

CONJUGATE HARDY'S INEQUALITIES WITH DECREASING WEIGHTS

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ABSTRACT. We prove that for a decreasing weight ω on \mathbf{R}^+ , the conjugate Hardy transform is bounded on $L_p(\omega)$ ($1 \leq p < \infty$) if and only if it is bounded on the cone of all decreasing functions of $L_p(\omega)$. This property does not depend on p .

1. INTRODUCTION

All functions are assumed to be measurable on $\mathbf{R}^+ = (0, \infty)$, which is endowed with Lebesgue measure. A decreasing function will be a non-negative and non-increasing function, and ω will be a weight, i.e., a non-negative function.

If $1 \leq p < \infty$, we denote

$$L_p(\omega) = \left\{ f; \|f\|_{L_p(\omega)} = \left(\int_0^\infty |f(t)|^p \omega(t) dt \right)^{1/p} < \infty \right\}$$

and

$$L_p(\omega)^d = \{f \in L_p(\omega); f \text{ decreasing}\}.$$

Results by Muckenhoupt [2] state that the Hardy operator

$$Pf(x) = \frac{1}{x} \int_0^x f(t) dt$$

is bounded on $L_p(\omega)$ if and only if

$$(1) \quad \sup_{r>0} \left(\int_r^\infty \frac{\omega(s)}{s^p} ds \right)^{1/p} \left(\int_0^r \omega(s)^{-p'/p} ds \right)^{1/p'} < \infty$$

in the case $1 < p < \infty$, and, for $p = 1$, if and only if

$$(2) \quad \int_t^\infty \frac{\omega(s)}{s} ds \leq C\omega(t).$$

The corresponding condition for the conjugate Hardy operator

$$Qf(x) = \int_x^\infty f(t) \frac{dt}{t}$$

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is, if $1 < p < \infty$,

$$(3) \quad \sup_{r>0} \left(\int_0^r \omega(s) ds \right)^{1/p} \left(\int_r^\infty \frac{\omega(s)^{-p'/p}}{s^{p'}} ds \right)^{1/p'} < \infty$$

and, in the case $p = 1$,

$$(4) \quad \frac{1}{r} \int_0^r \omega(s) ds \leq C\omega(r).$$

It is a result by Ariño and Muckenhoupt [1] that the operator

$$P : L_p(\omega)^d \longrightarrow L_p(\omega)$$

is bounded on $L_p(\omega)^d$ if and only if ω satisfies the so-called B_p -condition:

$$(5) \quad t^p \int_t^\infty \frac{\omega(s)}{s^p} ds \leq C \int_0^t \omega(s) ds.$$

More recently, Neugebauer [3] has shown that Q is bounded on $L_p(\omega)^d$ if and only if

$$(6) \quad \int_0^t P\omega(s) ds \leq C \int_0^t \omega(s) ds.$$

Thus, the boundedness of Q on $L_p(\omega)^d$ does not depend on $p \geq 1$.

If ω is decreasing, we prove that conditions (3), (4) and (6) are equivalent. Hence, Q is bounded on $L_p(\omega)$ if and only if the decreasing weight ω satisfies (6). For decreasing ω , (3), (4) and (6) are essentially a rate of decrease condition; this is proved in Proposition 2.

By $h_1(t) \simeq h_2(t)$ we mean that $h_1(t) \leq C_1 h_2(t)$ and $h_2(t) \leq C_2 h_1(t)$.

2. THE BOUNDEDNESS THEOREM FOR THE CONJUGATE HARDY OPERATOR

Theorem 1. *Let ω be a decreasing weight. If Q is bounded on $L_p(\omega)^d$ ($1 \leq p < \infty$), it is also bounded on $L_p(\omega)$.*

Proof. We start by showing that (3) and (4) are equivalent.

Obviously, (3) implies (4), since ω is decreasing and then

$$\int_0^r \omega(s) ds \leq C_1 \omega(r) \left(\int_r^\infty \frac{ds}{s^{p'}} \right)^{-p'/p} = Cr\omega(r).$$

Assume now that ω has property (4). Since ω is decreasing,

$$\int_r^\infty \frac{\omega(s)^{-p'/p}}{s^{p'}} ds = \int_r^\infty \frac{\omega(s)}{(s\omega(s))^{p'}} ds \leq C \int_r^\infty \frac{\omega(s)}{\left(\int_0^\infty \omega(t) dt \right)^{p'}} ds,$$

and on performing the integration in the last expression we obtain

$$\int_r^\infty \frac{\omega(s)^{-p'/p}}{s^{p'}} ds \leq \frac{C}{p' - 1} \left(\int_0^r \omega(t) dt \right)^{-p'/p},$$

which is condition (3).

Obviously, (4) implies (6). To prove that (6) implies (4) we first observe that, for $a > 1$,

$$\frac{1}{r} \int_0^r \omega(s) ds = \frac{1}{r \log a} \int_r^{ar} \frac{\int_0^r \omega(s) ds}{t} dt \leq \frac{1}{r \log a} \int_r^{ar} P\omega(t) dt.$$

Now, (6) and the monotonicity of ω show that

$$\frac{1}{r} \int_0^r \omega(s) ds \leq \frac{C}{r \log a} \int_0^{ar} \omega(s) ds \leq \frac{C}{r \log a} \int_0^r \omega(s) ds + \frac{C(a-1)}{\log a} \omega(r),$$

whence

$$\frac{1}{r} \int_0^r \omega(s) ds \left(1 - \frac{C}{\log a}\right) \leq \frac{C(a-1)}{\log a} \omega(r),$$

and (4) follows taking $a = e^{2C}$. □

Property (4) is a decrease condition on ω :

Proposition 1. *For a decreasing weight w , the conditions*

$$(7) \quad \frac{1}{s} \int_0^s \omega(t) dt \simeq \omega(s) \quad (i.e., \frac{1}{s} \int_0^s \omega(t) dt \leq C\omega(s))$$

and

$$(8) \quad \inf_{x>0} \frac{\omega(rx)}{\omega(x)} > \frac{1}{r} \text{ for some constant } r > 1$$

are equivalent.

Proof. Assume that (8) holds. Then $\omega(rx) \geq a\omega(x)$ with $ar > 1$, and $\omega(r^{-n}x)a^n \leq \omega(x)$. Hence, from

$$\int_0^s \omega(t) dt = \sum_{n=0}^{\infty} \int_{s/r^{n+1}}^{s/r^n} \omega(t) dt \leq \sum_{n=0}^{\infty} \omega(sr^{-(n+1)})sr^{-n}$$

it follows that

$$\int_0^s \omega(t) dt \leq s\omega(s) \sum_{n=0}^{\infty} \frac{1}{r^n a^{n+1}}$$

with $C = (1/a) \sum_{n=0}^{\infty} (ra)^{-n} < \infty$.

Assume now that (7) holds and observe that, if $\tau \geq 1$,

$$\omega(\tau x) \geq \frac{1}{C\tau x} \int_0^{\tau x} \omega \geq \frac{1}{C\tau x} \int_0^x \omega \geq \frac{\omega(x)}{C\tau}.$$

Let us see that (7) holds if $r > \exp(C^2)$. From the above remark we have

$$\int_1^r \omega(\tau x) d\tau \geq \omega(x) \frac{\log r}{C}$$

and

$$\int_1^r \omega(\tau x) d\tau \leq \int_0^r \omega(\tau x) d\tau = r \left(\frac{1}{rx} \int_0^{rx} \omega(s) ds \right) \leq Cr\omega(rx).$$

Thus, putting together both estimates,

$$\frac{\omega(rx)}{\omega(x)} \geq \frac{1}{r} \frac{\log r}{C^2} > \frac{1}{r}. \quad \square$$

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