

SOME RESULTS ON FINITE DRINFELD MODULES

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ABSTRACT. Let K be a global function field, ∞ a degree one prime divisor of K and let A be the Dedekind domain of functions in K regular outside ∞ . Let H be the Hilbert class field of A , B the integral closure of A in H . Let ψ be a rank one normalized Drinfeld A -module and let \mathfrak{P} be a prime ideal in B . We explicitly determine the finite A -module structure of $\psi(B/\mathfrak{P}^N)$. In particular, if $K = \mathbb{F}_q(t)$, q is an odd prime number and ψ is the Carlitz $\mathbb{F}_q[t]$ -module, then the finite $\mathbb{F}_q[t]$ -module $\psi(\mathbb{F}_q[t]/\mathfrak{P}^N)$ is always cyclic.

1. INTRODUCTION

Recall that $G_m(\mathbb{Z}/p^N\mathbb{Z}) = (\mathbb{Z}/p^N\mathbb{Z})^\times$ is always cyclic except for the case that $p = 2$ and $N \geq 3$; if $N \geq 3$, then $(\mathbb{Z}/2^N\mathbb{Z})^\times$ is the direct product of two cyclic groups, one of order 2, the other of order 2^{N-2} . Let X be a smooth, projective, geometrically connected curve defined over the finite field \mathbb{F}_q with q elements and let ∞ be a rational point on X . We set K to be the function field of X over \mathbb{F}_q and $A \subset K$ to be the Dedekind domain of functions regular outside ∞ . We will consider the Drinfeld A -modules. From the view point of class field theory, these modules are interesting arithmetic objects over function fields. In particular, the rank one Drinfeld A -modules play a role entirely analogous to the important role played by G_m over number fields. This naturally leads us to explore an analogous phenomenon for rank one Drinfeld A -modules.

Let K_∞ be the completion of K with respect to ∞ and let C_∞ be the completion of the algebraic closure of K_∞ with respect to ∞ . Let $C_\infty\{\tau\} = \text{End}_{\mathbb{F}_q}(G_a/C_\infty)$ be the twisted polynomial ring in the q^{th} power Frobenius mapping τ . A rank one Drinfeld A -module ψ over C_∞ is an injective ring homomorphism $\psi : A \rightarrow C_\infty\{\tau\}$ such that the constant coefficient of $\psi_a(\tau)$ is equal to a and $\deg_\tau \psi_a(\tau) = -\text{Ord}_\infty a$ for all $a \in A$. A sign-function (cf. [5] and [3]) $\text{sgn} : K_\infty^\times \rightarrow \mathbb{F}_q^\times$ is a co-section of the inclusion map $\mathbb{F}_q^\times \hookrightarrow K_\infty^\times$ such that $\text{sgn}(\alpha) = 1$ for all $\alpha \in K_\infty^\times$ with $\alpha - 1$ vanishing at ∞ . A Drinfeld A -module ψ of rank one over C_∞ is said to be *sgn-normalized* if the leading coefficient of $\psi_a(\tau)$ in τ is equal to $\text{sgn}(a)$ for all $0 \neq a \in A$. It is known [5] that any Drinfeld A -module of rank one over C_∞ is isomorphic to a *sgn-normalized* A -module ψ over H , where H is the Hilbert class field of A , i.e., H is the maximal abelian extension of K such that the extension H/K completely splits over ∞ and is unramified over every finite place of K .

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Let B be the integral closure of A in H and let ψ be a rank one sgn-normalized Drinfeld A -module. Let \mathfrak{M} be an ideal in B . Via the action of $\psi \pmod{\mathfrak{M}}$, B/\mathfrak{M} becomes a finite A -module. We denote this by $\psi(B/\mathfrak{M})$. This finite A -module plays the role of $G_m(\mathbb{Z}/p^N\mathbb{Z})$. The purpose of this note is to determine the A -module structure of $\psi(B/\mathfrak{M})$. It is sufficient to consider the case $\psi(B/\mathfrak{P}^N)$, where \mathfrak{P} is a prime ideal in B . The structure of $\psi(B/\mathfrak{P}^N)$ is obtained in Theorems 2.1 and 2.2. In particular, if ψ is the Carlitz $\mathbb{F}_q[t]$ -module ([3], chapter 3) and $q \neq 2$, then $\psi(B/\mathfrak{P}^N)$ is always cyclic (cf. Corollaries 2.1 and 2.2). The Carlitz case is completely analogous to the classical case.

In section 3, we discuss the relations between $B_{\mathfrak{P}}$ and $\varprojlim \psi(B/\mathfrak{P}^N)$ via the exponential and logarithm functions of the sgn-normalized Drinfeld module ψ .

2. THE STRUCTURE OF $\psi(B/\mathfrak{P}^N)$

Let the notation $X, \mathbb{F}_q, \infty, K, A, H, B$ and sgn be as in the introduction. If M is a commutative \mathbb{F}_q -algebra, we let $M\{\tau\}$ denote the composition ring of Frobenius polynomials in τ , where τ is the q^{th} power mapping. From now on, we let ψ be a sgn-normalized rank one Drinfeld A -module over H , i.e., $\psi : A \rightarrow H\{\tau\}$ is a rank one Drinfeld A -module over H such that for any $a \in A$, the leading coefficient of $\psi_a(\tau)$ is equal to $\text{sgn}(a)$. It is known that $\psi_a(\tau) \in B\{\tau\}$ for all $a \in A$. Thus, via ψ , B becomes an A -module. We denote this module by $\psi(B)$ and denote the action $\psi_a(\tau)(b)$ by b^a for all $a \in A, b \in B$. Let \mathfrak{M} be an ideal in B . Since $\psi(A) \subset B\{\tau\}$, it follows that, via $\psi \pmod{\mathfrak{M}}$, B/\mathfrak{M} becomes a finite A -module. We denote this finite module by $\psi(B/\mathfrak{M})$. If the decomposition of \mathfrak{M} is equal to $\mathfrak{P}_1^{N_1}\mathfrak{P}_2^{N_2} \cdots \mathfrak{P}_L^{N_L}$, where $\mathfrak{P}_1, \mathfrak{P}_2, \dots, \mathfrak{P}_L$ are prime ideals in B , then, by the Chinese remainder theorem, we have

$$\psi(B/\mathfrak{M}) \cong \bigoplus_{i=1}^L \psi(B/\mathfrak{P}_i^{N_i}).$$

Thus to determine the A -module structure of $\psi(B/\mathfrak{M})$, it is sufficient to consider the case $\psi(B/\mathfrak{P}^N)$, where \mathfrak{P} is a prime ideal in B . Let $\wp = \mathfrak{P} \cap A$ and let f be the dimension of the vector space B/\mathfrak{P} over A/\wp . It follows from class field theory that $\text{Norm}_K^H \mathfrak{P} = \wp^f$ is a principal ideal in A . We let $\wp^f = (\pi_\wp)$ for the unique element $\pi_\wp \in A$ with $\text{sgn}(\pi_\wp) = 1$. It is known that $\psi(B/\mathfrak{P})$ is a cyclic A -module with Euler-Poincaré characteristic $\pi_\wp - 1$ (cf. [3], chapter 4), i.e., as A -module,

$$\psi(B/\mathfrak{P}) \cong A/(\pi_\wp - 1).$$

We let $\psi_\wp(\tau)$ be the monic generator of the left ideal of $H\{\tau\}$ generated by $\psi_a(\tau)$ for all $a \in \wp$. We also denote the polynomial $\psi_\wp(\tau)(x)$ in x by x^\wp for all $x \in H$. The important property of the polynomial $\psi_\wp(\tau)$, $\wp \subset A$ a prime ideal, is the following (cf. [5], Proposition 11.4):

$$f(x) = \psi_\wp(\tau)(x)/x = x^\wp/x$$

is an Eisenstein polynomial over B at any prime ideal \mathfrak{P} above \wp . Let $c_\wp = f(0) \in B$. Then we have $\text{deg}_{\mathfrak{P}} c_\wp = 1$. If \wp is a rational point on the curve X (i.e., \wp is a prime ideal in A of degree one) defined over \mathbb{F}_2 , then $\psi_\wp(\tau) = c_\wp\tau^0 + \tau^1 \in B\{\tau\}$, i.e., $x^\wp = \psi_\wp(\tau)(x) = c_\wp x + x^2$, where $c_\wp \in \mathfrak{P}$ but $c_\wp \notin \mathfrak{P}^2$.

Lemma 2.1. *Suppose that N is a positive integer, \mathfrak{P} and \wp are as above. Then for any $b_1, b_2 \in \psi(B)$, if $b_1 \equiv b_2 \pmod{\mathfrak{P}^N}$, then $b_1^\wp \equiv b_2^\wp \pmod{\mathfrak{P}^{N+1}}$.*

Proof. We may write $b_1 = b_2 + x$ for some $x \in \mathfrak{P}^N$. Then, by the Eisenstein polynomial property,

$$b_1^\varphi = (b_2 + x)^\varphi = b_2^\varphi + x^\varphi \equiv b_2^\varphi \pmod{\mathfrak{P}^{N+1}}. \quad \square$$

Lemma 2.2. *Suppose that \mathfrak{P} , φ and c_φ are as above. Then:*

- (1) *If $N \geq 2$ is a positive integer, then $x^{\varphi^{N-2}} \equiv x \cdot c_\varphi^{N-2} \pmod{\mathfrak{P}^N}$ for all $x \in \mathfrak{P}$ except for the case that φ is a rational point on the curve X defined over \mathbb{F}_2 .*
- (2) *If φ is a rational point on the curve X defined over \mathbb{F}_2 (i.e., φ is a prime ideal of A of degree one) and $N \geq 3$ is a positive integer, then*

$$x^{\varphi^{N-3}} \equiv x \cdot c_\varphi^{N-3} \pmod{\mathfrak{P}^N}$$

for all $x \in \mathfrak{P}^2$.

- (3) *If φ is a rational point on the curve X defined over \mathbb{F}_2 , then for all $x \in \mathfrak{P}, x \not\equiv 0, c_\varphi \pmod{\mathfrak{P}^2}$, we have $x^\varphi \in \mathfrak{P}^2$ but $x^\varphi \notin \mathfrak{P}^3$.*

Proof. Assertion (1) is obvious for $N = 2$. Now suppose that it is true for $N \geq 2$, i.e.,

$$x^{\varphi^{N-2}} \equiv x \cdot c_\varphi^{N-2} \pmod{\mathfrak{P}^N}.$$

We show that it is true for $N + 1$. Applying Lemma 2.1 and the Eisenstein polynomial property, we have

$$(x^{\varphi^{N-2}})^\varphi \equiv (x \cdot c_\varphi^{N-2})^\varphi \equiv c_\varphi \cdot (x \cdot c_\varphi^{N-2}) + (x \cdot c_\varphi^{N-2})^{q^{\deg \varphi}} \pmod{\mathfrak{P}^{N+1}}.$$

If $q \neq 2$ or $q = 2$ and φ is not a rational point on the curve X over \mathbb{F}_2 (i.e., $q^{\deg \varphi} \geq 3$), then we have $(x \cdot c_\varphi^{N-2})^{q^{\deg \varphi}} \in \mathfrak{P}^{N+1}$. Hence

$$x^{\varphi^{N-1}} \equiv xc_\varphi^{N-1} \pmod{\mathfrak{P}^{N+1}}.$$

This completes the proof of (1). The proof of (2) is similar.

To prove (3), since $x \in \mathfrak{P}$ and $\deg_{\mathfrak{P}} c_\varphi = 1$, $x^\varphi = c_\varphi \cdot x + x^2 = x(x + c_\varphi) \in \mathfrak{P}^2$. The assertion $x^\varphi \notin \mathfrak{P}^3$ follows from the facts that the characteristic of B is 2 and $x \not\equiv 0, c_\varphi \pmod{\mathfrak{P}^2}$. □

Lemma 2.3. *Suppose that \mathfrak{P} and φ are as above. Then:*

- (1) *If $N \geq 2$ is a positive integer and $x \in \mathfrak{P}, x \notin \mathfrak{P}^2$, then the submodule $\langle \bar{x} \rangle$ of $\psi(B/\mathfrak{P}^N)$ generated by $\bar{x} = x \pmod{\mathfrak{P}^N} \in \psi(B/\mathfrak{P}^N)$ is isomorphic to A/φ^{N-1} except for the case when φ is a rational point on the curve X defined over \mathbb{F}_2 .*
- (2) *Suppose that φ is a rational point on the curve X defined over \mathbb{F}_2 . If $N \geq 3$ is a positive integer and $x \in \mathfrak{P}^2, x \notin \mathfrak{P}^3$, then the submodule $\langle \bar{x} \rangle$ of $\psi(B/\mathfrak{P}^N)$ generated by $\bar{x} = x \pmod{\mathfrak{P}^N} \in \psi(B/\mathfrak{P}^N)$ is isomorphic to A/φ^{N-2} .*
- (3) *Suppose that φ is a rational point on the curve X defined over \mathbb{F}_2 and $N \geq 2$. Then for all $x \in \mathfrak{P}, x \not\equiv 0, c_\varphi \pmod{\mathfrak{P}^2}$, the submodule $\langle \bar{x} \rangle$ of $\psi(B/\mathfrak{P}^N)$ generated by $\bar{x} = x \pmod{\mathfrak{P}^N} \in \psi(B/\mathfrak{P}^N)$ is isomorphic to A/φ^{N-1} .*

Proof. By Lemma 2.2 (1), $x^{\varphi^{N-1}} \equiv x \cdot c_\varphi^{N-1} \pmod{\mathfrak{P}^{N+1}}$. Since $\deg_{\mathfrak{P}} c_\varphi = 1$ and $x \in \mathfrak{P}$, we have

$$x^{\varphi^{N-1}} \equiv 0 \pmod{\mathfrak{P}^N}.$$

Again by Lemma 2.2 (1), $\deg_{\mathfrak{P}} c_{\wp} = 1$ and $x \in \mathfrak{P}, x \notin \mathfrak{P}^2$,

$$x^{\wp^{N-2}} \equiv x \cdot c_{\wp}^{N-2} \not\equiv 0 \pmod{\mathfrak{P}^N}.$$

Since $\langle \bar{x} \rangle$ is a cyclic A-module, $\langle \bar{x} \rangle$ is isomorphic to A/\wp^{N-1} . The proof of (2) is similar, and the proof of (3) follows from Lemma 2.2 (3), (2). \square

Let $[H : K] = fr$ be the class number of A, where $f = [B/\mathfrak{P} : A/\wp]$. Then the main result is

Theorem 2.1. *Suppose that $\mathfrak{P}, \wp, \pi_{\wp}$ and f are as above. Then*

$$\psi(B/\mathfrak{P}^N) \cong \begin{cases} A/(\pi_{\wp} - 1) & \text{if } N = 1, \\ A/(\pi_{\wp} - 1) \oplus (A/\wp^{N-1})^f & \text{if } N > 1, \end{cases}$$

except for the case when \wp is a rational point on the curve X defined over \mathbb{F}_2 .

Proof. The case $N = 1$ follows from the theory of Drinfeld modules over finite fields (cf. [3], Chapter 4). We suppose that $N \geq 2$. Given $x \in \psi(B)$. Since $\psi(B/\mathfrak{P}) \cong A/(\pi_{\wp} - 1)$, it follows that $x^{\pi_{\wp}-1} \equiv 0 \pmod{\mathfrak{P}}$. By Lemma 2.2 (1), we have

$$(x^{\pi_{\wp}-1})^{\wp^{N-1}} \equiv 0 \pmod{\mathfrak{P}^N}.$$

This implies that the Euler-Poincaré characteristic of any A-cyclic submodule of $\psi(B/\mathfrak{P}^N)$ divides $(\pi_{\wp} - 1)\wp^{N-1}$.

Since $\psi(A) \subset B\{\tau\}$, $\psi(\mathfrak{P}/\mathfrak{P}^N)$ is a submodule of $\psi(B/\mathfrak{P}^N)$. We have

$$\dim_{\mathbb{F}_q} \psi(\mathfrak{P}/\mathfrak{P}^N) = (N - 1) \dim_{\mathbb{F}_q} B/\mathfrak{P} = f(N - 1) \dim_{\mathbb{F}_q} A/\wp.$$

Since $x^{\wp^{N-1}} \equiv 0 \pmod{\mathfrak{P}^N}$ for all $x \in \mathfrak{P}$ (by Lemma 2.2 (1)), as A-module

$$\psi(\mathfrak{P}/\mathfrak{P}^N) \cong \bigoplus_{i=1}^l A/\wp^{n_i}$$

for suitable positive integers $1 \leq n_1 \leq n_2 \leq \dots \leq n_l \leq N - 1$ such that

$$n_1 + n_2 + \dots + n_l = (N - 1)f.$$

By Lemma 2.2 (1) and Lemma 2.3 (1), the subset of elements \bar{x} in $\psi(\mathfrak{P}/\mathfrak{P}^N)$ such that $x^{\wp^{N-2}} \equiv 0 \pmod{\mathfrak{P}^N}$ is equal to $\psi(\mathfrak{P}^2/\mathfrak{P}^N)$. Counting cardinalities, we must have

$$l = f, \quad n_1 = n_2 = \dots = n_f = N - 1.$$

Thus we get

$$\psi(\mathfrak{P}/\mathfrak{P}^N) \cong (A/\wp^{N-1})^f.$$

Next, we take $g \in B$ such that $g \pmod{\mathfrak{P}}$ is a generator of $\psi(B/\mathfrak{P})$. We let $\langle \bar{g} \rangle$ be the A-submodule of $\psi(B/\mathfrak{P}^N)$ generated by $\bar{g} = g \pmod{\mathfrak{P}^N}$ in $\psi(B/\mathfrak{P}^N)$. We define the A-module homomorphism $\chi : \langle \bar{g} \rangle \rightarrow \psi(B/\mathfrak{P})$ by $\chi(x \pmod{\mathfrak{P}^N}) = x \pmod{\mathfrak{P}}$ for all $x \pmod{\mathfrak{P}^N}$ in $\langle \bar{g} \rangle$. Since $g \pmod{\mathfrak{P}}$ is a generator of $\psi(B/\mathfrak{P})$ and $\chi(g \pmod{\mathfrak{P}^N}) = g \pmod{\mathfrak{P}}$, χ is a surjective homomorphism of $\langle \bar{g} \rangle$ onto $\psi(B/\mathfrak{P})$. This implies that $(\pi_{\wp} - 1)$ divides the Euler-Poincaré characteristic of $\psi(B/\mathfrak{P}^N)$, because $\langle \bar{g} \rangle$ is a submodule of $\psi(B/\mathfrak{P}^N)$. Combining these, we obtain that $(\pi_{\wp} - 1)\wp^{f(N-1)}$ divides the Euler-Poincaré characteristic of $\psi(B/\mathfrak{P}^N)$; this implies that the Euler-Poincaré characteristic of $\psi(B/\mathfrak{P}^N)$ is equal to

$(\pi_\varphi - 1)\varphi^{f(N-1)}$; hence $\langle \bar{g} \rangle$ (or $\psi(\mathbb{B}/\mathfrak{P}^N)$) contains an A -submodule which is isomorphic to $A/(\pi_\varphi - 1)$. Therefore, we obtain that

$$\psi(\mathbb{B}/\mathfrak{P}^N) \cong A/(\pi_\varphi - 1) \oplus (A/\varphi^{N-1})^f. \quad \square$$

Theorem 2.2. *Suppose that φ is a rational point on the curve X defined over \mathbb{F}_2 . Then*

$$\psi(\mathbb{B}/\mathfrak{P}^N) \cong \begin{cases} A/(\pi_\varphi - 1), & \text{if } N = 1; \\ A/(\pi_\varphi - 1) \oplus (A/\varphi)^f, & \text{if } N = 2; \\ A/(\pi_\varphi - 1) \oplus A/\varphi \oplus (A/\varphi^{N-1})^{f-1} \oplus A/\varphi^{N-2}, & \text{if } N \geq 3. \end{cases}$$

Proof. The case $N = 1$ is standard. We suppose that $N \geq 2$. Using Lemmas 2.2 and 2.3, the proof is almost the same as the proof of Theorem 2.1. We obtain that the Euler-Poincaré characteristic of any A -cyclic submodule of $\psi(\mathbb{B}/\mathfrak{P}^N)$ divides $(\pi_\varphi - 1)\varphi^{N-1}$, the finite A -module $\psi(\mathfrak{P}/\mathfrak{P}^N)$ is annihilated by φ^{N-1} , and $\psi(\mathbb{B}/\mathfrak{P}^N)$ contains an A -submodule which is isomorphic to $A/(\pi_\varphi - 1)$. From these, we deduce that the Euler-Poincaré characteristic of $\psi(\mathbb{B}/\mathfrak{P}^N)$ is equal to $(\pi_\varphi - 1)\varphi^{f(N-1)}$ and

$$\psi(\mathbb{B}/\mathfrak{P}^N) \cong A/(\pi_\varphi - 1) \oplus \psi(\mathfrak{P}/\mathfrak{P}^N).$$

Next, we deal with the A -module structure of $\psi(\mathfrak{P}/\mathfrak{P}^N)$. For $N = 2$, since $\psi(\mathfrak{P}/\mathfrak{P}^2)$ is annihilated by φ , counting the dimension of $\psi(\mathfrak{P}/\mathfrak{P}^2)$ over A/φ , we obtain that $\psi(\mathfrak{P}/\mathfrak{P}^2) \cong (A/\varphi)^f$. For $N \geq 3$, since φ is a rational point on the curve X defined over \mathbb{F}_2 and $[\mathbb{B}/\mathfrak{P} : A/\varphi] = f$, as abelian group $\mathfrak{P}/\mathfrak{P}^2 \cong (A/\varphi)^f \cong \mathbb{F}_2^f$. Let S be the A -submodule of $\psi(\mathfrak{P}/\mathfrak{P}^N)$ generated by elements $x \pmod{\mathfrak{P}^N}$ is such that $x \in \mathfrak{P}$ but $x \notin \mathfrak{P}^2$. We define the abelian group homomorphism $\chi : S \rightarrow \psi(\mathfrak{P}^2/\mathfrak{P}^3)$ by $\chi(x \pmod{\mathfrak{P}^N}) = x^\varphi \pmod{\mathfrak{P}^3}$ for all $x \pmod{\mathfrak{P}^N} \in S$. From Lemma 2.2 (3), we know that if $x \pmod{\mathfrak{P}^N} \in S$ is such that $\chi(x \pmod{\mathfrak{P}^N}) = 0$, then $x \equiv 0, c_\varphi \pmod{\mathfrak{P}^2}$. This implies that $\dim_{\mathbb{F}_2} \chi(S) = f - 1$. Combining this with Lemma 2.3 (3) and the fact that S is annihilated by φ^{N-1} , we obtain that

$$S \cong A/\varphi \oplus (A/\varphi^{N-1})^{f-1}.$$

Since $\chi(c_\varphi \pmod{\mathfrak{P}^N}) = 0$ and $c_\varphi \pmod{\mathfrak{P}^N} \in S$, there exists an element $x \in \mathfrak{P}^2, x \notin \mathfrak{P}^3$, such that $x \pmod{\mathfrak{P}^3} \notin \chi(S)$. By Lemma 2.2 (2), we know that $\langle x \pmod{\mathfrak{P}^N} \rangle$ is a submodule of $\psi(\mathfrak{P}/\mathfrak{P}^N)$ which is isomorphic to A/φ^{N-2} . Combining these and counting the dimension of $\psi(\mathfrak{P}/\mathfrak{P}^N)$ over $A/\varphi \cong \mathbb{F}_2$, we obtain that

$$\psi(\mathbb{B}/\mathfrak{P}^N) \cong A/(\pi_\varphi - 1) \oplus A/\varphi \oplus (A/\varphi^{N-1})^{f-1} \oplus A/\varphi^{N-2}.$$

This completes the proof. □

As an application, we let $A = \mathbb{F}_q[t]$ and let ϕ be the Carlitz A -module, i.e., $\phi : A \rightarrow \mathbb{F}_2(t)\{\tau\}$ is given by

$$\phi_t(\tau) = t\tau^0 + \tau^1.$$

Then we have

Corollary 2.1. *If N is a positive integer and $\varphi = (p)$ is a prime ideal in A generated by the monic polynomial p , then the finite A -module*

$$\phi(A/\varphi^N) \cong A/(p^N - p^{N-1})$$

is cyclic except for the case when \mathbb{F}_q equals \mathbb{F}_2 and $p \mid t(t + 1)$.

Corollary 2.2. *If N is a positive integer, $A = \mathbb{F}_2[t]$, $\wp = (p)$ with $p = t$ or $t + 1 \in A$, then the finite A -module $\phi(A/\wp^N)$ is isomorphic to*

$$\begin{cases} A/(p-1), & \text{if } N = 1; \\ A/(t^2+t), & \text{if } N = 2; \\ A/(t^2+t) \oplus A/(p^{N-2}) & \text{if } N \geq 3. \end{cases}$$

3. PASSAGE TO THE LIMIT

Let the notation $X, \mathbb{F}_q, \infty, K, A, H, B, f, \pi_\wp$ and sgn be as before. Let ψ be a sgn -normalized rank one Drinfeld A -module over H . Suppose that \mathfrak{P} and \wp are as in section 2 and \wp does not correspond to a rational point on X if $q = 2$. It is well-know that there exists a lattice $\mathfrak{A}\zeta$, $\zeta \in C_\infty, \mathfrak{A}$ an ideal of A , of rank one such that ψ is determined by this lattice. The exponential function e_ψ associated to $\mathfrak{A}\zeta$ is defined to be

$$e_\psi(x) = z \prod_{a \in \mathfrak{A}} \left(1 - \frac{x}{a \cdot \zeta} \right) \in H\{\{\tau\}\}.$$

Let $H_{\mathfrak{P}}$ (resp. K_\wp) be the completion fields associated to \mathfrak{P} (resp. \wp). Let $B_{\mathfrak{P}} \subset H_{\mathfrak{P}}$ and $A_\wp \subset K_\wp$ be the rings of integers.

It follows from theorem 2.1 that as A -module

$$\begin{aligned} \psi(B_{\mathfrak{P}}) &= \psi(\varprojlim B/\mathfrak{P}^N) \\ &= \varprojlim \psi(B/\mathfrak{P}^N) \\ &= \varprojlim A/(\pi_\wp - 1) \oplus (A/\wp^{N-1})^f \\ &= A/(\pi_\wp - 1) \oplus A_\wp^f \\ &= A/(\pi_\wp - 1) \oplus B_{\mathfrak{P}}. \end{aligned}$$

We know that the coefficients of e_ψ are in H and these coefficients are obtained by solving a recursion equation via any ψ_a , $a \in A, a \notin \mathbb{F}_q$ (cf. [3], Lemma 4.6.5). We can deduce from this recursion that e_ψ converges in a neighborhood of 0. Thus there exist element $\alpha \in H_{\mathfrak{P}}$ such that $e(x) = e_\psi(\alpha \cdot x)$ is an analytic injective function of $B_{\mathfrak{P}}$ into $B_{\mathfrak{P}}$. By the property of the exponential function e_ψ , we obtain that $e(ax) = \psi_a(e(x))$ for all $a \in A$. Combining these, we have

Theorem 3.1. *As A -module,*

$$\psi(B_{\mathfrak{P}}) \cong \begin{cases} A/(\pi_\wp - 1) \oplus A/\wp \oplus B_{\mathfrak{P}} & \text{if } q = 2 \text{ and } \wp \text{ is a rational point;} \\ A/(\pi_\wp - 1) \oplus B_{\mathfrak{P}} & \text{otherwise.} \end{cases}$$

Moreover, one has an analytic map $e : B_{\mathfrak{P}} \rightarrow \psi(B_{\mathfrak{P}})$ satisfying the following commutative diagram:

$$\begin{array}{ccc} B_{\mathfrak{P}} & \xrightarrow{e} & \psi(B_{\mathfrak{P}}) \\ a \downarrow & & \downarrow \psi_a \\ B_{\mathfrak{P}} & \xrightarrow{e} & \psi(B_{\mathfrak{P}}) \end{array} .$$

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