

## ON THE MERGELYAN APPROXIMATION PROPERTY ON PSEUDOCONVEX DOMAINS IN $\mathbb{C}^n$

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ABSTRACT. Let  $\Omega$  be a smoothly bounded pseudoconvex domain of finite type in  $\mathbb{C}^n$ . We prove the Mergelyan approximation property in various topologies on  $\Omega$  when the estimates for  $\bar{\partial}$ -equation are known in the corresponding topologies.

### 1. INTRODUCTION

Let  $\Omega$  be a bounded domain in  $\mathbb{C}^n$  and let  $H(\Omega)$  denote the functions holomorphic on  $\Omega$ . We say  $\Omega$  has the Mergelyan property if every  $f \in H(\Omega) \cap C(\bar{\Omega})$  can be approximated uniformly on  $\bar{\Omega}$  by functions in  $H(\bar{\Omega})$ , where  $H(\bar{\Omega})$  denotes the functions holomorphic in a neighborhood of  $\bar{\Omega}$ .

For  $n > 1$ , Henkin [9], Kerzman [10] and Lieb [11] proved the Mergelyan property on strongly pseudoconvex domains in  $\mathbb{C}^n$ . The key technical tools needed to prove the Mergelyan property were:

- (1) existence of a Stein neighborhood basis for  $\bar{\Omega}$ , and
- (2) Lipschitz estimates and their stability for  $\bar{\partial}$  on this Stein neighborhood basis.

Notice that the author has constructed a Stein neighborhood basis of  $\bar{\Omega}$  when  $\Omega$  is a smoothly bounded pseudoconvex domain of finite type in  $\mathbb{C}^n$  [3].

Let  $\Lambda^\alpha(\Omega)$  denote the usual Lipschitz space of order  $\alpha \geq 0$  with norm  $\|\cdot\|_{\Lambda^\alpha(\Omega)}$  and let  $L_k^p(\Omega)$  denote the space of functions on  $\Omega$  that are in  $L^p(\Omega)$  along with all their derivatives up to order  $k$ , with norm denoted by  $\|\cdot\|_{L_k^p(\Omega)}$ . Lipschitz estimates for  $\bar{\partial}$  on pseudoconvex domains in  $\mathbb{C}^n$  are known for some special kinds of domains; that is, smoothly bounded pseudoconvex domains of finite type in  $\mathbb{C}^2$  [2], [7], and smoothly bounded pseudoconvex domain  $\Omega$  of finite type in  $\mathbb{C}^n$  such that the Levi-form of  $b\Omega$  is diagonalizable [8], etc. However, it is difficult and sometimes tedious to prove the stability of the estimates for  $\bar{\partial}$  even though the estimates are known [6], [12].

In this paper, we will present a new method to prove the Mergelyan property in various topologies. That is, if  $\bar{\Omega}$  has a Stein neighborhood basis with the estimates for  $\bar{\partial}$  in  $\Lambda^\alpha(\Omega)$ ,  $\alpha \geq 0$ , or in  $L_k^p(\Omega)$ ,  $1 < p < \infty$ ,  $k \geq 0$ , we will prove the Mergelyan

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property in the corresponding topologies. Here we are not assuming the stability of the estimates for  $\bar{\partial}$  in the corresponding spaces. Nevertheless, we will use the known result of the stability of estimates for  $\bar{\partial}$  in  $L_k^2(\Omega)$  spaces [4].

We will state and prove our results only on smoothly bounded pseudoconvex domains of finite type in  $\mathbb{C}^2$ . The same (or similar) results hold for the domains of finite type in  $\mathbb{C}^n$  with estimates for  $\bar{\partial}$  in the corresponding spaces.

**Theorem 1.** *Let  $\Omega$  be a smoothly bounded pseudoconvex domain of finite type in  $\mathbb{C}^2$ . Assume  $f \in H(\Omega) \cap L_k^p(\Omega)$ , where  $1 < p < \infty$  and  $k$  is a non-negative integer. Then there is a sequence  $\{g_n\} \subset H(\bar{\Omega})$  such that*

$$\lim_{n \rightarrow \infty} \|g_n - f\|_{L_k^p(\Omega)} = 0.$$

Let  $\mathbb{N}$  denote the set of natural numbers.

**Theorem 2.** *Let  $\Omega$  be as in Theorem 1 and assume  $f \in H(\Omega) \cap \Lambda^\alpha(\Omega)$ . Then for each  $\alpha' < \alpha$  ( $\alpha' = \alpha$  if  $\alpha \in \{0\} \cup \mathbb{N}$ ) arbitrarily given, there is a sequence  $\{g_n\} \subset H(\bar{\Omega})$  such that*

$$\lim_{n \rightarrow \infty} \|g_n - f\|_{\Lambda^{\alpha'}(\Omega)} = 0.$$

*Remark 3.* In [5], the author proved that every  $f \in H(\Omega) \cap L_k^2(\Omega)$  can be approximated by functions in  $H(\bar{\Omega})$  in the  $L_k^2(\Omega)$  topologies,  $k \geq 0$ , when  $\Omega$  is a smoothly bounded pseudoconvex domain of finite type in  $\mathbb{C}^n$ . When  $n = 2$ , Cho and others [6] also proved the Mergelyan property in Lipschitz spaces  $\Lambda^\alpha(\Omega)$ ,  $0 \leq \alpha < 1/m$ , for  $f \in H(\Omega) \cap \Lambda^\alpha(\Omega)$ . Here  $m$  is the type of  $b\Omega$ . Both of these results depend on the solvability and stability of the estimates for  $\bar{\partial}$  in the corresponding topologies [4], [6].

The key ingredients needed to prove Theorem 1 and Theorem 2 are the  $L_k^p(\Omega)$  and Lipschitz estimates for  $\bar{\partial}$  on  $\Omega \Subset \mathbb{C}^2$  [2], [7], [8], and the smooth bumping theorem for pseudoconvex domains of finite type in  $\mathbb{C}^n$  [2], [3], [4].

## 2. APPROXIMATION BY SMOOTH FUNCTIONS

In this section, we prove that any holomorphic function in  $L_k^p(\Omega)$  or  $\Lambda^\alpha(\Omega)$  can be approximated by smooth functions on  $\bar{\Omega}$  in appropriate topologies.

Let  $U_j$ ,  $j = 0, 1, \dots, N$ , be a finite collection of open sets with the following properties:

- (a)  $\bar{\Omega} \subset \bigcup_{j=0}^N U_j$ .
- (b)  $U_0 \Subset \Omega$ .
- (c) On each  $U_j$ ,  $j = 1, 2, \dots, N$ , there are holomorphic coordinates  $z_1^j, z_2^j$  with the property that  $\partial r / \partial x_2^j > 0$ , where  $z_2^j = x_2^j + iy_2^j$ .

Let  $\zeta_j$ ,  $j = 0, 1, \dots, N$ , be a partition of unity subordinate to the covering  $\{U_j\}$ . For all sufficiently small  $\delta > 0$ , and for a given function  $f \in H(\Omega)$ , let  $f_\delta$  be given by

$$(1) \quad f_\delta(z) = \zeta_0(z)f(z) + \sum_{j=1}^N \zeta_j(z)f(z_1^j, z_2^j - \delta).$$

Observe that  $f_\delta \in C^\infty(\bar{\Omega})$ .

**Proposition 4.** *Suppose that  $f \in H(\Omega) \cap L^p_k(\Omega)$ ,  $1 < p < \infty$ ,  $k \geq 0$ . Then  $\|f_\delta - f\|_{L^p_k(\Omega)} \rightarrow 0$  and  $\|\bar{\partial}f_\delta\|_{L^p_k(\Omega)} \rightarrow 0$  as  $\delta \rightarrow 0$ .*

*Proof.* Note that  $D^\alpha(f - f_\delta) \in L^p(\Omega)$ ,  $|\alpha| \leq k$ . So  $\|f_\delta - f\|_{L^p_k(\Omega)}$  converges to zero as  $\delta$  does, by the Lebesgue dominated convergence theorem. Also note that

$$D^\alpha(\bar{\partial}f_\delta) = D^\alpha(\bar{\partial}(f_\delta - f)) = D^\alpha\left(\sum_{j=1}^N (\bar{\partial}\zeta_j)(f(z_1^j, z_2^j - \delta) - f(z_1^j, z_2^j))\right).$$

So  $\|\bar{\partial}f_\delta\|_{L^p_k(\Omega)} \rightarrow 0$  as  $\delta \rightarrow 0$  by the same reasoning. □

The functions  $f_\delta \in C^\infty(\bar{\Omega})$  defined in (1) also approximate  $f$  in the  $\Lambda^\alpha(\Omega)$ -topology:

**Proposition 5.** *Suppose that  $f \in H(\Omega) \cap \Lambda^\alpha(\Omega)$ . Then for each  $\alpha' < \alpha$  ( $\alpha' = \alpha$  if  $\alpha \in \{0\} \cup \mathbb{N}$ ),  $\|f_\delta - f\|_{\Lambda^{\alpha'}(\Omega)}$  and  $\|\bar{\partial}f_\delta\|_{\Lambda^{\alpha'}(\Omega)}$  converge to zero as  $\delta$  does.*

*Proof.* It is clear that the conclusion holds if  $\alpha \in \{0\} \cup \mathbb{N}$ . Without loss of generality, we may assume that  $0 < \alpha' < \alpha < 1$ . If  $x, y$  satisfy  $\delta \leq |x - y|$ , we use the Lipschitz continuity condition on  $(f_\delta - f)(x)$  and  $(f_\delta - f)(y)$ , and if  $\delta > |x - y|$ , we use the same condition on  $f(x) - f(y)$  and  $f_\delta(x) - f_\delta(y)$ . In both cases, we will get  $\|f_\delta - f\|_{\Lambda^\alpha(\Omega)} \lesssim \delta^{\alpha - \alpha'}$ . Since  $\alpha' < \alpha$ , this proves that  $\|f_\delta - f\|_{\Lambda^\alpha(\Omega)}$  converges to zero as  $\delta$  does. Similarly, we can prove that  $\|\bar{\partial}f_\delta\|_{\Lambda^\alpha(\Omega)}$  converges to zero as  $\delta$  does. □

*Remark 6.* If  $\alpha \notin \mathbb{N} \cup \{0\}$ , the functions  $f_\delta$  defined in (1) does not converge to  $f$  in the  $\Lambda^\alpha(\Omega)$  norm in general.

**Definition 7.** Let  $\Omega \subset \mathbb{C}^n$  be a smoothly bounded pseudoconvex domain with  $C^\infty$  defining function  $r$ . By a smooth bumping family of  $\Omega$  we mean a family of smoothly bounded pseudoconvex domains  $\{\Omega_\tau\}_{0 \leq \tau \leq 1}$  satisfying the following properties:

- (1)  $\Omega_0 = \Omega$ ,
- (2)  $\Omega_{\tau_1} \Subset \Omega_{\tau_2}$  if  $\tau_1 < \tau_2$ ,
- (3)  $\{b\Omega_\tau\}_{0 \leq \tau \leq 1}$  is a  $C^\infty$  family of real hypersurfaces in  $\mathbb{C}^n$ ,
- (4) the boundary defining functions  $r_\tau$  of  $\Omega_\tau$  vary smoothly with respect to  $\tau$ , and  $r_\tau \rightarrow r$  as  $t \rightarrow 0$  in the  $C^\infty$  topology.

In [3], the author constructed a smooth bumping family of  $\Omega$  if  $b\Omega$  is of finite type.

For the final remark of this section, we state the following theorem which gives the stability of  $L^2$ -estimates for  $\bar{\partial}$  on  $\Omega$  [4].

**Theorem 8.** *Let  $\{\Omega_\tau\}_{0 \leq \tau \leq 1}$  be a smooth bumping family of pseudoconvex domains in  $\mathbb{C}^n$ . Then for each  $m$  there exists a constant  $C_m$ , independent of  $\tau$ , such that*

$$\|f^\tau\|_m \leq C_m \|\square_\tau f^\tau\|_m,$$

for all  $f^\tau \in \text{Dom}(\square_\tau) \cap C^\infty(\bar{\Omega}_\tau)$  with  $f^\tau \perp H^{0,1}(\bar{\Omega}_\tau)$ . Here  $\square_\tau$  denotes the complex Laplacian on  $\Omega_\tau$ .

3. APPROXIMATION BY HOLOMORPHIC FUNCTIONS

Let  $\Omega$  be a smoothly bounded pseudoconvex domain of finite type in  $\mathbb{C}^2$ , and let  $N$  be the Neumann operator associated with the  $\bar{\partial}$ -Neumann problem. Then the main results in [2], [7] show that

$$(2) \quad \bar{\partial}^* N : L_k^p(\Omega) \longrightarrow L_k^p(\Omega), \quad 1 < p < \infty, \quad k \geq 0,$$

and

$$(3) \quad \bar{\partial}^* N : \Lambda^\alpha(\Omega) \longrightarrow \Lambda^\alpha(\Omega), \quad \alpha \geq 0,$$

are bounded operators on the corresponding spaces.

Now let us prove Theorem 1 and Theorem 2. Here we will only present a proof of Theorem 2. The proof of Theorem 1 follows the same (or similar) lines. Let us state Theorem 2 again:

**Theorem 2.** *Let  $\Omega$  be a smoothly bounded pseudoconvex domain of finite type in  $\mathbb{C}^2$ . Assume  $f \in H(\Omega) \cap \Lambda^\alpha(\Omega)$ . Then for each  $\alpha' < \alpha$  ( $\alpha' = \alpha$  if  $\alpha \in \{0\} \cup \mathbb{N}$ ) arbitrarily given, there is a sequence  $\{g_n\} \subset H(\bar{\Omega})$  such that*

$$\lim_{n \rightarrow \infty} \|g_n - f\|_{\Lambda^{\alpha'}(\Omega)} = 0.$$

*Proof.* Let  $f_\delta$  be defined by (1). By Proposition 5,  $f_\delta \in C^\infty(\bar{\Omega})$  converge to  $f$  in the  $\Lambda^{\alpha'}(\Omega)$  topology as  $\delta$  goes to zero. For each fixed  $\delta > 0$ , let us solve  $\bar{\partial}u_\delta = \bar{\partial}f_\delta$  on  $\Omega$ . Then by Catlin’s global regularity theorem for the  $\bar{\partial}$ -equation on pseudoconvex domains of finite type in  $\mathbb{C}^n$  [1], it follows that  $u_\delta \in C^\infty(\bar{\Omega})$ . Also, by (3),  $u_\delta$  satisfies

$$\|u_\delta\|_{\Lambda^{\alpha'}(\Omega)} \leq C_\alpha \|\bar{\partial}f_\delta\|_{\Lambda^{\alpha'}(\Omega)},$$

where  $C_\alpha$  does not depend on  $\delta$ . Since  $\|\bar{\partial}f_\delta\|_{\Lambda^{\alpha'}(\Omega)}$  converges to zero as  $\delta$  does, it follows that  $\|u_\delta\|_{\Lambda^{\alpha'}(\Omega)} \rightarrow 0$  as  $\delta \rightarrow 0$ . Set

$$h_\delta = f_\delta - u_\delta.$$

Then  $h_\delta \in H(\Omega) \cap C^\infty(\bar{\Omega})$  and

$$(4) \quad \lim_{\delta \rightarrow 0} \|h_\delta - f\|_{\Lambda^{\alpha'}(\Omega)} = 0.$$

Let  $\{\Omega_\tau\}$  be a smooth pseudoconvex bumping family of  $\Omega$ . We extend  $h_\delta$  to  $\Omega_\tau$  and set it equal to  $h_\delta^\tau$  on  $\Omega_\tau$ . Notice that  $\bar{\partial}h_\delta^\tau \equiv 0$  on  $\Omega$ , and  $\bar{\partial}h_\delta^\tau \in C^\infty(\bar{\Omega})$ . Hence  $\bar{\partial}h_\delta^\tau$  vanishes to order infinity on  $b\Omega$  as  $\tau \rightarrow 0$ . Set  $k = [\alpha] + 4$ , where  $[\alpha]$  denotes the greatest integer less than or equal to  $\alpha$ . Let us solve the following  $\bar{\partial}$ -equation in  $L_k^2(\Omega)$  spaces (with weighted estimates of  $\bar{\partial}$ ):

$$\bar{\partial}p_\delta^\tau = \bar{\partial}h_\delta^\tau \quad \text{on } \Omega_\tau.$$

From Theorem 8 (stability of  $L_k^2(\Omega_\tau)$ -estimates of  $\bar{\partial}$ -equation), it follows that

$$(5) \quad \|p_\delta^\tau\|_{L_k^2(\Omega_\tau)} \leq C_\alpha \|\bar{\partial}h_\delta^\tau\|_{L_k^2(\Omega_\tau)},$$

where  $C_\alpha$  does not depend on  $\tau$ . Set

$$g_\delta^\tau = p_\delta^\tau - h_\delta^\tau \in H(\Omega_\tau).$$

Then (4), (5) and the Sobolev embedding theorem imply that

$$\|g_\delta^\tau - f\|_{\Lambda^{\alpha'}(\Omega)} \longrightarrow 0 \quad \text{as } \delta, \tau \rightarrow 0.$$

This proves Theorem 2. □

## REFERENCES

1. Catlin, D.W., *Global regularity of the  $\bar{\partial}$ -Neumann problem*, Proc. Sympos. Pure Math. **41** (1984), 39–49. MR **85j**:32033
2. Chang, D.C., Nagel, A. and Stein, E., *Estimates for the  $\bar{\partial}$ -Neumann problem in pseudoconvex domains of finite type in  $\mathbb{C}^2$* , Acta Math. **169** (1992), 153–228. MR **93k**:32025
3. Cho, S., *Extension of complex structures on weakly pseudoconvex compact complex manifolds with boundary*, Math. Z. **211** (1992), 105–119. MR **94h**:32030
4. H. R. Cho, S. Cho, and K. H. Shon, *Stability of the estimates for  $\bar{\partial}$ -equation on compact pseudoconvex complex manifolds*, Kyushu J. Math. **48-1** (1994), 19–34. MR **95k**:32017
5. Cho, H.R. and Cho, S., *Holomorphic approximation on compact pseudoconvex complex manifolds*, J. of Geom. Anal. (to appear).
6. Cho, S., Ahn, H. and Kim, S., *Stability of Hölder estimates for  $\bar{\partial}$  on pseudoconvex domains of finite type in  $\mathbb{C}^2$* , Nagoya Math. J. (to appear).
7. Fefferman, C. and Kohn, J.J., *Hölder estimates on domains of complex dimension two and on three dimensional CR manifolds*, Adv. Math. **69** (1988), 233–303. MR **89g**:32027
8. Fefferman, C., Kohn, J.J. and Machedon, M., *Hölder estimates on CR manifolds with diagonalizable Levi form*, Adv. Math. **84** (1990), 1–90. MR **92a**:32019
9. G. M. Henkin, *Integral representations of functions which are holomorphic in strictly pseudoconvex regions, and some applications*, Math. Sb. **78** (1969), 611–632; , Engl. Transl.: Math. USSR. Sb. **7** (1969), 597–616. MR **40**:2902
10. N. Kerzman, *Hölder and  $L^p$  estimates for solutions of  $\bar{\partial}u = f$  in strongly pseudoconvex domains*, Comm. Pure Appl. Math. **24** (1971), 301–379. MR **43**:7658
11. I. Lieb, *Ein Approximationssatz auf streng pseudokonvexen Gebieten*, Math. Ann. **184** (1969), 56–60. MR **41**:7146
12. Range, R.M., *Integral kernels and Hölder estimates for  $\bar{\partial}$  on pseudoconvex domains of finite type in  $\mathbb{C}^2$* , Math. Ann. **288** (1990), 63–74. MR **91j**:32018

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