 0$. Since $\mathbf{D}^*(M)$ is dense in the Mackey topology $m(\mathbf{X}^*, \mathbf{X})$, we have

$$(3.7) \quad s(x^*, F(\omega)) = s(x^*, G(\omega)), \omega \in N^c, \forall x^* \in \mathbf{X}^*$$

Thus $F(\omega) = G(\omega), \omega \in N^c$. □

The following lemma concerns martingales having three items.

Lemma 3.3. *If $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3 \subset \mathcal{F}$ and $\{F_n, \mathcal{F}_n, n = 1, 2, 3\}$ is an \mathcal{L}^1_{wkc} -valued martingale and there is a sequence $\{f^k, k \geq 1\} \subset S^1_{\mathcal{F}_3}(\mathcal{F})$ such that*

$$(3.8) \quad \begin{aligned} F_3 &= cl\{f^k; k \geq 1\}, a.s. \\ F_1 &= cl\{\mathbf{E}(f^k | \mathcal{F}_1); k \geq 1\}, a.s., \end{aligned}$$

then $F_2 = cl\{\mathbf{E}(f^k | \mathcal{F}_2); k \geq 1\}, a.s.$

Proof. Let $G = cl\{\mathbf{E}(f^k | \mathcal{F}_2); k \geq 1\}$. From Theorem 5.1 in Hiai-Umegaki [20] we have

$$(3.9) \quad S^1_G(\mathcal{F}_2) \subset cl\{\mathbf{E}(f | \mathcal{F}_2); f \in S^1_{\mathcal{F}_3}(\mathcal{F}_3)\} = S^1_{F_2}(\mathcal{F}_2),$$

where the closure is in the norm of the space $L^1(\Omega, \mathbf{X})$. So $G \subset F_2, a.s.$

On the other hand,

$$(3.10) \quad \begin{aligned} F_1 &= cl\{\mathbf{E}(\mathbf{E}(f^k | \mathcal{F}_2) | \mathcal{F}_1); k \geq 1\} \\ &\subset \mathbf{E}(G | \mathcal{F}_1) \subset \mathbf{E}(F_2 | \mathcal{F}_1) = F_1 a.s. \end{aligned}$$

Therefore $\mathbf{E}(G | \mathcal{F}_1) = \mathbf{E}(F_2 | \mathcal{F}_1), a.s.$, thus $\mathbf{E}(G) = \mathbf{E}(F_2)$. By Lemma 3.2, we have $G = F_2, a.s.$ □

Remark 3.1. In Lemma 3.2 and 3.3, we can drop the multifunction set \mathcal{L}^1_{wkc} to \mathcal{L}^1_{fc} at the expense of introducing the strong separability hypothesis on the dual space \mathbf{X}^* .

Now we prove the main result of this section.

Theorem 3.2. *Let $\{F_t, \mathcal{F}_t, t \in \mathbf{R}_+\}$ be an \mathcal{L}^1_{fc} -valued martingale. If \mathbf{X}^* is strongly separable or $F_t \in \mathcal{L}^1_{wkc}, \forall t \geq 0$, then there exists a sequence of point-valued martingales $\{f_t^k, \mathcal{F}_t, t \in \mathbf{R}_+, k \geq 1\}$ in $MS(F_t)$ such that for all $t \geq 0$ and all $\omega \in \Omega$, $F_t(\omega) = cl\{f_t^k(\omega); k \in \mathbf{J}\}$.*

Proof. Using Luu [22, Proposition 1.4] or Hess [17, Corollary 3.3], we can see that for the sampling sequence $\{F_n, \mathcal{F}_n, n \in \bar{\mathbf{J}}\}$, there exists a sequence of point-valued martingales $\{g_n^k, \mathcal{F}_n, n \in \bar{\mathbf{J}}, k \geq 1\}$ in $MS(F_n)$ such that for any $n \in \bar{\mathbf{J}}, F_n(\omega) = cl\{g_n^k(\omega); k \geq 1\}, \forall \omega \in \Omega$.

If $t \in \bar{\mathbf{J}}$, we define $f_t^k = g_t^k$ for every $k \geq 1$.

Now for a fixed $t \in \mathbf{R}_+ \setminus \bar{\mathbf{J}}$, there exists an $n \in \bar{\mathbf{J}}$ such that $n < t < n + 1$. Then the three-term martingale $\{F_s, \mathcal{F}_s, s = n, t, n + 1\}$ satisfies the hypothesis of Lemma 3.3. So we have

$$(3.11) \quad F_t = cl\{\mathbf{E}(f_{n+1}^k | \mathcal{F}_t); k \geq 1\}, a.s.$$

where the exceptional set is denoted as $N_t, \mathbf{P}(N_t) = 0$. Also F_t has its Castaing representation $(g_t^k, k \geq 1) \subset S^1_{F_t}(\mathcal{F}_t)$ such that

$$(3.12) \quad F_t(\omega) = cl\{g_t^k(\omega), k \geq 1\}, \forall \omega \in \Omega.$$

Defining

$$(3.13) \quad f_t^k(\omega) = \begin{cases} \mathbf{E}(f_{n+1}^k | \mathcal{F}_t)(\omega), & \omega \in N_t^c, \\ g_t^k(\omega), & \omega \in N_t, \end{cases}$$

we conclude that $(f_t^k, k \geq 1) \subset S_{F_t}^1(\mathcal{F}_t)$ and, for all $\omega \in \Omega, F_t(\omega) = cl\{f_t^k(\omega); k \geq 1\}$ and $\{f_t^k, \mathcal{F}_t, t \in \mathbf{R}_+\}$ is a point-valued martingale for every $k \geq 1$. \square

§4. CONVERGENCE RESULTS

In this section, we give some convergence results to be used in the next section. The first one is a general convergent result concerning an adapted process $\{X_t, \mathcal{F}_t, t \in \mathbf{R}_+\}$ on a metric space (\mathbf{E}, ρ) . We can find similar results in Frangos [13] when \mathbf{E} is a Banach space.

Theorem 4.1. *Let $\{X_t, \mathcal{F}_t, t \in \mathbf{R}_+\}$ be an \mathbf{E} -valued process.*

1. *If for every bounded stopping time τ , there exists an $\Omega_\tau, \mathbf{P}(\Omega_\tau) = 1$, such that for any $\omega \in \Omega_\tau, \rho - \lim_{s \downarrow \tau, s \in \mathbf{S}} X_s(\omega)$ exists, then for almost all $\omega, \rho - \lim_{s \downarrow t, s \in \mathbf{S}} X_s(\omega)$ exists for all $t \geq 0$,*
2. *Suppose that $(X_t, t \in \mathbf{R}_+)$ has a right limit for every path at $t \geq 0$ and for any bounded predictable time τ announced by a sequence in $\mathbf{T}(\mathbf{S})$, $\rho - \lim_{s \uparrow \tau, s \in \mathbf{S}} X_s(\omega)$ exists for any $\omega \in \Omega_\tau$, where $\mathbf{P}(\Omega_\tau) = 1$. Then for almost all $\omega, \rho - \lim_{s \uparrow t, s \in \mathbf{S}} X_s(\omega)$ exists for any $t \in \mathbf{R}_+$.*

The proof of this theorem (but not the statement) is just like that in Choi-Sucheston [8, Theorems 2.3 and 3.2]. We need only change $\|X_s - X_t\|$ there into $\rho(X_s, X_t)$.

Next we turn our attention to the convergence of closed multivalued martingales. Dam [9] and Wang-Xue [34] proved the K-M convergence theorems for right-closed martingales in the ascending case. Recently, Papageorgiou [29] extended the result in Wang-Xue [34] to multivalued uniform amarts.

In the sequel, we discuss discrete parameter processes adapted to an increasing σ -field filtration $(\mathcal{F}_n, n \in \mathbf{Z})$. Denote $\mathcal{F}_\infty = \sigma(\bigcup_{n=1}^\infty \mathcal{F}_n), \mathcal{F}_{-\infty} = \bigcap_{n=1}^\infty \mathcal{F}_{-n}$.

Theorem 4.2. *Suppose that $F \in \mathcal{L}_{fc}^1$ and $F_n = \mathbf{E}(F | \mathcal{F}_n), -\infty \leq n \leq +\infty$, and suppose either of the following assumptions holds: (a) there exists a \mathcal{K} -valued multifunction G such that for all $\omega \in \Omega$ and $1 \leq n \leq +\infty$ we have $F_n(\omega) \subset H(\omega)$ (resp. $F_{-n}(\omega) \subset H(\omega)$), or (b) \mathbf{X}^* is strongly separable. Then $F_n \xrightarrow{\text{K-M}} F_\infty$ a.s. (resp. $F_{-n} \xrightarrow{\text{K-M}} F_{-\infty}$ a.s.), as $n \rightarrow \infty$.*

Proof. We apply techniques similar to those used in the proof of Papageorgiou [29, Theorem 14] to prove the ascending case. First, by Theorem 3.1, there exists a sequence $(f^i, i \geq 1) \subset S_F^1$, such that $F(\omega) = cl\{f^i(\omega), i \geq 1\}$ for all $\omega \in \Omega$ and

$$(4.1) \quad F_n = cl\{\mathbf{E}(f^i | \mathcal{F}_n); i \geq 1\}, \quad a.s.$$

for all $1 \leq n \leq +\infty$. By the convergence of vector-valued right-closed martingales, for all $i \geq 1$,

$$(4.2) \quad \lim_{n \rightarrow \infty} \|\mathbf{E}(f^i | \mathcal{F}_n) - \mathbf{E}(f^i | \mathcal{F}_\infty)\| = 0 \quad a.s.$$

Thus there exists an $N_1 \in \mathcal{F}_\infty, \mathbf{P}(N_1) = 0$, such that for all $\omega \in N_1^c$,

$$(4.3) \quad \lim_{n \rightarrow \infty} \mathbf{E}(f^i | \mathcal{F}_n)(\omega) = \mathbf{E}(f^i | \mathcal{F}_\infty)(\omega), \quad i \in \mathbf{J},$$

and

$$(4.4) \quad F_n(\omega) = cl\{\mathbf{E}(f^i \mid \mathcal{F}_n)(\omega); i \in \mathbf{J}\}, \forall n \in \mathbf{J}$$

Hence

$$(4.5) \quad F_\infty(\omega) \subset s - \underline{\lim}_{n \rightarrow \infty} F_n(\omega), \quad \omega \in N_1^c.$$

On the other hand, if assumption (a) holds, then from the convergence of right-closed real-valued martingales, for all $x^* \in \mathbf{X}^*$

$$(4.6) \quad \lim_{n \rightarrow \infty} s(x^*, F_n) = s(x^*, F_\infty), \quad a.s.$$

So there is an $N_2 \in \mathcal{F}_\infty$ satisfying $\mathbf{P}(N_2) = 0$ and such that for all $\omega \in N_2^c$,

$$(4.7) \quad \lim_{n \rightarrow \infty} s(x^*, F_n(\omega)) = s(x^*, F_\infty(\omega)), \quad \forall x^* \in \mathbf{D}^*(M).$$

Hence from Lemma 2.1, we have

$$(4.8) \quad F_n(\omega) \xrightarrow{w} F_\infty(\omega) (n \rightarrow +\infty).$$

This implies that

$$(4.9) \quad w - \overline{\lim}_{n \rightarrow \infty} F_n(\omega) \subset F_\infty(\omega), \quad \forall \omega \in N_2^c,$$

by N.S.Papageorgiou [26, Proposition 4.1].

If assumption (b) holds, there is also a null set $N_3 \in \mathcal{F}_\infty$ such that for all $\omega \in N_3^c$,

$$(4.10) \quad \lim_{n \rightarrow \infty} s(x^*, F_n(\omega)) = s(x^*, F_\infty(\omega)), \quad \forall x^* \in \mathbf{D}^*$$

Now for all $1 \leq n \leq +\infty$, $|F_n| \leq \mathbf{E}(|F| \mid \mathcal{F}_n), a.s.$ Here $\{\mathbf{E}(|F| \mid \mathcal{F}_n), \mathcal{F}_n, 1 \leq n \leq +\infty\}$ is a right-closed real-valued martingale, so by the Maximal Lemma of supermartingales (cf. C.Dellacherie and P.A.Meyer [11, p.14]), we have

$$(4.11) \quad \sup_{1 \leq n \leq \infty} |F_n| \leq \sup_{1 \leq n \leq \infty} \mathbf{E}(|F| \mid \mathcal{F}_n) < \infty \quad a.s.$$

Thus there exists another null set $N_4 \in \mathcal{F}_\infty$ including N_3 such that (4.11) is true for all $\omega \in N_4^c$. By the usual density narrative, for all $x^* \in \mathbf{X}^*$ we have

$$(4.12) \quad \lim_{n \rightarrow \infty} s(x^*, F_n(\omega)) = s(x^*, F_\infty(\omega)), \forall \omega \in N_4^c.$$

Hence also by Proposition 4.1 in Papageorgiou [26],

$$(4.13) \quad w - \overline{\lim}_{n \rightarrow \infty} F_n(\omega) \subset F_\infty(\omega), \quad \forall \omega \in N_4^c.$$

Combining (4.5) and (4.9) (or (4.13)), we obtain the proof.

For the reversed martingales, the proof is similar. □

Remark 4.1. Combining Wang-Xue [34, Theorems 2.1 and 3.2], we can also obtain the result for the ascending parameter case.

We have an analogous result for convergence in the Hausdorff metric for reversed martingales with $F \in \mathbf{L}_c^1$.

Theorem 4.3. *Suppose that $F \in \mathbf{L}_c^1$. Let $F_{-n} = \mathbf{E}(F \mid \mathcal{F}_{-n}), n \in \mathbf{J}$. Then $F_{-n} \xrightarrow{h} F_{-\infty} (n \rightarrow \infty)$ a.s. and $\Delta(F_{-n}, F_{-\infty}) \rightarrow 0$, where $F_{-\infty} = \mathbf{E}(F \mid \mathcal{F}_{-\infty})$.*

Proof. Without loss of generality, we may assume F is \mathcal{F}_{-1} -measurable. First suppose that F_{-1} itself is a simple function in \mathcal{L}_{fc}^1 , i.e. $F_{-1} = \sum_{k=1}^K H_k I_{A_k}$, where $\bigcup_{k=1}^K A_k = \Omega, A_k A_j = \emptyset, k \neq j$, and $H_k \in \mathcal{C}$. For any $F_1, F_2, G_1, G_2 \in \mathcal{C}$, it is known that

$$(4.14) \quad h(F_1 + F_2, G_1 + G_2) \leq h(F_1, G_1) + h(F_2, G_2).$$

Thus by the convergence of real reversed martingales

$$(4.15) \quad \begin{aligned} h(F_n, F_{-\infty}) &= h\left(\sum_{k=1}^K H_k \mathbf{E}(I_{A_k} \mid \mathcal{F}_{-n}), \sum_{k=1}^K H_k \mathbf{E}(I_{A_k} \mid \mathcal{F}_{-\infty})\right) \\ &\leq \sum_{k=1}^K h(H_k \mathbf{E}(I_{A_k} \mid \mathcal{F}_{-n}), H_k \mathbf{E}(I_{A_k} \mid \mathcal{F}_{-\infty})) \\ &\leq \max_{1 \leq i \leq K} |H_i| \cdot \sum_{k=1}^K |\mathbf{E}(I_{A_k} \mid \mathcal{F}_{-n}) - \mathbf{E}(I_{A_k} \mid \mathcal{F}_{-\infty})| \\ &\longrightarrow 0 \quad a.s. (n \rightarrow \infty). \end{aligned}$$

Now for a general $F_{-1} \in \mathbf{L}_c^1$ and for any $\epsilon > 0$, there is a simple function $H \in \mathcal{L}_{fc}^1$ such that H is \mathcal{F}_{-1} -measurable and $\Delta(F_{-1}, H) < \epsilon^2$. Assume that $H = \sum_{k=1}^K H_k I_{A_k}$, where $\bigcup_{k=1}^K A_k = \Omega, A_k A_j = \emptyset, k \neq j$, and $H_k \in \mathcal{C}$; denote $H_{-n} = \mathbf{E}(H \mid \mathcal{F}_{-n}), n \geq 1$, and $H_{-\infty} = \mathbf{E}(H \mid \mathcal{F}_{-\infty}), n \geq 1$. Then by the preceding proof, $h(H_{-n}, H_{-\infty}) \rightarrow 0$ a.s. So there exists a $N \in \mathbf{J}$ such that the probability $\mathbf{P}(\sup_{n \geq N} (h(H_{-n}, H_{-\infty})) > \epsilon) < \epsilon$. For any $n \geq 1$, by Hiai [21, Lemma 2.6]

$$(4.16) \quad \begin{aligned} h(F_{-n}, H_{-n}) &= h(\mathbf{E}(F_{-1} \mid \mathcal{F}_{-n}), \mathbf{E}(H \mid \mathcal{F}_{-n})) \\ &\leq \mathbf{E}(h(F_{-1}, H) \mid \mathcal{F}_{-n}) := h_{-n}; \end{aligned}$$

$$(4.17) \quad h(F_{-\infty}, H_{-\infty}) \leq \mathbf{E}(h(F_{-1}, H) \mid \mathcal{F}_{-\infty}) := h_{-\infty}.$$

Since $\{h_k, \mathcal{F}_k, -n \leq k \leq -1\}$ is a real martingale for any $n \in \mathbf{J}$, by the Maximal Lemma of supermartingales

$$(4.18) \quad \mathbf{P}\left(\max_{-n \leq k \leq -1} h_k > \epsilon\right) \leq \frac{1}{\epsilon} \mathbf{E}h_{-1} = \frac{1}{\epsilon} \Delta(F_{-1}, H) < \epsilon.$$

Let $n \rightarrow \infty$; we have $\mathbf{P}(\sup_{n \geq 1} h_{-n} > \epsilon) \leq \epsilon$. We also have

$$(4.19) \quad \mathbf{P}(h(F_{-\infty}, H_{-\infty}) > \epsilon) \leq \frac{1}{\epsilon} \mathbf{E}h_{-\infty} \leq \frac{1}{\epsilon} \mathbf{E}h_{-1} < \epsilon.$$

Therefore

$$(4.20) \quad \begin{aligned} \mathbf{P}(\sup_{n \geq N} h(F_{-n}, F_{-\infty}) > 3\epsilon) &\leq \mathbf{P}(\sup_{n \geq N} h(F_{-n}, H_{-n}) > \epsilon) \\ &+ \mathbf{P}(\sup_{n \geq N} h(H_{-n}, H_{-\infty}) > \epsilon) \\ &+ \mathbf{P}(h(H_{-\infty}, F_{-\infty}) > \epsilon) < 3\epsilon. \end{aligned}$$

Hence

$$F_{-n} \xrightarrow{h} F_{-\infty} (n \rightarrow \infty) \quad a.s.$$

Since $h(F_{-n}, F_{-\infty}) \leq |F_{-n}| + |F_{-\infty}| \leq \mathbf{E}(|F_{-1}| \mid \mathcal{F}_{-n}) + \mathbf{E}(|F_{-1}| \mid \mathcal{F}_{-\infty})$, we have

$$(4.21) \quad \Delta(F_{-n}, F_{-\infty}) = \mathbf{E}h(F_{-n}, F_{-\infty}) \longrightarrow 0 \quad (n \longrightarrow \infty).$$

□

Remark 4.2. Theorem 4.3 was obtained by Hiai-Umegaki [20, Thm. 6.2] under the additional assumption that \mathbf{X} is reflexive or $F(\omega)$ is compact for *a.s.* ω . Papageorgiou [29, Thm. 20] proved the convergence of amarts with compact values. Wang-Xue [34, Thm. 3.1] proved the ascending indexed case.

§5. REGULARITY

Let (\mathcal{K}, ρ) be the metric space defined in Section 2.

Theorem 5.1. *Suppose that $\{F_t, \mathcal{F}_t, t \in \mathbf{R}_+\}$ is an \mathcal{L}_{wkc}^1 -valued martingale. If there exists a multifunction H with weakly compact values such that $F_t(\omega) \subset H(\omega)$ for all $\omega \in \Omega$ and $t \geq 0$, then the following assertions are true:*

1. *For any bounded stopping time τ , $(\text{K-M}) \lim_{t \downarrow \tau, t \in \mathbf{S}} F_t(\omega)$ exists almost surely (the exceptional null set depends on τ).*
2. *For any bounded predictable time τ announced by a sequence in $\mathbf{T}(\mathbf{S})$, $(\text{K-M}) \lim_{t \uparrow \tau, t \in \mathbf{S}} F_t(\omega)$ exists almost surely (the exceptional null set depends on τ).*

Proof. (1) We can arbitrarily select a bounded sequence $(\tau_n, n \in \mathbf{J}) \subset \mathbf{T}(\mathbf{S})$ recalling τ . Let $b = \max_{\omega \in \Omega} \tau_1(\omega)$. Then $b < \infty$ and $\tau_n \leq b$ for $n \geq 1$. Denote $\mathbf{S}_b = \{s \in \mathbf{S} : s \leq b\}$; then $\{F_s, \mathcal{F}_s, s \in \mathbf{S}_b\}$ is an \mathcal{L}_{wkc}^1 -valued martingale right closed by F_b . This is also true for $\{F_{\tau_n}, \mathcal{F}_{\tau_n}, n \in \mathbf{J}\}$ by Papageorgiou [29, Theorem 12]. So by Theorem 4.2, we have

$$(5.1) \quad (\text{K-M}) \lim_{n \rightarrow \infty} F_{\tau_n} = G := \mathbf{E}(F_b \mid \mathcal{F}_{\tau_+}) = \mathbf{E}(F_b \mid \mathcal{F}_{\tau}), \quad a.s.;$$

the exceptional null set depends on τ . Let

$$(5.2) \quad \xi_t(\omega) = \begin{cases} \rho(F_t(\omega), G(\omega)), & t > \tau(\omega), \\ 0, & t \leq \tau(\omega). \end{cases}$$

Then $\{\xi_t, \mathcal{F}_t, t \in \mathbf{R}_+\}$ is an adapted process and

$$(5.3) \quad \lim_{n \rightarrow \infty} \xi_{\tau_n} = 0, \quad a.s.$$

Since the sequence (τ_n) is chosen arbitrarily, we know by Frangos [13, Theorem 2.2] that

$$(5.4) \quad \lim_{t \downarrow \tau, t \in \mathbf{S}} \xi_t = 0, \quad a.s.;$$

the exceptional null set depends on τ .

(2) Similarly, for a fixed bounded predictable time τ announced by any sequence $(\tau_n) \subset \mathbf{T}(\mathbf{S})$, we can prove that

$$(5.5) \quad (\text{K-M}) \lim_{n \rightarrow \infty} F_{\tau_n} = G_1 := \mathbf{E}(F_{b_1} \mid \mathcal{F}_{\tau_-}), \quad a.s.,$$

where $b_1 = \sup_{\omega \in \Omega} \tau(\omega) < \infty$ and the exceptional null set depends on τ and $G_1 \in \mathcal{L}_{wkc}^1$ by Papageorgiou [29, Proposition 3]. This means that for any fixed

$x \in \mathbf{D}$ and $x^* \in \mathbf{D}^*(M)$

- (a) $\lim_{n \rightarrow \infty} d(x, F_{\tau_n}) = d(x, G_1), \quad a.s.,$
- (b) $\lim_{n \rightarrow \infty} s(x^*, F_{\tau_n}) = s(x^*, G_1), \quad a.s.$

Also from Frangos [13, Thm. 2.2], there exist two exceptional null sets $N(x)$ and $N(x^*)$ s.t.

- (a') $\lim_{t \uparrow \tau(\omega), t \in \mathbf{S}} d(x, F_{\tau_n}(\omega)) = d(x, G_1(\omega)), \quad \omega \in N(x)^c,$
- (b') $\lim_{t \uparrow \tau(\omega), t \in \mathbf{S}} s(x^*, F_{\tau_n}(\omega)) = s(x^*, G_1(\omega)), \quad \omega \in N(x^*)^c.$

Let $N = (\bigcup_{x \in \mathbf{D}} N(x)) \cup (\bigcup_{x^* \in \mathbf{D}^*(M)} N(x^*)) \in \mathcal{F}_{\tau-}$. Then $\mathbf{P}(N) = 0$, and N only depends on τ . By Lemma 2.2

$$(5.6) \quad (\text{K-M}) \lim_{t \uparrow \tau(\omega), t \in \mathbf{S}} F_t(\omega) = G_1(\omega), \quad \omega \in N^c.$$

We obtain the proof. □

The above result and Theorem 4.1 imply the following theorem.

Theorem 5.2. *If the hypotheses of Theorem 5.1 hold, then for almost all $\omega \in \Omega$*

- 1. $\forall t \geq 0, \quad (\text{K-M}) \lim_{s \downarrow t, s \in \mathbf{S}} F_s(\omega)$ exists,
- 2. $\forall t > 0, \quad (\text{K-M}) \lim_{s \uparrow t, s \in \mathbf{S}} F_s(\omega)$ exists.

Now we prove one of the main results of this section.

Theorem 5.3. *If the hypotheses of Theorem 5.1 hold, then there exists an adapted process $\{\tilde{F}_t, \mathcal{F}_t, t \in \mathbf{R}_+\}$ with values in \mathcal{L}_{wkc}^1 such that*

- 1. $(\tilde{F}_t, t \in \mathbf{R}_+)$ is K-M right continuous and, for almost all ω ,

$$(5.7) \quad \tilde{F}_t(\omega) = (\text{K-M}) \lim_{s \downarrow t, s \in \mathbf{S}} F_s(\omega), \quad t \geq 0,$$

- 2. for almost all ω , $\tilde{F}_{t-}(\omega) = (\text{K-M}) \lim_{s \uparrow t, s \in \mathbf{R}_+} \tilde{F}_s(\omega)$ exists for every $t > 0$ and $\tilde{F}_{t-}(\omega) = (\text{K-M}) \lim_{s \uparrow t, s \in \mathbf{S}} F_s(\omega)$,
- 3. $\forall t \geq 0, \quad \tilde{F}_t = F_t, a.s.,$
- 4. $\{\tilde{F}_t, \mathcal{F}_t, t \in \mathbf{R}_+\}$ is a multivalued martingale.

Proof. (1) For N the exceptional null set in Theorem 5.2, we define

$$(5.8) \quad \tilde{F}_t(\omega) = \begin{cases} (\text{K-M}) \lim_{s \downarrow t, s \in \mathbf{S}} F_s(\omega), & \omega \in N^c, \\ \{\mathbf{0}\}, & \omega \in N; \end{cases}$$

then $\{\tilde{F}_t, \mathcal{F}_t, t \in \mathbf{R}_+\}$ is an \mathcal{L}_{wkc}^1 -valued adapted process. For a fixed $t \geq 0$, when $\omega \in N$, $\tilde{F}_t(\omega)$ is apparently K-M right continuous. Now for $\omega \notin N, \forall \epsilon > 0$, there exists a $\delta > 0$, such that for any $s \in \mathbf{S}$ satisfying $0 < s - t < \delta, \rho(\tilde{F}_t(\omega), F_s(\omega)) < \epsilon$. Thus for $r > t$ satisfying $r - t < \delta$, we have

$$(5.9) \quad \rho(\tilde{F}_t(\omega), \tilde{F}_r(\omega)) = \lim_{s \downarrow r, s \in \mathbf{S}} \rho(\tilde{F}_t(\omega), F_s(\omega)) \leq \epsilon.$$

Hence $\tilde{F}_t(\omega)$ is K-M right continuous.

(2) For $t > 0, \omega \in N^c, (\text{K-M}) \lim_{s \uparrow t, s \in \mathbf{S}} F_s(\omega)$ exists. The rest of the proof is similar to (1).

(3) $\forall t \geq 0$, by Theorem 4.2

$$(5.10) \quad \tilde{F}_t = \mathbf{E}(F_{t+1} | \mathcal{F}_{t+}) = \mathbf{E}(F_{t+1} | \mathcal{F}_t) = F_t, \quad a.s.$$

(4) This is apparently true from (3). \square

Using Theorem 3.3, Wang-Xue [34, Thm. 3.1], and the similar proofs of Theorem 5.1 to Theorem 5.3, we have the following regularity result for multivalued martingales with \mathbf{L}_c^1 values in convergence in the Hausdorff metric.

Theorem 5.4. *Suppose that $\{F_t, \mathcal{F}_t, t \in \mathbf{R}_+\}$ is an \mathbf{L}_c^1 -valued martingale. Then there exists an adapted process $\{\tilde{F}_t, \mathcal{F}_t, t \in \mathbf{R}_+\}$ with values in \mathbf{L}_c^1 such that*

1. $(\tilde{F}_t, t \in \mathbf{R}_+)$ is Hausdorff right-continuous and, for almost all ω ,

$$(5.11) \quad \tilde{F}_t(\omega) = h - \lim_{s \uparrow t, s \in \mathbf{S}} F_s(\omega), \quad t \geq 0,$$

2. for almost all ω , $\tilde{F}_{t-}(\omega) = h - \lim_{s \uparrow t, s \in \mathbf{R}_+} \tilde{F}_s(\omega)$ exists for every $t > 0$ and

$$\tilde{F}_{t-}(\omega) = h - \lim_{s \uparrow t, s \in \mathbf{S}} F_s(\omega),$$

3. $\forall t \geq 0$, $\tilde{F}_t = F_t, a.s.$,

4. $\{\tilde{F}_t, \mathcal{F}_t, t \in \mathbf{R}_+\}$ is a multivalued martingale.

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