

**A GENERALIZATION
OF CARLEMAN'S UNIQUENESS THEOREM
AND A DISCRETE PHRAGMÉN-LINDELÖF THEOREM**

B. KORENBLUM, A. MASCULLI, AND J. PANARIELLO

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ABSTRACT. Let $d\mu \geq 0$ be a Borel measure on $[0, \infty)$ and $A_n = \int_0^\infty t^n d\mu(t) < \infty$ ($n = 0, 1, 2, \dots$) be its moments. T. Carleman found sharp conditions on the magnitude of $\{A_n\}_0^\infty$ for $d\mu$ to be uniquely determined by its moments. We show that the same conditions ensure a stronger property: if $A'_n = \int_0^\infty t^n d\mu_1(t)$ are the moments of another measure, $d\mu_1 \geq 0$, with $\limsup_{n \rightarrow \infty} |A_n - A'_n|^{\frac{1}{n}} = \rho < \infty$, then the measure $d\mu - d\mu_1$ is supported on the interval $[0, \rho]$. This result generalizes both the Carleman theorem and a theorem of J. Mikusiński. We also present an application of this result by establishing a discrete version of a Phragmén-Lindelöf theorem.

§1. PRELIMINARIES

Definition 1.1. A sequence $\{A_n\}_0^\infty$ of positive numbers is called *logarithmically convex* if $A_n \leq \sqrt{A_{n-1}A_{n+1}}$ ($n = 1, 2, \dots$).

Clearly, the moments $A_n = \int_0^\infty t^n d\mu(t)$ of any Borel measure $d\mu \geq 0$, with $A_n < \infty$, form a logarithmically convex sequence.

Definition 1.2. If $\{A_n\}_0^\infty$ is an arbitrary sequence of positive numbers, then the *convex regularization of $\{A_n\}_0^\infty$ by means of logarithms*, denoted $\{A_n^c\}_0^\infty$, is formed by setting

$$A_n^c = \sup\{B_n : \{B_\nu\}_0^\infty \text{ is logarithmically convex, and } B_\nu \leq A_\nu \ (\nu = 0, 1, \dots)\}.$$

Proposition 1.3 ([Ma]). *Given a sequence $\{A_n\}_0^\infty$, $A_n > 0$, the following three conditions are equivalent:*

Carleman's condition:

$$(C) \quad \sum_{n=0}^{\infty} \frac{1}{\beta_n} = \infty, \quad \text{where} \quad \beta_n = \inf_{k \geq n} A_k^{\frac{1}{k}};$$

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Ostrowski's condition:

$$(O) \quad \int_1^\infty \frac{\log T(r) dr}{r^2} = \infty, \quad \text{where} \quad T(r) = \sup_{n \geq 0} \frac{r^n}{A_n};$$

Mandelbrojt's condition:

$$(M) \quad \text{either} \quad \liminf_{n \rightarrow \infty} A_n^{\frac{1}{n}} < \infty \quad \text{or} \quad \sum_{n=0}^\infty \frac{A_n^c}{A_{n+1}^c} = \infty.$$

We shall refer to the above as the (COM) condition.

Definition 1.4. We say a measure, $d\mu$, on $[0, \infty)$ is *supported on* $[0, \rho]$, if the total variation of $d\mu$ on (ρ, ∞) is 0, and ρ is the smallest number having this property.

Theorem 1.5 ([C] Carleman's Uniqueness Theorem). *If $A_n = \int_0^\infty t^n d\mu(t)$, $d\mu \geq 0$, and $\{\sqrt{A_n}\}_0^\infty$ satisfies the (COM) condition, then no other measure $d\mu_1 \geq 0$ has the same moments, A_n , for $n = 0, 1, \dots$*

Proposition 1.6 ([K] Analytic Quasianalyticity). *Let $\{A_n\}_0^\infty$ be a sequence of positive numbers. Let $C\{A_n\}$ be the class of functions, $f(z)$, analytic in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and infinitely differentiable on $\overline{\mathbb{D}}$ such that $\max_{z \in \mathbb{D}} |f^{(n)}(z)| \leq C_f A_n$. Then the following are equivalent:*

- (1) *The only $f \in C\{A_n\}$ vanishing with all of its derivatives at a point $\zeta_0 \in \partial\mathbb{D}$ is $f(z) \equiv 0$*
- (2) *$\{\sqrt{A_n}\}_0^\infty$ satisfies the (COM) condition.*

§2. THE MAIN THEOREM

Theorem 2.1. *Let $d\mu \geq 0$ and $d\mu_1 \geq 0$ be two Borel measures on $[0, \infty)$ with moments*

$$A_n = \int_0^\infty t^n d\mu(t) \quad \text{and} \quad A'_n = \int_0^\infty t^n d\mu_1(t) \quad (n = 0, 1, \dots).$$

If $\{\sqrt{A_n}\}_0^\infty$ satisfies the (COM) condition, and

$$(3) \quad \limsup_{n \rightarrow \infty} |A_n - A'_n|^{\frac{1}{n}} = \rho < \infty,$$

then the measure $d\mu - d\mu_1$ is supported on $[0, \rho]$. Conversely, if $\{A_n\}_0^\infty$ is a sequence of positive numbers such that $\{\sqrt{A_n}\}_0^\infty$ does not satisfy the (COM) condition, then there are distinct measures $d\mu \geq 0$ and $d\mu_1 \geq 0$ with

$$\int_0^\infty t^n d\mu(t) = \int_0^\infty t^n d\mu_1(t) \leq A_n \quad (n = 0, 1, \dots).$$

Remark 2.2. The uniqueness case of the above theorem corresponds to $\rho = 0$:

$$(4) \quad \lim_{n \rightarrow \infty} |A_n - A'_n|^{\frac{1}{n}} = 0.$$

The theorem then asserts that if $\{\sqrt{A_n}\}_0^\infty$ and $\{\sqrt{A'_n}\}_0^\infty$ satisfy the (COM) condition, and (4) holds as well, then $A_n = A'_n$ for $n = 1, 2, \dots$ ($A_0 \neq A'_0$ is possible).

Remark 2.3. If we replace the hypothesis that $\{\sqrt{A_n}\}_0^\infty$ satisfies (COM) by the much stronger requirement

$$\limsup_{n \rightarrow \infty} A_n^{\frac{1}{n}} < \infty,$$

then we obtain a theorem of Mikusiński [Mi].

Corollary 2.4. *Let $d\mu$ be a signed Borel measure on $[0, \infty)$ with*

$$B_n = \int_0^\infty t^n |d\mu(t)| < \infty \quad (n = 0, 1, \dots)$$

($|d\mu|$ is the variation measure of μ). If $\{\sqrt{B_n}\}_0^\infty$ satisfies the (COM) condition, then

$$\rho = \limsup_{n \rightarrow \infty} \left| \int_0^\infty t^n d\mu(t) \right|^{\frac{1}{n}} = \limsup_{n \rightarrow \infty} B_n^{\frac{1}{n}}$$

and $d\mu$ is supported on $[0, \rho]$ ($\rho = +\infty$ is not excluded).

§3. PROOF OF THEOREM 2.1

Define the signed measure, $d\sigma$, by $d\sigma = d\mu - d\mu_1$. We then define

$$M_n = A_n - A'_n = \int_0^\infty t^n d\sigma(t).$$

Also, let $B_n = \int_0^\infty t^n |d\sigma(t)| \leq A_n + A'_n$. Define the function

$$\Phi(z) = \int_0^\infty e^{tz} d\sigma(t) \quad \text{for } z \in \overline{\mathbb{C}_-} = \{z : \operatorname{Re}(z) \leq 0\}.$$

Φ is easily seen to be analytic in \mathbb{C}_- , and differentiable in $\overline{\mathbb{C}_-}$, with

$$\Phi^{(n)}(z) = \int_0^\infty t^n e^{tz} d\sigma(t).$$

We see that $\Phi^{(n)}(0) = M_n$ and $|\Phi^{(n)}(z)| \leq B_n$, for all $n = 0, 1, 2, \dots$ and for all $z \in \overline{\mathbb{C}_-}$.

Define another function, $F(z)$, via

$$F(z) = \sum_{n=0}^\infty \frac{M_n}{n!} z^n.$$

F is an entire function, and is of exponential type ρ (i.e. $|F(z)| \leq C_\epsilon e^{(\rho+\epsilon)|z|}$ for all $\epsilon > 0$).

Now, we observe that $F^{(n)}(0) = \Phi^{(n)}(0) = M_n$ ($n = 0, 1, 2, \dots$). We know that $|\Phi^{(n)}(z)| \leq B_n$. Also,

$$F^{(n)}(z) = \sum_{k=0}^\infty \frac{M_{n+k}}{k!} z^k,$$

and therefore

$$|F^{(n)}(z)| \leq C_R R^n \quad (n = 0, 1, \dots) \quad \text{for all } R > \rho \text{ and } |z| \leq 2.$$

Hence, we can conclude that on the closed disk $\{z : |z + 1| \leq 1\}$,

$$|\Phi^{(n)}(z) - F^{(n)}(z)| \leq B_n + C_R R^n \leq C'_R B_n R^n.$$

By Proposition 1.6, $\Phi - F \equiv 0$ on $\{z : |z + 1| \leq 1\}$; that is, F is an entire extension of Φ .

We next let $F_\delta(z) = e^{-(\rho+\delta)z}F(z)$. This function is bounded on the imaginary axis (by B_0), bounded on the non-negative real axis (by a constant dependent on δ) and of exponential type ρ . Hence, applying the Phragmén-Lindelöf theorem [Mar, vol. 2, p. 214], we can conclude that $|F_\delta(z)|$ is bounded on all of $\overline{\mathbb{C}_+} = \{z : \operatorname{Re}(z) \geq 0\}$ by B_0 . Taking limits (as $\delta \rightarrow 0^+$), we see that

$$|F(z)| \leq B_0 e^{\rho \operatorname{Re}(z)} \quad \text{on } \overline{\mathbb{C}_+}.$$

Define the function $G(z)$ by setting

$$G(z) = \frac{F(z)e^{-\rho z}}{1+z}.$$

Clearly, G is analytic on \mathbb{C}_+ and is square summable on the imaginary axis. Thus, we can apply the Paley-Wiener theorem [PW, p. 8, Theorem V] to conclude that

$$G(z) = \int_{-\infty}^0 \Psi(t)e^{tz} dt \quad \text{for some } \Psi \in L^2((-\infty, 0)).$$

We will assume that Ψ is defined for all real numbers (by setting $\Psi(t) = 0$ for all $t > 0$), and we will also extend our signed measure, $d\sigma$, to the entire real line (by requiring $|d\sigma((-\infty, 0))| = 0$).

On the imaginary axis, we have two representations for G :

$$\int_{-\infty}^{\infty} \Psi(t)e^{iyt} dt = G(iy) = \frac{e^{-i\rho y}F(iy)}{1+iy}.$$

So, we can conclude that

$$\int_{-\infty}^{\infty} \Psi(t - \rho)e^{iyt} dt = \frac{1}{1+iy} \int_{-\infty}^{\infty} e^{iyt} d\sigma(t).$$

Using the notation \tilde{f} for the Fourier transform of f (i.e. $\tilde{f}(x) = \int_{-\infty}^{\infty} f(y)e^{iyx} dy$), and letting $\Psi_\rho(t) = \Psi(t - \rho)$, we arrive at

$$\tilde{\Psi}_\rho(y) = \tilde{\gamma}(y)\tilde{d\sigma}(y),$$

where γ is the function

$$\gamma(x) = \begin{cases} e^x & \text{if } x \leq 0, \\ 0 & \text{if } x > 0. \end{cases}$$

Hence,

$$\Psi_\rho(y) = (\gamma * d\sigma)(y),$$

so

$$\Psi(t - \rho) = \int_t^\infty e^{t-x} d\sigma(x).$$

Thus, for all $t > \rho$, we have $\int_t^\infty e^{-x} d\sigma(x) = 0$, which implies that the total variation of $d\sigma$ on (ρ, ∞) is 0; i.e., $d\sigma = d\mu - d\mu_1$ is supported on $[0, \rho]$.

Conversely, if we are given a logarithmically convex sequence of positive numbers, $\{A_n\}_0^\infty$, with $\sum_{n=0}^\infty \sqrt{\frac{A_n}{A_{n+1}}} < \infty$, we set $A_{-1} = A_0$. A version of Proposition 1.6 for the half-plane $\overline{[K]}$ shows that there is some function, $f \not\equiv 0$, analytic in \mathbb{C}_+ and continuous on $\overline{\mathbb{C}_+}$, with

$$\sup_{x \geq 0} \int_{-\infty}^\infty |f^{(n)}(x + iy)|^2 dy \leq A_{n-1}^2 \quad (n = 0, 1, 2, \dots)$$

and $f^{(n)}(0) = 0$ ($n = 0, 1, 2, \dots$). Applying the same Paley-Wiener theorem, we obtain

$$f(z) = \int_0^\infty \phi(t) e^{-tz} dt \quad \text{for some } \phi \in L^2((0, \infty)),$$

as well as

$$f^{(n)}(z) = \int_0^\infty \phi(t) (-t)^n e^{-tz} dt.$$

Applying Plancherel's theorem, we find

$$\int_{-\infty}^\infty |f^{(n)}(iy)|^2 dy = 2\pi \int_{-\infty}^\infty t^{2n} |\phi(t)|^2 dt.$$

So, $\int_0^\infty t^{2n} |\phi(t)|^2 dt \leq A_{n-1}^2$. Hence, by the Cauchy-Schwarz inequality,

$$\int_0^\infty t^n |\phi(t)| dt = \int_0^1 t^n |\phi(t)| dt + \int_1^\infty t^{n+1} |\phi(t)| \frac{dt}{t} \leq A_{n-1} + A_n \leq K A_n$$

for some constant K .

Letting $d\sigma = \frac{1}{K} \phi(t) dt$, and then defining $d\mu = d\sigma^+$ and $d\mu_1 = d\sigma^-$, we get the desired measures. □

§4. SOME APPLICATIONS

In this section we examine some consequences of Theorem 2.1. The first result is a discrete Phragmén-Lindelöf type theorem.

Theorem 4.1. *Let $f(z)$ be analytic in $\mathbb{C}_+ = \{z : \operatorname{Re}(z) > 0\}$ and continuous in $\overline{\mathbb{C}_+}$. Define*

$$A_n = \sup_{0 \leq x \leq n} |f(x + iy)| \quad \text{for } n=1,2,\dots$$

Assume $A_n < \infty$ for all n .

If $\{\sqrt{A_n}\}_0^\infty$ satisfies the (COM) condition, and $\sup_{n \in \mathbb{N}} |f(n)| < \infty$, then $f(z)$ is bounded on $\overline{\mathbb{C}_+}$.

Proof. Consider the function

$$g(z) = f(z) \left(\frac{1 - e^{-z}}{z} \right)$$

and let

$$B_n = \sup_{0 \leq x \leq n} \left(\int_{-\infty}^{\infty} |g(x + iy)|^2 dy \right)^{\frac{1}{2}}.$$

We find that $B_n \leq 4A_n$. Using the Paley-Wiener theorem for the strip [PW, p. 7, Theorem IV], we have

$$g(x + iy) = \int_{-\infty}^{\infty} \phi(\sigma) e^{\sigma(x+iy)} d\sigma,$$

where this integral converges for all $x > 0$. Applying Plancherel's theorem yields

$$\int_{-\infty}^{\infty} |g(n + iy)|^2 dy = 2\pi \int_{-\infty}^{\infty} e^{2n\sigma} |\phi(\sigma)|^2 d\sigma \leq B_n^2 \leq 16A_n^2.$$

Setting $\Psi(t) = \phi(\log t)$ we get

$$g(1 + z) = \int_0^\infty t^z \Psi(t) dt \quad (\operatorname{Re} z > -1).$$

Also ($n \geq 0$),

$$\begin{aligned} C_n &= \int_0^\infty t^n |\Psi(t)| dt = \int_{-\infty}^\infty e^{(n+1)t} |\phi(t)| dt \\ &= \int_{-\infty}^0 e^{nt} |\phi(t)| e^t dt + \int_0^\infty e^{(n+2)t} |\phi(t)| e^{-t} dt \\ &\leq \left(\frac{1}{2} \int_{-\infty}^0 e^{2nt} |\phi(t)|^2 dt \right)^{\frac{1}{2}} + \left(\frac{1}{2} \int_0^\infty e^{2(n+2)t} |\phi(t)|^2 dt \right)^{\frac{1}{2}} \\ &\leq \frac{B_n + B_{n+2}}{2\sqrt{\pi}} \leq \frac{4}{\sqrt{\pi}} A_{n+2}. \end{aligned}$$

Thus $\{\sqrt{C_n}\}_0^\infty$ also satisfies the (COM) condition. Since $\sup_{n \in \mathbb{N}} |g(n)| \leq \sup_{n \in \mathbb{N}} |f(n)|$, Corollary 2.4 yields $\Psi(t) = 0$ ($t > 1$) and $\phi(t) = 0$ ($t > 0$). Therefore

$$f(z) = \frac{z}{1 - e^{-z}} \int_{-\infty}^0 \phi(\sigma) e^{\sigma z},$$

which is bounded on every strip $\{0 < \operatorname{Re} z < a\}$, satisfies $|f(z)| = O(|z|)$ ($|z| \rightarrow \infty$). Applying the Phragmén-Lindelöf theorem [Mar, vol. 2, p. 214], we see that f is bounded on $\overline{\mathbb{C}_+}$. □

Theorem 4.2. Let $f(z)$ be analytic in \mathbb{C}_+ and continuous on $\overline{\mathbb{C}_+}$. Assume

$$|f(iy)| \leq M \text{ for all real } y \quad \text{and} \quad \sup_{n \in \mathbb{N}} |f(n)| < \infty.$$

If $|f(z)| \leq Ce^{p|z|\log|z|}$ for some $p < 2$ and some constant C , then $|f(z)| \leq M$ for all $z \in \overline{\mathbb{C}_+}$.

Proof. Consider the function

$$h_1(z) = f(z)e^{i\frac{p\pi}{2}z}z^{-pz}$$

in the first quadrant: $Q_1 = \{z : 0 \leq \text{Arg}(z) \leq \frac{\pi}{2}\}$. This function is bounded on ∂Q_1 , and in Q_1

$$|h_1(z)| = O(e^{|z|^{1+\epsilon}}) \quad \text{for } |z| \rightarrow \infty, \epsilon > 0.$$

Applying the same Phragmén-Lindelöf theorem [Mar] gives that $h_1(z)$ is bounded in Q_1 . Using similar estimates on $h_2(z) = f(z)e^{-i\frac{p\pi}{2}z}z^{-pz}$ in the fourth quadrant $Q_4 = \{z : -\frac{\pi}{2} \leq \text{Arg}(z) \leq 0\}$ gives that $h_2(z)$ is bounded in Q_4 . These estimates yield

$$|f(x+iy)| \leq C \exp\{px \log|x+iy| - (py)\tan^{-1}\left(\frac{y}{x}\right) + \frac{p}{2}\pi|y|\}$$

in \mathbb{C}_+ . Now, consider the function

$$g(z) = f(z)e^{-\delta z^{\frac{p}{2}}} \quad \text{for any } \delta > 0.$$

If

$$D_n = \sup_{0 \leq x \leq n} |g(z)|,$$

then by a straightforward computation it is not hard to see that

$$D_n \leq C_\delta^n e^{\pi n} n^{2n}.$$

So, in fact, the sequence $\{\sqrt{D_n}\}_0^\infty$ satisfies the (COM) condition. This implies that $|g(z)|$ is bounded in $\overline{\mathbb{C}_+}$ by M . Letting $\delta \rightarrow 0^+$, we get

$$|f(z)| \leq M \text{ on } \overline{\mathbb{C}_+}. \quad \square$$

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DEPARTMENT OF MATHEMATICS AND STATISTICS, STATE UNIVERSITY OF NEW YORK AT ALBANY,
ALBANY, NEW YORK 12222