PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 126, Number 8, August 1998, Pages 2351–2353
S 0002-9939(98)04288-9

MEASURES WITH NATURAL SPECTRA ON LOCALLY COMPACT ABELIAN GROUPS

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(Communicated by J. Marshall Ash)

Abstract. Every bounded regular Borel measure on noncompact LCA groups is a sum of an absolutely continuous measure and a measure with natural spectrum. The set of bounded regular Borel measures with natural spectrum on a nondiscrete LCA group G whose Fourier-Stieltjes transforms vanish at infinity is closed under addition if and only if G is compact.

Let G be a locally compact abelian group and Γ its dual group. Let $M(G)$ be the measure algebra on G, and $M_0(G)$ the subalgebra of $M(G)$ which consists of measures whose Fourier-Stieltjes transforms vanish at infinity. For every $\mu \in M(G)$ we denote by $\check{\mu}$ the Gelfand transform of μ. X denotes the maximal ideal space of $M(G)$. We denote by $M_{00}(G)$ the algebra of all $\mu \in M(G)$ whose Gelfand transforms vanish off Γ.

Note that $L^1(G) \subset M_{00}(G)$. Note also that $\Gamma \subset X$ and $\hat{\mu} = \check{\mu}$ on Γ for every $\mu \in M(G)$, where $\hat{\mu}$ is the Fourier-Stieltjes transform of μ. Let $\mathrm{sp}(\mu)$ denote the spectrum of μ. Then $\mathrm{sp}(\mu) = \check{\mu}(X)$. We denote by $\mathrm{NS}(G)$ the set of all measures with natural spectra, that is, $\mathrm{NS}(G) = \{\mu \in M(G) : \mathrm{sp}(\mu) = \overline{\check{\mu}(\Gamma)}\}$. Williamson [6] proved that $\mathrm{NS}(G)$ is a proper subset of $M(G)$ for every non-discrete G. Put $\mathrm{NS}_0(G) = \mathrm{NS}(G) \cap M_0(G)$. Then $M_{00}(G) \subset \mathrm{NS}_0(G)$ holds for every G. Neumann [3, Theorem 9] proved, as a generalization of a theorem of Zafran [7, Theorem 3.2], that $\mathrm{NS}_0(G) = M_{00}(G) = \mathrm{Reg}\, M_0(G) = \mathrm{Dec}\, M_0(G)$ if G is compact, where $\mathrm{Reg}\, M_0(G)$ is the greatest regular closed subalgebra of $M_0(G)$ and $\mathrm{Dec}\, M_0(G)$ is the Apostol algebra of $M_0(G)$.

Rudin [4] for $G = \mathbb{R}$ and Varopoulos [5] for an arbitrary non-discrete G proved that $\mathrm{NS}_0(G)$ is a proper subset of $M_0(G)$. Eschmeier, Laursen and Neumann [1] gave examples of measures in $\mathrm{NS}_0(G) \setminus M_{00}(G)$ for certain non-compact G. In this note we show that for every non-compact G and for every $\mu \in M(G)$ (resp. $M_0(G)$) there exists an $f \in L^1(G)$ such that $\mu - f \in \mathrm{NS}(G)$ (resp. $\mathrm{NS}_0(G)$). It follows that $\mathrm{NS}_0(G) \setminus M_{00}(G) \neq \emptyset$ for every non-compact and non-discrete G, which is a solution to the question posed by Eschmeier, Laursen and Neumann [1, p.288].

Theorem 1. *Let G be a non-compact locally compact abelian group. Then we have*

$$\mathrm{NS}(G) + L^1(G) = M(G).$$

Received by the editors September 19, 1996 and, in revised form, January 20, 1997.
1991 *Mathematics Subject Classification.* Primary 43A10, 43A25.
Key words and phrases. Natural spectrum, locally compact abelian group.
The author was partially supported by the Grants-in Aid for Scientific Research, The Ministry of Education, Science and Culture, Japan.

Proof. Let Γ be the dual group of G and X the maximal ideal space of $M(G)$. By [2, Theorem 41.5, Theorem 41.13] there exists a Helson set $K \subset \Gamma$ which is homeomorphic with Cantor's ternary set H, since Γ is non-discrete. Let π be a homeomorphism from K onto H. Let c be the restriction to H of Cantor's function defined on the unit interval I. Then $c(H) = I$. Let p be a continuous function defined on I onto the square $\Delta = \{z \in C : |\operatorname{Re} z| \leq 1,\ |\operatorname{Im} z| \leq 1\}$ (Peano curve!). Then $p \circ c \circ \pi$ is continuous function defined on K onto Δ. Since K is a Helson set, there exits a function $f \in L^1(G)$ such that $\hat{f} = p \circ c \circ \pi$ on K. Let h be a function in $L^1(G)$ such that $\hat{h} = 1$ on K.

Let μ be a measure in $M(G)$. Put $\nu = \mu - \mu * h$. Let r be the spectral radius of ν. Then the measure $\nu_1 = \nu - rf$ satisfies $\mu = \nu_1 + rf + \mu * h$, where $rf + \mu * h \in L^1(G)$. We show that $\nu_1 \in \mathrm{NS}(G)$. Let $p \in X \setminus \Gamma$. Then $\check{\nu}(p) = \check{\nu}_1(p)$ since $r\hat{f}(p) = 0$; hence $|\check{\nu}_1(p)| \leq r$. Since $r\hat{f}(K) = r\Delta$, there is $x \in K$ such that $r\hat{f}(x) = -\check{\nu}_1(p)$. Since $\hat{\nu} = 0$ on K, we have $\hat{\nu}_1(x) = \check{\nu}_1(p)$. It follows that $\check{\nu}_1(X) = \hat{\nu}_1(\Gamma)$. Thus we have $\nu_1 \in \mathrm{NS}(G)$. □

Zafran [7, Example 3.4] pointed out that $\mathrm{NS}(G)$ is not closed under addition if G is an I-group. By Theorem 1 and a theorem of Williamson we see that $\mathrm{NS}(G)$ is not closed under addition if G is non-discrete and non-compact, since $L^1(G) \subset \mathrm{NS}(G)$.

Next we consider that case of $M_0(G)$.

Corollary 2. *Let G be a non-compact locally compct abelian group. Then we have*

$$\mathrm{NS}_0(G) + L^1(G) = M_0(G).$$

Proof. Let μ be a measure in $M_0(G)$. Then by Theorem 1 there are $\nu \in \mathrm{NS}(G)$ and $f \in L^1(G)$ such that $\mu = \nu + f$. Since the Fourier-Stieltjes transforms of μ and f vanish at infinity, we see that $\nu \in \mathrm{NS}_0(G)$. □

Eschmeier, Laursen and Neumann [1, Proposition 14] proved for a locally compact abelian group G that $\mathrm{NS}_0(G) = M_{00}(G)$ if and only if $\mathrm{NS}_0(G)$ is closed under addition. For a discrete G the situation is simple, since $M(G) = L^1(G)$. In particular, $\mathrm{NS}_0(G) = M_0(G)$ holds if G is discrete. For a non-discrete G we have the following.

Corollary 3. *Let G be a non-discrete locally compact abelian group. Then the following are equivalent.*

(1) $\mathrm{NS}_0(G)$ *is closed under addition.*

(2) $\mathrm{NS}_0(G) = M_{00}(G)$.

(3) G *is compact.*

Proof. The equivalence between (1) and (2) is proved by Eschmeier, Laursen and Neumann [1, Proposition 14]. Suppose that G is compact. Neumann [3, Theorem 9] proved that $\mathrm{NS}_0(G) = M_{00}(G)$.

We show that (3) follows from (1). Suppose that (1) holds. If G is not compact, then $\mathrm{NS}_0(G) + \mathrm{NS}_0(G) = M_0(G)$ by Corollary 2. Hence we have $\mathrm{NS}_0(G) = M_0(G)$, which is a contradiction (cf. [4], [5]). Thus we see that G is compact. □

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