

## BORDISM OF TWO COMMUTING INVOLUTIONS

PEDRO L. Q. PERGHER

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ABSTRACT. In this paper we obtain conditions for a Whitney sum of three vector bundles over a closed manifold,  $\varepsilon_1 \oplus \varepsilon_2 \oplus \varepsilon_3 \rightarrow F$ , to be the fixed data of a  $(Z_2)^2$ -action; these conditions yield the fact that if  $(\varepsilon_1 \oplus R) \oplus \varepsilon_2 \oplus \varepsilon_3 \rightarrow F$  is the fixed data of a  $(Z_2)^2$ -action, where  $R \rightarrow F$  is the trivial one dimensional bundle, then the same is true for  $\varepsilon_1 \oplus \varepsilon_2 \oplus \varepsilon_3 \rightarrow F$ . The results obtained, together with techniques previously developed, are used to obtain, up to bordism, all possible  $(Z_2)^2$ -actions fixing the disjoint union of an even projective space and a point.

### 1. INTRODUCTION

From [4, 28.1] one has an exact sequence

$$0 \rightarrow \mathcal{N}_n^{Z_2} \xrightarrow{F} \bigoplus_{j=0}^n \mathcal{N}_{n-j}(BO(j)) \xrightarrow{\partial} \mathcal{N}_{n-1}(BO(1)) \rightarrow 0$$

which shows that the equivariant bordism class of an involution  $(M, T)$  is determined by the bordism class of its fixed point data  $\eta \rightarrow F_T$ ; additionally, it yields the fact that a real vector bundle  $\eta \rightarrow F$  is fixed data of an involution if and only if  $\lambda \rightarrow RP(\eta)$  bounds in  $\mathcal{N}_{n-1}(BO(1))$ , where  $\lambda$  is the usual line bundle over the projective space bundle  $RP(\eta)$ . The first task of this paper is to obtain the analogue of this fact for  $(Z_2)^2$ -actions; specifically, given a Whitney sum of vector bundles over a closed manifold,  $\varepsilon_1 \oplus \varepsilon_2 \oplus \varepsilon_3 \rightarrow F$ , we want to determine conditions in terms of Whitney numbers for this sum to be the fixed data of a  $(Z_2)^2$ -action (Theorem 1). The extension of the above sequence for  $(Z_2)^k$ -actions is the “exact sequence of bordism of  $((Z_2)^k, q)$ -manifolds-bundles”, introduced by Stong in [10] and which shows that the stationary point structure of a  $(Z_2)^k$ -action determines the bordism class. Looking at the case  $k = 1$ , it is then natural to appeal to this sequence to handle the problem in question. For  $k = 2$  and  $q = 0$  it is

$$0 \rightarrow \mathcal{N}_n((Z_2)^2, 0) \xrightarrow{F} \bigoplus \mathcal{N}_{r,s}(Z_2, 1) \xrightarrow{S} \widehat{\mathcal{N}}_{n-1,s}((Z_2)^2, 0) \rightarrow 0.$$

By iteration this sequence yields

$$0 \rightarrow \mathcal{N}_n((Z_2)^2, 0) \xrightarrow{FF} \bigoplus \mathcal{N}_{r',r'',s',s''}((Z_2)^0, 3) \xrightarrow{S} \widehat{\mathcal{N}}_{r-1,s}((Z_2)^2, 0) \rightarrow 0.$$

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Here, for a given  $(Z_2)^2$ -action  $(M, \Phi)$ , one has  $FF[M, \Phi] = [\varepsilon_1 \oplus \varepsilon_2 \oplus \varepsilon_3 \rightarrow F]$ . Unfortunately, since iteration destroys exactness, the condition  $S[\varepsilon_1 \oplus \varepsilon_2 \oplus \varepsilon_3] = 0$  is a necessary but not sufficient condition for this sum be the fixed data of a  $(Z_2)^2$ -action. This will be bypassed simply by putting together that sequence with some additional facts extracted from [10] and [9]. In a certain sense, the mentioned conditions serve as a complement of [6] and [5] in the case  $k = 2$ ; in these works it was shown how to construct the equivariant bordism class of a manifold with  $(Z_2)^k$ -action from its fixed data.

Next we recall from [4] that for a given vector bundle  $\eta \rightarrow F$  we have

$$\Delta[\lambda \rightarrow RP(\eta \oplus R)] = [\lambda \rightarrow RP(\eta)],$$

where  $\Delta : \mathcal{N}_n(BO(1)) \rightarrow \mathcal{N}_{n-1}(BO(1))$  is the Smith homomorphism and  $R \rightarrow F$  denotes the trivial 1-dimensional bundle over  $F$ . This yields particularly the fact that if  $\eta \oplus R \rightarrow F$  is fixed data of an involution, then the same is true for  $\eta \rightarrow F$ . The second step of this paper will consist of using the above conditions to obtain the analogue of this fact for  $(Z_2)^2$ -actions; specifically, we will show that if  $\varepsilon_1, \varepsilon_2$  and  $\varepsilon_3$  are vector bundles over  $F$  such that  $(\varepsilon_1 \oplus R) \oplus \varepsilon_2 \oplus \varepsilon_3 \rightarrow F$  is fixed data of some  $(Z_2)^2$ -action, then the same is valid for  $\varepsilon_1 \oplus \varepsilon_2 \oplus \varepsilon_3 \rightarrow F$  (Theorem 2).

Finally, together with facts of [7], the above result will be used to obtain, up to bordism, all possible  $(Z_2)^2$ -actions fixing  $RP(2n) \cup \{p\}$ ,  $p = \text{point}$  (Theorem 3). Referring to this we observe that in [7] we have developed a method to analyse the bordism classes of  $(Z_2)^k$ -actions fixing  $V^n \cup \{p\}$ , where  $V^n$  is a connected closed  $n$ -dimensional manifold, and where the starting point is the knowledge of the set  $\mathcal{A}$  of all equivariant bordism classes of involutions containing a representative  $(M, T)$  with  $M$  connected and with the above fixed set. This method has shown itself particularly effective when  $\mathcal{A}$  has a single element; in this case, if  $(M, \Phi)$  is a  $(Z_2)^k$ -action fixing  $V^n \cup \{p\}$ , we have seen that the fixed data of  $(M, \Phi)$  bears, in terms of bordism, a strong resemblance to the fixed data of  $\sigma\Gamma_t^k(W, \tau)$ , where  $[(W, \tau)]$  is the only element of  $\mathcal{A}$  and  $\sigma\Gamma_t^k$  denotes certain operations which produce special  $(Z_2)^k$ -actions from a given involution. In particular, we have proved further that  $(M, \Phi)$  is bordant to some action of type  $\sigma\Gamma_t^k(W, \tau)$  when  $V^n = S^p \times S^q$ ,  $S^n$  or  $RP(2p+1)$  (see [7] and [8]). However, if  $\mathcal{A}$  has more than one element, the classification seems more difficult because of the fact that the results of [7] indicate the possibility of occurrence of other classes added to those produced by the operations  $\sigma\Gamma_t^k$ . In fact, as it will be seen, this is what surprisingly happens when  $V^n = RP(2r)$  (one knows from [3] that in this case the set  $\mathcal{A}$  has more than one element). So in this case the results of [7] are not definitively enough to determine the classification. In this way the above results constitute, at least in the case  $k = 2$ , an additional tool to handle this question.

## 2. CONDITIONS FOR A WHITNEY SUM OF VECTOR BUNDLES TO BE THE FIXED DATA OF A $(Z_2)^2$ -ACTION

As usual,  $(Z_2)^2$  is considered as the group generated by two commuting involutions  $T_1, T_2$ . Given a  $(Z_2)^2$ -action  $(M, \Phi)$ ,  $\Phi = (T_1, T_2)$ , one may consider the fixed data of  $\Phi$  as being  $\varepsilon_1 \oplus \varepsilon_2 \oplus \varepsilon_3 \rightarrow F_\Phi$ , where  $F_\Phi$  denotes the set of stationary points of  $\Phi$  and  $\varepsilon_1, \varepsilon_2$  and  $\varepsilon_3$  are the normal bundles of  $F_\Phi$  in  $F_{T_2}, F_{T_1 T_2}$  and  $F_{T_1}$ , respectively (this establishes a standard order for the fixed data which can be naturally extended for  $(Z_2)^k$ -actions). We will be using  $\lambda_i, i = 1, 2, 3$ , to denote the line

bundle over  $RP(\varepsilon_i)$  (or occasionally over  $RP(\varepsilon_i \oplus R)$ ) and will also be suppressing all bundle maps.

One starts then with a Whitney sum of real vector bundles over a closed manifold,  $\varepsilon_1 \oplus \varepsilon_2 \oplus \varepsilon_3 \rightarrow F$ , and wishes that this sum be the fixed data of a  $(Z_2)^2$ -action  $(M^n, \Phi)$ . Here  $F$  is not assumed to be connected and for each component  $F_i$  one must have

$$\dim(\varepsilon_{1|F_i}) + \dim(\varepsilon_{2|F_i}) + \dim(\varepsilon_{3|F_i}) + \dim(F_i) = n$$

For each  $0 \leq l \leq n$  one has the Stong's sequence of  $(Z_2, 1)$ -manifolds-bundles before mentioned:

$$0 \rightarrow \mathcal{N}_{l,n-l}(Z_2, 1) \xrightarrow{F_l} \bigoplus_{l', l'', g', g''} \mathcal{N}_{l', l'', g', g''}(Z_2^0, 3) \xrightarrow{S_l} \widehat{\mathcal{N}}_{l-1, n-l}(Z_2, 1) \rightarrow 0$$

where the sum is over all sequences with  $l' + l'' = l$ ,  $g' + g'' = n - l$ . Taking the sum over  $0 \leq l \leq n$  one obtains the exact sequence

$$0 \rightarrow \bigoplus_l \mathcal{N}_{l,n-l}(Z_2, 1) \xrightarrow{\oplus F_l} \bigoplus_l \bigoplus_{l', l'', g', g''} \mathcal{N}_{l', l'', g', g''}(Z_2^0, 3) \xrightarrow{\oplus S_l} \bigoplus_l \widehat{\mathcal{N}}_{l-1, n-l}(Z_2, 1) \rightarrow 0$$

and  $\varepsilon_1 \oplus \varepsilon_2 \oplus \varepsilon_3 \rightarrow F$  represents a bordism class  $\alpha$  of the middle term. Suppose  $\oplus S_l(\alpha) = 0$ ; as it can be seen by using the definition of  $S_l$  and [9, 8.7], this means that  $\lambda_1 \oplus (\varepsilon_2 \oplus (\varepsilon_3 \otimes \lambda_1)) \rightarrow RP(\varepsilon_1)$  bounds as an element of

$$\bigoplus_l \mathcal{N}_{l-1}(BO(1) \times BO(n-l)).$$

Then there is  $\beta \in \bigoplus_l \mathcal{N}_{l,n-l}(Z_2, 1)$  such that  $\oplus F_l(\beta) = \alpha$ ;  $\beta$  is the bordism class of a 4-tuple  $(V, T; \eta, \overline{T})$ , where  $T$  is an involution on the closed manifold  $V$ ,  $\eta$  is a vector bundle over  $V$  and  $\overline{T}$  is an involution of  $\eta$  by a bundle map covering  $T$  (we will be omitting the indication about the variation of dimension over the several components, but this should always be assumed). According to a remark after Proposition 8 of [9, page 67] one can exhibit an explicit representative for  $\beta$ : denoting a typical element of the total space of the bundle  $\varepsilon_2 \oplus (\varepsilon_3 \otimes \lambda_1) \rightarrow RP(\varepsilon_1 \oplus R)$  by  $((u, w \otimes r), [v, s]_p)$ , where  $(u, w \otimes r)$  belongs to the fiber of  $\varepsilon_2 \oplus (\varepsilon_3 \otimes \lambda_1)$  over  $[v, s]_p$ , which in turn denotes an element of the total space of  $RP(\varepsilon_1 \oplus R)$  belonging to the fiber over the point  $p \in F$ , the correspondence

$$((u, w \otimes r), [v, s]_p) \rightarrow ((u, -w \otimes r), [-v, s]_p)$$

defines an involution  $T$  on the total space of  $\varepsilon_2 \oplus (\varepsilon_3 \otimes \lambda_1)$  recovering the involution  $t$  on  $RP(\varepsilon_1 \oplus R)$  given by  $t[v, s]_p = [-v, s]_p$ . The mentioned representative is then

$$(RP(\varepsilon_1 \oplus R), t; \varepsilon_2 \oplus (\varepsilon_3 \otimes \lambda_1), T).$$

Considering the Stong's sequence relative to  $k = 2, q = 0$ ,

$$0 \rightarrow \mathcal{N}_n((Z_2)^2, 0) \xrightarrow{F} \bigoplus_l \mathcal{N}_{l,n-l}(Z_2, 1) \xrightarrow{S} \widehat{\mathcal{N}}_{n-1}((Z_2)^2, 0) \rightarrow 0$$

all that remains is to analyse more closely the condition  $S(\beta) = 0$ ; in fact, this condition will guarantee the existence of a  $(Z_2)^2$ -action  $(M^n, \Phi)$  with  $\bigoplus_l F_l(F[M^n, \Phi]) =$

$\alpha$ , that is, with fixed data bordant to  $\varepsilon_1 \oplus \varepsilon_2 \oplus \varepsilon_3 \rightarrow F$ . But a standard equivariant construction yields in this case a  $(Z_2)^2$ -action having precisely this fixed data.

We remark that  $\widehat{\mathcal{N}}_{n-1}((Z_2)^2, 0)$  is the bordism group of  $(Z_2)^2$ -actions for which the first involution acts freely. Using the definition of  $S$  we see that a representative of  $S(\beta)$  is  $(S(\varepsilon_2 \oplus (\varepsilon_3 \otimes \lambda_1)), A, T)$ , where  $S(\ )$  means sphere bundle construction,  $A$  is the antipodal involution and  $T$  is restriction of the previous  $T$ . Now one knows from [10, Prop.3] that  $S$  maps the summand  $\mathcal{N}_{n-1,1}(Z_2, 1)$  isomorphically onto  $\widehat{\mathcal{N}}_{n-1}((Z_2)^2, 0)$  and that the inverse for  $S$  on this summand is given by

$$[(W; T_1, T_2)] \rightarrow [(W/T_1, T_2^*; \lambda, \overline{T}_2^*)]$$

where  $\lambda$  is the line bundle of the double cover of  $W/T_1$  by  $W$  and  $T_2^*, \overline{T}_2^*$  are induced by  $T_2$ . Hence we must analyse the class of

$$(RP(\varepsilon_2 \oplus (\varepsilon_3 \otimes \lambda_1)), T^*; \lambda', \overline{T}^*)$$

in  $\mathcal{N}_{n-1,1}(Z_2, 1)$ , where  $\lambda'$  is the line bundle over  $RP(\varepsilon_2 \oplus (\varepsilon_3 \otimes \lambda_1))$  (which fibers over  $RP(\varepsilon_1 \oplus R)$ ) and  $T^*, \overline{T}^*$  are, as above described, induced by the previous  $T$ . To make this analysis one simply looks at the fixed point information; according to [10, Prop. 3], for a given element  $[(W, T; \eta, \overline{T})] \in \mathcal{N}_{l,p}(Z_2, 1)$ , this information is given by the class of  $\mu \oplus \eta^+ \oplus \eta^- \rightarrow F_T$  in

$$\bigoplus_{i=0}^l \bigoplus_{p'+p''=p} \mathcal{N}_i(BO(l-i) \times BO(p') \times BO(p''))$$

where  $\mu \rightarrow F_T$  is the normal bundle of  $F_T$  in  $W$  and  $\eta|_{F_T} = \eta^+ \oplus \eta^-$ ,  $\eta^+$  and  $\eta^-$  denoting the subbundles of  $\eta|_{F_T}$  where  $\overline{T}$  acts as  $+1$  and  $-1$ , respectively.

Adopting the type of notation previously established one sees that  $\overline{T}^*$  is given by

$$([(u, w \otimes r), [v, s]_p], x) \rightarrow([(u, -w \otimes r), [-v, s]_p], x).$$

The fixed set of  $T^*$  must be taken over the fixed set of  $t$ , which is the disjoint union of  $RP(\varepsilon_1)$  and  $RP(R) = F$ .

1) Over  $RP(\varepsilon_1)$  one has

$$\begin{aligned} T^*[(u, w \otimes r), [v, 0]_p] &= [(u, -w \otimes r), [-v, 0]_p] = [(u, -w \otimes -r), [v, 0]_p] \\ &= [(u, w \otimes r), [v, 0]_p], \end{aligned}$$

that is,  $T^*$  acts trivially. Hence  $RP(\varepsilon_2 \oplus (\varepsilon_3 \otimes \lambda_1))$  fibering over  $RP(\varepsilon_1)$  is a component of the fixed set of  $T^*$ , and the normal bundle is

$$\lambda_1 \rightarrow RP(\varepsilon_2 \oplus (\varepsilon_3 \otimes \lambda_1)).$$

Over this component  $\overline{T}^*$  acts trivially, so  $\eta^+ \oplus \eta^-$  is  $\lambda' \oplus 0$  restricted to this component.

2) Over  $RP(R) = F$  one has  $\lambda_1 = R$ , hence  $RP(\varepsilon_2 \oplus (\varepsilon_3 \otimes \lambda_1)) = RP(\varepsilon_2 \oplus \varepsilon_3)$ ; further,  $T^*$  acts in this case as

$$[(u, w \otimes r), [0, s]_p] \rightarrow [(u, -(w \otimes r)), [0, s]_p].$$

Therefore one has two components of fixed points:

- a)  $RP(\varepsilon_2)$  fibering over  $F$  with normal bundle  $(\varepsilon_3 \otimes \lambda_2) \oplus \varepsilon_1$ ;
- b)  $RP(\varepsilon_3)$  fibering over  $F$  with normal bundle  $(\varepsilon_2 \otimes \lambda_3) \oplus \varepsilon_1$ .

Over the component of a)  $\overline{T}^*$  acts trivially and  $\lambda' = \lambda_2$ , hence  $\eta^+ \oplus \eta^- = \lambda_2 \oplus 0$ . Over the component of b) one has

$$\overline{T}^*([(0, w \otimes r), [0, s]_p], x) =([(0, -w \otimes r), [0, s]_p], x) =([(0, w \otimes r), [0, s]_p], -x)$$

and  $\lambda' = \lambda_3$ , hence  $\eta^+ \oplus \eta^- = 0 \oplus \lambda_3$ .

One concludes that  $S(\beta) = 0$  if and only if

$$(\lambda_1 \oplus \lambda' \oplus 0 \rightarrow RP(\varepsilon_2 \oplus (\varepsilon_3 \otimes \lambda_1))) \cup (((\varepsilon_3 \otimes \lambda_2) \oplus \varepsilon_1) \oplus \lambda_2 \oplus 0 \rightarrow RP(\varepsilon_2)) \\ \cup (((\varepsilon_2 \otimes \lambda_3) \oplus \varepsilon_1) \oplus 0 \oplus \lambda_3 \rightarrow RP(\varepsilon_3))$$

bounds in

$$\left(\bigoplus_{l=0}^{n-1} \mathcal{N}_l(BO(n-1-l) \times BO(1) \times BO(0))\right) \\ \oplus \left(\bigoplus_{l=0}^{n-1} \mathcal{N}_l(BO(n-1-l) \times BO(0) \times BO(1))\right),$$

where the first base space fibers over  $RP(\varepsilon_1)$  and the remaining fiber over  $F$  (note that no contribution is made by the components  $F_i$  with dimension= $n$ , which can be ignored with no loss). Putting together with  $\oplus S_l(\alpha) = 0$  and taking into account the dimension of the bordism groups involved, one obtains the following

**Theorem 1.** *A Whitney sum  $\varepsilon_1 \oplus \varepsilon_2 \oplus \varepsilon_3 \rightarrow F$  is fixed data of a  $(Z_2)^2$ -action if and only if the following conditions hold:*

a)  $\lambda_1 \oplus (\varepsilon_2 \oplus (\varepsilon_3 \otimes \lambda_1)) \rightarrow RP(\varepsilon_1)$  bounds in  $\bigoplus_{l=0}^{n-1} \mathcal{N}_l(BO(1) \times BO(n-1-l))$ ,

where  $RP(\varepsilon_1)$  fibers over  $F$ ;

b)  $(\lambda_1 \oplus \lambda' \oplus 0 \rightarrow RP(\varepsilon_2 \oplus (\varepsilon_3 \otimes \lambda_1))) \cup (((\varepsilon_3 \otimes \lambda_2) \oplus \varepsilon_1) \oplus \lambda_2 \oplus 0 \rightarrow RP(\varepsilon_2))$  bounds in  $\bigoplus_{l=0}^{n-1} \mathcal{N}_l(BO(n-1-l) \times BO(1) \times BO(0))$ , where the base spaces fiber over

$RP(\varepsilon_1)$  and  $F$ , respectively;

c)  $((\varepsilon_2 \otimes \lambda_3) \oplus \varepsilon_1) \oplus 0 \oplus \lambda_3 \rightarrow RP(\varepsilon_3)$  bounds in

$$\bigoplus_{l=0}^{n-1} \mathcal{N}_l(BO(n-1-l) \times BO(0) \times BO(1)),$$

with  $RP(\varepsilon_3)$  fibering over  $F$ .

Since the conditions obtained are detectable through Whitney numbers calculation, our first purpose is established.

*Remark.* Looking closer at the dimensions of the bordism groups involved, one observes that condition (b) can alternatively be replaced by the two following conditions:

b1)

$$(\lambda_1 \oplus \lambda' \oplus 0 \rightarrow RP(\varepsilon_2 \oplus (\varepsilon_3 \otimes \lambda_1))) \\ \cup \left( \bigcup_{\dim(\varepsilon_1) + \dim(\varepsilon_3) = 1} ((\varepsilon_3 \otimes \lambda_2) \oplus \varepsilon_1) \oplus \lambda_2 \oplus 0 \rightarrow RP(\varepsilon_2) \right)$$

bounds in  $\mathcal{N}_{n-2}(BO(1) \times BO(1) \times BO(0))$ ;

b2)

$$\bigcup_{\dim(\varepsilon_1)+\dim(\varepsilon_3)\neq 1} ((\varepsilon_3 \otimes \lambda_2) \oplus \varepsilon_1) \oplus \lambda_2 \oplus 0 \rightarrow RP(\varepsilon_2)$$

bounds in  $\bigoplus_{l=0}^{n-1} \mathcal{N}_l(BO(n-1-l) \times BO(1) \times BO(0))$ .

### 3. A TECHNICAL RESULT

The result to be proved in this section is:

**Theorem 2.** *Suppose that  $\varepsilon_1, \varepsilon_2$  and  $\varepsilon_3$  are vector bundles over  $F$  such that  $(\varepsilon_1 \oplus R) \oplus \varepsilon_2 \oplus \varepsilon_3 \rightarrow F$  is fixed data of a  $(Z_2)^2$ -action. Then  $\varepsilon_1 \oplus \varepsilon_2 \oplus \varepsilon_3 \rightarrow F$  is fixed data of a  $(Z_2)^2$ -action.*

*Proof.* It is easy to check that if a Whitney sum of the form  $R \oplus \varepsilon \oplus 0$  is fixed data of a  $(Z_2)^2$ -action, then  $\lambda \rightarrow RP(\varepsilon)$  bounds; hence we can assume  $\dim(\varepsilon_1) + \dim(\varepsilon_3) > 0$ , that is,  $\dim(\varepsilon_1 \oplus R) + \dim(\varepsilon_3) > 1$ . But then the remark after Theorem 1 implies that

- i)  $\lambda_1 \oplus (\varepsilon_2 \oplus (\varepsilon_3 \otimes \lambda_1)) \rightarrow RP(\varepsilon_1 \oplus R)$ ,
- ii)  $\lambda_1 \oplus \lambda' \oplus 0 \rightarrow RP(\varepsilon_2 \oplus (\varepsilon_3 \otimes \lambda_1))$ ,
- iii)  $((\varepsilon_3 \otimes \lambda_2) \oplus (\varepsilon_1 \oplus R)) \oplus \lambda_2 \oplus 0 \rightarrow RP(\varepsilon_2)$  and
- iv)  $((\varepsilon_2 \otimes \lambda_3) \oplus (\varepsilon_1 \oplus R)) \oplus 0 \oplus \lambda_3 \rightarrow RP(\varepsilon_3)$

bound in the appropriate bordism groups, with ii) fibering over  $RP(\varepsilon_1 \oplus R)$  and the remaining over  $F$ . From [9, Section 8, Lemma 4] one has a commutative diagram

$$\begin{array}{ccc} \bigoplus \mathcal{N}_l(BO(r) \times BO(s) \times BO(t)) & \xrightarrow{S} & \mathcal{N}_{l+r-1}(BO(1) \times BO(s+t)) \\ I_* \downarrow & & \uparrow \Delta \\ \bigoplus \mathcal{N}_l(BO(r+1) \times BO(s) \times BO(t)) & \xrightarrow{S} & \mathcal{N}_{l+r}(BO(1) \times BO(s+t)) \end{array}$$

where  $I_*$  adds a trivial line bundle to the first factor,  $\Delta$  is the Smith homomorphism and the sum is over  $(l, r, s, t)$  with  $l + r + s + t = n$  and  $0 \leq l \leq n - 1$ . Since  $SI^*[\varepsilon_1 \oplus \varepsilon_2 \oplus \varepsilon_3]$  is the class of i), the class obtained from i) by replacing  $\varepsilon_1 \oplus R$  by  $\varepsilon_1$  is zero. Now the fact that the classes obtained from iii) and iv) by replacing  $\varepsilon_1 \oplus R$  by  $\varepsilon_1$  are also zero can be checked trivially by looking at the characteristic classes. Noting that if the classes obtained from i), ii), iii) and iv) by replacing  $\varepsilon_1 \oplus R$  by  $\varepsilon_1$  are zero, then conditions (a), (b) and (c) of Theorem 1 will be valid for  $\varepsilon_1 \oplus \varepsilon_2 \oplus \varepsilon_3$ , all that remains is to prove the following

**Lemma.** *Let  $\eta \rightarrow F$  and  $\mu \rightarrow RP(\eta \oplus R)$  be vector bundles such that  $\xi \otimes \lambda \rightarrow RP(\mu)$  bounds in  $\mathcal{N}_*(BO(1) \times BO(1))$ , where  $\xi \rightarrow RP(\mu)$  is the line bundle and  $\lambda \rightarrow RP(\mu)$  is the pullback of the line bundle  $\lambda \rightarrow RP(\eta \oplus R)$ . Then  $\xi' \oplus \lambda' \rightarrow RP(\mu')$  bounds in  $\mathcal{N}_{*-1}(BO(1) \times BO(1))$ ; here,  $\mu' = \mu|_{RP(\eta)}$ ,  $\xi' = \xi|_{RP(\mu')}$  and  $\lambda' = \lambda|_{RP(\mu')}$ .*

*Proof.* Denote the total Stiefel-Whitney class of  $\eta$  by  $W(\eta) = 1 + v_1 + v_2 + \dots + v_r$  and let  $c, c', d$  and  $d'$  be, respectively, the characteristic classes of  $\lambda, \lambda', \xi$  and  $\xi'$ . Let  $j : RP(\mu') \rightarrow RP(\mu)$  be the inclusion and  $\theta$  be the vector bundle tangent to the fibers of  $RP(\mu) \rightarrow RP(\eta \oplus R)$ ; set  $\theta' = \theta|_{RP(\mu')}$ . Denote yet  $W(RP(\mu)) = 1 + W_1 + W_2 + \dots + W_n$ ,  $W(RP(\mu')) = 1 + w_1 + w_2 + \dots + w_{n-1}$ ,  $W(\theta) = 1 + L_1 + L_2 + \dots + L_s$ ,

$W(\theta') = 1 + L'_1 + L'_2 + \dots + L'_s$ . Clearly  $j^*(c) = c', j^*(d) = d'$  and  $j^*(L_i) = L'_i$ . From [1] one has

$$W(RP(\mu')) = W(F) \left( \sum_{i=0}^r (1 + c')^{r-i} v_i \right) (1 + L'_1 + \dots + L'_s)$$

and

$$W(RP(\mu)) = (1 + c)W(F) \left( \sum_{i=0}^r (1 + c)^{r-i} v_i \right) (1 + L_1 + \dots + L_s)$$

where the classes  $v_i, W(F) \in H^*(F, Z_2)$  are being considered, with the same notation, as classes in  $H^*(RP(\mu))$  or  $H^*(RP(\mu'))$  via bundle maps. Hence  $j^*(W_i) = w_i + c'w_{i-1}$  for  $1 \leq i \leq n$ . We will show that all Whitney numbers

$$c'^p d'^q w_{i_1} w_{i_2} \dots w_{i_t} [RP(\mu')]$$

are zero; this will be made by using induction on  $l = i_1 + i_2 + \dots + i_t$ . For  $l = 0$  one has

$$c'^p d'^q [RP(\mu')] = j^*(c^p d^q) [RP(\mu')] = c^p d^q j_*([RP(\mu')]) = c^{p+1} d^q [RP(\mu)] = 0$$

by hypothesis; here we used the fact that  $RP(\mu')$  is the submanifold of  $RP(\mu)$  dual to  $c$ . Suppose the fact true for all  $l < i_1 + i_2 + \dots + i_t$ ; then

$$\begin{aligned} c'^p d'^q w_{i_1} \dots w_{i_t} [RP(\mu')] &= c'^p d'^q (j^*(W_{i_1}) + c'w_{i_1-1}) \dots (j^*(W_{i_t}) + c'w_{i_t-1}) [RP(\mu')] \\ &= c'^p d'^q j^*(W_{i_1}) \dots j^*(W_{i_t}) [RP(\mu')] + A = c^{p+1} d^q W_{i_1} \dots W_{i_t} [RP(\mu)] + A = A \end{aligned}$$

by hypothesis; here  $A$  is a sum of terms of the form

$$c'^a d'^q j^*(W_{b_1}) \dots j^*(W_{b_e}) w_{f_1} \dots w_{f_g} [RP(\mu')]$$

with  $b_1 + \dots + b_e + f_1 + \dots + f_g < i_1 + \dots + i_t$ . The replacement of each  $j^*(W_{b_i})$  by  $w_{b_i} + c'w_{b_i-1}$  converts  $A$  to a sum of terms of the form  $c'^m d'^q w_{h_1} \dots w_{h_u} [RP(\mu')]$  with  $h_1 + \dots + h_u < i_1 + \dots + i_t$ , and all these terms are zero by the induction hypothesis.  $\square$

*Remark.* We observe that if  $\varepsilon_1 \oplus \varepsilon_2 \oplus \varepsilon_3 \rightarrow F$  is fixed data of a  $(Z_2)^2$ -action, then the same is true for any permutation of the  $\varepsilon_{i'}$ s; it is then irrelevant, in Theorem 2, to which factor  $\varepsilon_i$  the trivial line bundle is added.

#### 4. $(Z_2)^2$ -ACTIONS FIXING $RP(2n) \cup \{p\}$

As announced in the introduction, we will obtain in this section, up to bordism, all possible  $(Z_2)^2$ -actions fixing  $RP(2n) \cup \{p\}$ .

The methods of [7] yield particularly specific information about  $(Z_2)^2$ -actions; we need first to describe this information. For a fixed smooth closed connected  $n$ -dimensional manifold  $V^n$ , denote by  $\mathcal{A}$  the collection of all equivariant bordism classes of involutions containing a representative  $(W, T)$  with  $W$  connected and  $V^n \cup \{p\}$  as fixed point set. Setting  $\mathcal{A} = \{[W_i^{n_i}, \tau_i]\}_i$ , let  $\eta_i \rightarrow V^n$  denote the normal bundle of  $V^n$  in each  $W_i^{n_i}$ . The results of [7] yield then the following

**Fact 1.** *Let  $(M, \Phi)$  be a  $(Z_2)^2$ -action with fixed set  $V^n \cup \{p\}$ , and let*

$$\left( \bigoplus_{j=1}^3 \varepsilon_j \rightarrow V^n \right) \cup \left( \bigoplus_{j=1}^3 \mu_j \rightarrow p \right)$$

denote the fixed data of  $\Phi$ . Then one of the following situations occurs:

i) there is exactly one bundle  $\varepsilon_{j_0}$  bordant to some  $\eta_i$ , and the remaining  $\varepsilon_j$ 's are the 0-dimensional bundle; in this case,  $\mu_{j_0}$  is the trivial  $n_i$ -dimensional bundle  $R^{n_i} \rightarrow p$ , and the remaining  $\mu_j$ 's are the 0-bundle.

ii) there are exactly two bundles  $\varepsilon_{j_1}, \varepsilon_{j_2}$  bordant to  $\eta_i$ 's, and the remaining  $\varepsilon_{j_0}$  is bordant to the tangent bundle  $T(V^n) \rightarrow V^n$ ; in this case, considering  $\varepsilon_{j_1}$  and  $\varepsilon_{j_2}$  bordant, respectively, to  $\eta_{i_1}$  and  $\eta_{i_2}$ , one has  $\mu_{j_1} = R^{n_{i_1}} \rightarrow p, \mu_{j_2} = R^{n_{i_2}} \rightarrow p$  and  $\mu_{j_0} = 0$ .

Next we focus our attention on the collection  $\mathcal{A}$  relative to  $V = RP(2n)$ . Consider the standard endomorphism  $\Gamma : \mathcal{N}_*^{Z_2} \rightarrow \mathcal{N}_*^{Z_2}$  of degree one (see, for example, [4]) and the augmentation  $\varepsilon : \mathcal{N}_*^{Z_2} \rightarrow \mathcal{N}_*$ ; consider yet the involution  $(RP(2n+1), \tau_0)$ , where  $\tau_0[x_0, x_1, \dots, x_{2n+1}] = [-x_0, x_1, \dots, x_{2n+1}]$ . The fixed data of this involution is  $(\lambda \rightarrow RP(2n)) \cup (R^{2n+1} \rightarrow p)$ , where  $\lambda$  is the canonical line bundle. Since  $RP(2n) \cup \{p\}$  does not bound, it follows from the strengthened Boardman  $\frac{5}{2}$ -theorem of [2] that there exists  $k_n \in Z^+$  such that  $\varepsilon \Gamma^{k_n}[RP(2n+1), \tau_0] \neq 0$  and  $\varepsilon \Gamma^i[RP(2n+1), \tau_0] = 0$  for all  $0 \leq i < k_n$ . Then for each  $0 \leq i \leq k_n$  the fixed data of  $\Gamma^i(RP(2n+1), \tau_0)$  can be considered, with no loss, as being  $(\lambda \oplus R^i \rightarrow RP(2n)) \cup (R^{2n+i+1} \rightarrow p)$ . In [3] Royster proved the following

**Fact 2.** *Let  $(W, T)$  be an involution fixing  $RP(2n) \cup \{p\}$ . Then*

$$[W, T] = \Gamma^i[RP(2n+1), \tau_0]$$

for some  $0 \leq i \leq k_n$ .

Suppose now that  $(M, \Phi)$  is a  $(Z_2)^2$ -action fixing  $RP(2n) \cup \{p\}$ , and let

$$\left( \bigoplus_{j=1}^3 \varepsilon_j \rightarrow RP(2n) \right) \cup \left( \bigoplus_{j=1}^3 \mu_j \rightarrow p \right)$$

denote the fixed data. Combining Facts 1 and 2 one has that one of the following situations occurs:

i) there is  $0 \leq i \leq k_n$  such that, up to permutation,  $\varepsilon_1$  is bordant to  $\lambda \oplus R^i$  and  $\varepsilon_2, \varepsilon_3$  are the 0-bundle;

ii) there are  $0 \leq i, j \leq k_n, i \leq j$ , such that, up to permutation,  $\varepsilon_1, \varepsilon_2$  and  $\varepsilon_3$  are bordant, respectively, to  $\lambda \oplus R^i, \lambda \oplus R^j$  and  $T(RP(2n))$ .

But a bundle  $\eta \rightarrow RP(2n)$  bordant to  $\lambda \oplus R^i$  has necessarily  $w_1(\eta) = \alpha$  and  $w_r(\eta) = 0$  for  $r > 1$ , where  $\alpha \in H^1(RP(2n), Z_2)$  is the generator; on the other hand, one knows from [11] that if  $\eta$  is bordant to  $T(RP(2n))$ , then  $W(\eta) = (1 + \alpha)^{2n+1}$ . These facts produce a simultaneous bordism; that is, they imply

that, up to permutation,  $(\bigoplus_{j=1}^3 \varepsilon_j) \cup (\bigoplus_{j=1}^3 \mu_j)$  is bordant to

$$((\lambda \oplus R^i) \oplus (\lambda \oplus R^j) \oplus T(RP(2n))) \cup (R^{2n+i+1} \oplus R^{2n+j+1} \oplus 0)$$

as elements of  $\bigoplus_p \mathcal{N}_p(BO(r) \times BO(s) \times BO(t))$ . In order to complete the classification our next task will consist then in exhibiting a list of  $(Z_2)^2$ -actions realizing all these possibilities. To do this, observe first that from a given involution  $(W, T)$  with fixed data  $\eta \rightarrow F$  one can construct the following special  $(Z_2)^2$ -actions:

i)  $(W \times W; D, S)$ , with  $D(x, y) = (T(x), T(y))$  and  $S(x, y) = (y, x)$ ; the fixed data is  $\eta \oplus \eta \oplus T(F) \rightarrow F$ ;

ii)  $(W; T, Id)$ , with fixed data  $\eta \oplus 0 \oplus 0 \rightarrow F$ .

We remark that the above actions are, respectively, the actions  $\Gamma_2^2(W, T)$  and  $\Gamma_1^2(W, T)$  mentioned in the introduction. Next observe that each automorphism  $\sigma : (Z_2)^2 \rightarrow (Z_2)^2$  gives rise, from a given  $(Z_2)^2$ -action  $(M, \Phi)$ ,  $\Phi = (T_1, T_2)$ , to a new action given by  $(M; \sigma(T_1), \sigma(T_2))$ ; we denote this action by  $\sigma(M, \Phi)$ . When  $\sigma$  varies in  $\text{Aut}((Z_2)^2)$ , the actions  $\sigma(M, \Phi)$  realize all possible permutations of the original fixed data.

Take now  $0 \leq i, j \leq k_n$  with  $i < j$ . By applying  $j - i$  times Theorem 2 to the fixed data of  $\Gamma_2^2 \Gamma^j(RP(2n+1), \tau_0)$  one obtains a  $(Z_2)^2$ -action  $(N_{i,j}, \varphi_{i,j})$  with fixed data

$$((\lambda \oplus R^i) \oplus (\lambda \oplus R^j) \oplus T(RP(2n))) \cup (R^{2n+i+1} \oplus R^{2n+j+1} \oplus 0)$$

(since a  $(Z_2)^2$ -action cannot fix precisely one point [4, 31.3],  $N_{i,j}$  is necessarily connected; such an action can be explicitly obtained by using the method of [5]).

The following  $(Z_2)^2$ -actions complete then our task:

- i)  $\sigma \Gamma_t^2 \Gamma^i(RP(2n+1), \tau_0)$ , with  $0 \leq i \leq k_n$ ,  $t = 1$  or  $2$  and  $\sigma \in \text{Aut}((Z_2)^2)$ ;
- ii)  $\sigma(N_{i,j}, \varphi_{i,j})$ , with  $0 \leq i, j \leq k_n$ ,  $i < j$  and  $\sigma \in \text{Aut}((Z_2)^2)$ .

Summarizing we obtain

**Theorem 3.** *A  $(Z_2)^2$ -action fixing  $RP(2n) \cup \{p\}$  is necessarily bordant to one of the above actions.*

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UNIVERSIDADE FEDERAL DE SÃO CARLOS, DEPARTAMENTO DE MATEMÁTICA, RODOVIA WASHINGTON LUIZ, KM. 235, 13.565-905, SÃO CARLOS, S.P., BRAZIL

*E-mail address:* pergher@power.ufscar.br