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AUTOMORPHIC-DIFFERENTIAL IDENTITIES AND ACTIONS OF POINTED COALGEBRAS ON RINGS

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ABSTRACT. In this paper, we prove the following two results which generalize the theorem concerning automorphic-differential endomorphisms asserted by J. Bergen. Let R be a ring, $R_{\mathcal{F}}$ its left Martindale quotient ring and $\mathfrak{A}$ a right ideal of R having no nonzero left annihilator. (1) Let C be a pointed coalgebra which measures R such that the group-like elements of C act as automorphisms of R. If R is prime and $\xi \cdot \mathfrak{A} = 0$ for $\xi \in R\#C$, then $\xi \cdot R = 0$. Furthermore, if the action of C extends to $R_{\mathcal{F}}$ and if $\xi \in R_{\mathcal{F}}\#C$ such that $\xi \cdot \mathfrak{A} = 0$, then $\xi \cdot R_{\mathcal{F}} = 0$. (2) Let f be an endomorphism of $R_{\mathcal{F}}$ given as a sum of composition maps of left multiplications, right multiplications, automorphisms and skew-derivations. If R is semiprime and $f(\mathfrak{A}) = 0$, then $f(R) = 0$.

1. INTRODUCTION

The motivation of this note is the following problem in [B, Question 4], posed by J. Bergen.

Question. *Suppose R is a ring, H a Hopf algebra acting on R, and let $f \in R\#H$. If R is special, such as being prime or a domain, and if $\lambda \neq 0$ is a special subset of R, such as being an ideal or a one-sided ideal, must $f \cdot R = 0$ whenever $f \cdot \lambda = 0$?*

This question is mentioned as a consequence of the theorem concerning a special mapping, called an automorphic-differential endomorphism, in prime rings. Let $\mathrm{End}(R, +)$ be the ring consisting of all additive endomorphisms of a ring R. For $a \in R$, T_a and L_a represent the right and left multiplication maps of R respectively. Let $\mathcal{A}$ be the subring of $\mathrm{End}(R, +)$ generated by $\{T_a, L_a | a \in R\}$, all automorphisms and derivations of R. An element of $\mathcal{A}$ is called an *automorphic-differential endomorphism* of R. For a subset $A \subseteq R$, we set $\ell(A) = \{a \in R | aA = 0\}$ and $r(A) = \{a \in R | Aa = 0\}$. Bergen proved the following theorem.

Theorem A ([B, Theorem 1]). *Let $\mathfrak{A}$ be a right ideal of a prime ring R with $\ell(\mathfrak{A}) = 0$ and f an automorphic-differential endomorphism of R. If $f(\mathfrak{A}) = 0$, then we have $f(R) = 0$.*

Our first aim is to extend this fact to the action of a pointed coalgebra on prime rings. For this purpose, we use the Taft-Wilson theorem, which determines the structure of the coradical filtration of a pointed coalgebra.

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Theorem 2. *Let R be a prime algebra and C a pointed coalgebra which measures R such that the group-like elements of C act as automorphisms of R. If $\xi \cdot \mathfrak{A} = 0$ for $\xi \in R\#C$ and a right ideal $\mathfrak{A}$ of R with $\ell(\mathfrak{A}) = 0$, then $\xi \cdot R = 0$. Furthermore, if the action of C extends to $R_{\mathcal{F}}$ and if $\xi \in R_{\mathcal{F}}\#C$ such that $\xi \cdot \mathfrak{A} = 0$, then $\xi \cdot R_{\mathcal{F}} = 0$.*

In [B, Question 1], Bergen also asked whether Theorem A holds even if R is semiprime, and in [O], A. Ouarit gave an affirmative answer to this problem. We extend this result to an endomorphism with skew-derivations, using a different proof from [O], in Section 4.

Theorem 6. *Let R be a semiprime ring and f an automorphic-skew-differential endomorphism. If $f(\mathfrak{A}) = 0$ for some right ideal $\mathfrak{A}$ with $\ell(\mathfrak{A}) = 0$, then $f(R) = 0$.*

2. Martindale quotient rings and actions of coalgebras

We recall some notions which we need in this paper. Let R be an arbitrary ring and $\mathcal{F}$ the set of all ideals I of R with $r(I) = \ell(I) = 0$. If R is a prime ring, $\mathcal{F}$ is the set of all nonzero ideals; and if R is semiprime, all essential ideals. We denote the *left Martindale quotient ring* of R by $R_{\mathcal{F}}$, the *symmetric quotient ring* of R by Q and the *extended centroid* by K. (See [K2, §1.4],[M2, p.97] for detailed definitions.) If R is semiprime, it is known that $R_{\mathcal{F}}$ and Q are semiprime rings and K is a von Neumann regular ring; and if R is prime, $R_{\mathcal{F}}$ and Q are prime rings and K is a field.

We have the following facts on $R_{\mathcal{F}}$ and Q.

1. For any $q \in R_{\mathcal{F}}$, there exists $I \in \mathcal{F}$ with $Iq \subseteq R$.
2. For any $q \in Q$, there exists $I \in \mathcal{F}$ with $qI \subseteq R$.

For nonempty subsets $A, B \subseteq R_{\mathcal{F}}$, we define $\ell_A(B) = \{a \in A | aB = 0\}$. Throughout, C represents a coalgebra over a field $\mathbf{k}$ with comultiplication Δ and counit ε. We use the sigma notation as follows : $\Delta(c) = \sum c_1 \otimes c_2$ for $c \in C$. Let R be a $\mathbf{k}$-algebra. We say that C measures R if an action $C \otimes R \ni c \otimes r \mapsto c \cdot r \in R$ satisfies the following conditions:

(1) for any $a, b \in R$ and $c \in C$, $c \cdot (ab) = \sum (c_1 \cdot a)(c_2 \cdot b)$, and
(2) $c \cdot 1 = \varepsilon(c)1$ for any $c \in C$.

We can define an (R, R)-bimodule $R\#C$ as follows:

(1) $R\#C = R \otimes C$ as a $\mathbf{k}$-space,
(2) the element $a \otimes c$ is denoted by $a\#c$, and
(3) the left (right) action of R is given by

$$a'(a\#c) = a'a\#c \qquad ((a\#c)a' = \sum a(c_1 \cdot a')\#c_2)$$

for $a, a' \in R$ and $c \in C$.

For $\xi = \sum a_i\#c_i \in R\#C$ and $r \in R$, we define $\xi \cdot r = \sum a_i(c_i \cdot r)$. It is easy to show that for $\xi \in R\#C$ and $a, r \in R$, we have $(\xi a) \cdot r = \xi \cdot (ar)$ and $(a\xi) \cdot r = a(\xi \cdot r)$.

3. On prime rings with actions of pointed coalgebras

We recall the coradical filtration of a coalgebra. For a coalgebra C, the coradical C_0 is the sum of all simple subcoalgebras of C. For $n \geq 1$,

$$C_n = \Delta^{-1}(C_0 \otimes C + C_{n-1} \otimes C).$$

C_n is a subcoalgebra of C for all n, and $C = \bigcup_{n=0}^{\infty} C_n$. If C is pointed, then $C_0 = \mathbf{k}G$, where G is the set of all group-like elements of C, i.e. $G = \{0 \neq \sigma \in C | \Delta(\sigma) = \sigma \otimes \sigma\}$. For $\sigma, \tau \in G$, we set

(1) $X_{\sigma,\tau,n} = \{x \in C_n | \Delta(x) \in \sigma \otimes x + x \otimes \tau + C_{n-1} \otimes C_{n-1}\}$ $(n \geq 1)$,
(2) if $\sigma = \tau$, then $X_{\sigma,\tau,0} = \mathbf{k}\sigma$; otherwise $X_{\sigma,\tau,0} = 0$,
(3) $X_{\sigma,\tau} = \sum_{n \geq 0} X_{\sigma,\tau,n}$.

We state the Taft-Wilson theorem.

Proposition 1 ([TW], [Ma]). *Let C be a pointed coalgebra and $\{C_n\}_{n=0}^{\infty}$ its coradical filtration. Set $C_{-1} = 0$. For each non-negative integer n, there exists a subspace X_n of C_n with $C_n = X_n \oplus C_{n-1}$ as a $\mathbf{k}$-space. Moreover, for the above $X_{\sigma,\tau,n}$*

(1) *$X_n = \sum_{\sigma,\tau} X_{\sigma,\tau,n}$, and*
(2) *$\Delta(X_{\sigma,\tau}) \subseteq \sigma \otimes X_{\sigma,\tau} + X_{\sigma,\tau} \otimes \tau + C_{n-1} \otimes C_{n-1}$.*

We are ready to prove our first main theorem.

Theorem 2. *Let R be a prime algebra and C a pointed coalgebra which measures R such that the group-like elements of C act as automorphisms of R. If $\xi \cdot \mathfrak{A} = 0$ for $\xi \in R\#C$ and a right ideal $\mathfrak{A}$ of R with $\ell(\mathfrak{A}) = 0$, then $\xi \cdot R = 0$. Furthermore, if the action of C extends to $R_{\mathcal{F}}$ and if $\xi \in R_{\mathcal{F}}\#C$ such that $\xi \cdot \mathfrak{A} = 0$, then $\xi \cdot R_{\mathcal{F}} = 0$.*

Proof. Let B denote either R or $R_{\mathcal{F}}$, depending upon whether we are proving the first or second part of the theorem. In any case, we can assume that C acts on B, $\xi \cdot \mathfrak{A} = 0$, and we need to show that $\xi \cdot B = 0$.

Suppose the contrary. This means that the set $D = \{\xi \in B\#C | \xi \cdot \mathfrak{A} = 0$ and $\xi \cdot B \neq 0\}$ is not empty. Let n be the smallest integer satisfying $D \cap (B\#C_n) \neq \emptyset$. From the Taft-Wilson theorem, we have a basis $\{x_i\}$ for X_n satisfying $\Delta(x_i) \in \sigma_i \otimes x_i + C \otimes C_{n-1}$ for $\sigma_i \in G$. An element of $D \cap (B\#C_n)$ can be written in the form $\sum a_i \# x_i + y$ for $a_i \in B$ and $y \in B\#C_{n-1}$. We take an element ξ for which the number of terms $a_i\#x_i$ $(a_i \neq 0)$ appearing in ξ is least among all elements of $D \cap (B\#C_n)$. Now, set $\xi = \sum_{i=1}^{k} a_i\#x_i + y$ $(0 \neq a_i \in B,\ y \in B\#C_{n-1})$.

If $a_1 \notin R$, multiplying by a suitable element of $I_1 \in \mathcal{F}$ with $I_1 a_1 \subseteq R$, we may assume that a_1 is contained in R.

By assumption, the mapping $g_1 : B \to B$ given by $q \mapsto \sigma_1 \cdot q$ is an automorphism of B with $g_1^{-1}(R) \subseteq R$.

There exists some $b \in B$ such that $\xi \cdot b \neq 0$. Let $I \neq 0$ be an ideal of R such that $I a_1 g_1(b) \subseteq R$. If we let $r \in g_1(\mathfrak{A})I$, then it is clear that $r a_1 g_1(b) \in g_1(\mathfrak{A})$, and so $g_1^{-1}(r a_1) b \in \mathfrak{A}$.

On the other hand, consider $\omega = a_1 r \xi - \xi(g_1^{-1}(r a_1))$. Since $g_1^{-1}(r) \in \mathfrak{A}$ and $a_1 \in R$, it is clear that $(\xi(g_1^{-1}(r a_1))) \cdot \mathfrak{A} = 0$ and $\omega \cdot \mathfrak{A} = 0$. It is easy to see that $\omega = \sum_{i=2}^{k} (a_1 r a_i - a_i(\sigma_i \cdot (g_1^{-1}(r a_1))))\#x_i + \widetilde{y}$ for some $\widetilde{y} \in B\#C_{n-1}$. By the assumptions on $\mathbf{k}$ and n, ω cannot be contained in D, so $\omega \cdot B = 0$. However, $(\xi(g_1^{-1}(r a_1))) \cdot b = \xi \cdot (g_1^{-1}(r a_1) b) \in \xi \cdot \mathfrak{A} = 0$; hence $a_1 r \xi \cdot b = 0$ for all $r \in g_1(\mathfrak{A})I$. It follows that $a_1 g_1(\mathfrak{A}) I(\xi \cdot b) = 0$ and $g_1^{-1}(a_1)\mathfrak{A} g_1^{-1}(I(\xi \cdot b)) = 0$. We note that $g_1^{-1}(a_1)\mathfrak{A} \neq 0$, as $\ell(\mathfrak{A}) = 0$. Since $g_1^{-1}(a_1)\mathfrak{A}$ is a nonzero right ideal of a prime ring, we have $g_1^{-1}(I(\xi \cdot b)) = 0$, which implies the contradiction $\xi \cdot b = 0$. Thus, we have the conclusion of the theorem. □

For a Hopf algebra H, we say that R is a left H-module algebra if (1) R is a left H-module and (2) H measures R. In this case, the group-like elements of H act as automorphisms of R. If H is pointed, the action can be extended uniquely to the action on $R_{\mathcal{F}}$ ([M1, Corollary 3.5(2)]) and we can define a *smash product algebra* $R_{\mathcal{F}}\#H$ ([M2, 4.1.3]). As a corollary of Theorem 2, we have the following.

Corollary 3. *Let R be a prime algebra and H a pointed Hopf algebra acting on R. If $\xi \cdot \mathfrak{A} = 0$ for $\xi \in R_{\mathcal{F}}\#H$ and a right ideal $\mathfrak{A}$ of R with $\ell(\mathfrak{A}) = 0$, then $\xi \cdot R_{\mathcal{F}} = 0$.*

4. Automorphic-skew-differential endomorphisms on semiprime rings

For any ring A, the set of all automorphisms of A is denoted by $\mathrm{Aut}(A)$. For $\sigma \in \mathrm{Aut}(A)$, an additive map $d : A \to A$ is called a *σ-derivation* if $d(ab) = d(a)b + \sigma(a)d(b)$ for all $a, b \in A$. The set of all σ-derivations is denoted by $\mathrm{Der}_\sigma(A)$. An element of $\bigcup_{\sigma \in \mathrm{Aut}(A)} \mathrm{Der}_\sigma(A)$ is called a *skew-derivation*.

Proposition 4. (1) *Any automorphism $\sigma : R \to R$ can be extended to an automorphism of $R_{\mathcal{F}}$, and in this case, $\sigma(Q) = Q$.*
(2) *Any σ-derivation $d : R \to R$ can be extended to a σ-derivation of $R_{\mathcal{F}}$, and in this case, $d(Q) \subseteq Q$.*

Proof. This is similar to [KP, Lemma 1]. □

According to Proposition 4, we can consider $\mathrm{Aut}(R) \subseteq \mathrm{Aut}(R_{\mathcal{F}})$ and $\mathrm{Der}_\sigma(R) \subseteq \mathrm{Der}_\sigma(R_{\mathcal{F}})$ for $\sigma \in \mathrm{Aut}(R)$.

We regard $\mathrm{End}(R_{\mathcal{F}}, +)$ as a K-module via $(cf)(x) = c(f(x))$ for $f \in \mathrm{End}(R_{\mathcal{F}}, +)$, $c \in K$ and $x \in R_{\mathcal{F}}$.

Let $\mathcal{B}$ be a subring of $\mathrm{End}(R_{\mathcal{F}})$ generated by $\{T_a, L_a | a \in R_{\mathcal{F}}\}$, $\mathrm{Aut}(R)$ and $\bigcup_{\sigma \in \mathrm{Aut}(R)} \mathrm{Der}_\sigma(R)$. We call an element of $\mathcal{B}$ an *automorphic-skew-differential endomorphism.*

Before proving the semiprime case, we apply Theorem 2 to the automorphic-skew-differential endomorphisms of prime rings.

For $f \in \mathcal{B}$, let $\{d_{ij} \in \mathrm{Der}_{\sigma_i}(R) | 1 \leq i \leq m, 1 \leq j \leq n_i\}$ be the set of all skew derivations and $\{g_k\}$ that of all automorphisms appearing in f. Removing the overlapping elements, we assume that $\{\sigma_i, g_k | 1 \leq i \leq m, 1 \leq k \leq t\}$ is the set of distinct automorphisms. Let B be the algebra with generators $\{T_{ij}, X_i, Y_k | 1 \leq i \leq m, 1 \leq j \leq n_i, 1 \leq k \leq t\}$. Then B will be a pointed bialgebra under the following operations:

$$\Delta(T_{ij}) = T_{ij} \otimes 1 + X_i \otimes T_{ij}, \Delta(X_i) = X_i \otimes X_i, \Delta(Y_k) = Y_k \otimes Y_k,$$
$$\varepsilon(T_{ij}) = 0, \varepsilon(X_i) = \varepsilon(Y_k) = 1.$$

We define the action of B on $R_{\mathcal{F}}$ by $T_{ij} \cdot x = d_{ij}(x)$, $X_i \cdot x = \sigma_i(x)$ and $Y_k \cdot x = g_k(x)$ for $x \in R_{\mathcal{F}}$. We note that R is B-stable.

Now, we have an element $\xi \in R_{\mathcal{F}}\#B$ corresponding to f. As $\xi \cdot r = f(r)$ for any $r \in R$ and B is pointed, we have the following result as an easy consequence of Theorem 2.

Theorem 5. *Let R be a prime ring and f an automorphic-skew-differential endomorphism. If $f(\mathfrak{A}) = 0$ for some right ideal $\mathfrak{A}$ with $\ell(\mathfrak{A}) = 0$, then $f(R) = 0$.*

The next theorem shows that Theorem 5 holds even if R is a semiprime ring. It is the main result of this section.

Theorem 6. *Let R be a semiprime ring and f an automorphic-skew-differential endomorphism. If $f(\mathfrak{A}) = 0$ for some right ideal $\mathfrak{A}$ with $\ell(\mathfrak{A}) = 0$, then $f(R) = 0$.*

We need several lemmas to prove this theorem. For the remainder of this note, we assume that R is a semiprime ring. The set of all idempotents of K is represented by E. Let e_1, e_2 be elements in E. The operation $\oplus$ defined by $e_1 \oplus e_2 = e_1 + e_2 - 2e_1e_2$ and the ordinary multiplication make E into a Boolean ring. Besides, we define the partial order $\leq$ of E by $e_1 \leq e_2 \Leftrightarrow e_1e_2 = e_1$. When E_0 is a nonempty subset of E, a nonzero element $e \in E_0$ is called minimal in E_0 if $e' \leq e$ for $0 \neq e' \in E_0$ implies $e' = e$.

A finite subset of E generates a finite Boolean subring.

For every nonempty subset S of $R_{\mathcal{F}}$, we can uniquely determine an idempotent $e = e(S) \in E$ satisfying (i) $es = s$ for all $s \in S$ and (ii) $ex = 0$ for any $x \in R_{\mathcal{F}}$ such that $SRx = 0$ ([K2, p.26]).

For $\sigma \in \mathrm{Aut}(R)$ and $q \in Q$, set $(ad_\sigma q)(x) = qx - \sigma(x)q$. Then $ad_\sigma q \in \mathrm{Der}_\sigma(R_{\mathcal{F}})$.

Proposition 7 ([K1, Lemma 10])**.** *Any $d \in \mathrm{Der}_\sigma(R)$ can be uniquely decomposed into the sum $ad_\sigma q + \mu_\sigma$ for some $q \in Q$ and some left and right E-linear σ-derivation μ_σ.*

For any $x \in R$, $a \in R_{\mathcal{F}}$, $g, \sigma \in \mathrm{Aut}(R)$ and σ-derivation d, we have the following relations.

(1) $dL_a(x) = d(ax) = d(a)x + \sigma(a)d(x) = (L_{d(a)} + L_{\sigma(a)}d)(x)$.
(2) $dT_a(x) = d(xa) = d(x)a + \sigma(x)d(a) = (T_a d + T_{d(a)}\sigma)(x)$.
(3) $gL_a(x) = g(ax) = g(a)g(x) = L_{g(a)}g(x)$.
(4) $gT_a(x) = g(xa) = g(x)g(a) = T_{g(a)}g(x)$.
(5) $ad_\sigma a(x) = ax - \sigma(x)a = (L_a - T_a\sigma)(x)$.
(6) $dg(x) = gg^{-1}dg(x)$.

It is easy to see that $g^{-1}dg$ is a skew-derivation in $\mathrm{Der}_{g^{-1}\sigma g}(R)$. Using the decomposition in Proposition 7, we may assume that an automorphic-skew-differential endomorphism is the sum of monomials of the form $T_a L_b g \Delta$ for $a, b \in R_{\mathcal{F}}$, $g \in \mathrm{Aut}(R)$ and a composition map of E-linear skew-derivations Δ.

Proposition 8 ([K2, p. 110])**.** *Let f be an automorphic-skew-differential endomorphism of R such that $f(\mathfrak{A}) = 0$ for some right ideal of R with $\ell(\mathfrak{A}) = 0$. For any $e \in E$, we have an idempotent $0 \neq e' \leq e$ such that $e'f$ can be decomposed into the sum $\sum_\lambda f_\lambda$ satisfying (1) $f_\lambda(\mathfrak{A}_\lambda) = 0$ for some $\mathfrak{A}_\lambda \lhd_r R$ with $\ell(\mathfrak{A}_\lambda) = 0$ and (2) for any idempotent $e'' \leq e'$ and automorphism g appearing in f_λ, $g(e'') = g_\lambda(e'')$ for some $g_\lambda \in \mathrm{Aut}(R)$.*

Proof. Let $\{g_\mu | \mu \in M\}$ be the set of all distinct automorphisms appearing in f.

If $g_\mu(e_1) = g_\nu(e_1)$ for any $\mu, \nu \in M$ and idempotent $e_1 \leq e$, we have the conclusion on setting $e' = e$. We assume that for some $e_1 \leq e$ and $\mu, \nu \in M$ we have $g_\mu(e_1) \neq g_\nu(e_1)$. Let M' be a subset of M so that $\{g_\mu(e_1) | \mu \in M'\}$ is the set of all distinct idempotents in $\{g_\mu(e_1) | \mu \in M\}$. Then $f(xe_1) = \sum_{\mu \in M'} f_\mu(x) g_\mu(e_1)$ for some $f_\mu \in \mathcal{B}$ with $f = \sum_\mu f_\mu$. Let e_2 be a minimal idempotent in the Boolean ring generated by $\{g_\mu(e_1)\}$ satisfying $\{\mu \in M' | e_2 g_\mu(e_1) \neq 0\} \neq \emptyset$ and $\{\mu \in M' | e_2 g_\mu(e_1) = 0\} \neq \emptyset$. Set $f_1(x) = f(x)e_2 - f(xe_1)e_2$ and $f_2(x) = f(xe_1)e_2$. Then $e_2 f = f_1 + f_2$ and $f_2 \neq 0$. Moreover, for $I \in \mathcal{F}$ with $Ie_1 \subseteq R$, we have $f_1(\mathfrak{A}I) = f_2(\mathfrak{A}I) = 0$, as $\mathfrak{A}Ie_1 \subseteq \mathfrak{A}$. It is easy to see that $\ell(\mathfrak{A}I) = 0$. As each number of the distinct automorphisms appearing in f_1 and f_2 is less than that of f, continuing this process, we get the conclusion of the proposition. □

Proposition 9. *Let f be an automorphic-skew-differential endomorphism of R. Assume that for any $e \in E$, there exists $e' \in E$ satisfying $0 \neq e' \leq e$ and $f(R)e' = 0$. Then we have $f(R) = 0$.*

Proof. See the proof of [K2, Theorem 2.3.6 (p.109)]. □

Here, we write $e'f = \sum f_\lambda$ as shown in Proposition 8. If we can show that $f_\lambda(R) = 0$ for each f_λ, it follows that $e'f(R) = 0$, and by Proposition 9 we obtain the conclusion of Theorem 6. Hence, we may assume that $g_\lambda(e') = e'$, as it is sufficient to show that $g_\lambda^{-1}(f_\lambda(R)) = 0$ for each λ. So, we assume that $f_\lambda(e''x) = e'' f_\lambda(x)$ for any idempotent $e'' \leq e'$. In this case, $f_\lambda(x) = e' f_\lambda(x) = f_\lambda(e'x)$, and thus our aim is to show that $f_\lambda(e'R) = 0$ for a semiprime ring $e'R$. From the above argument, setting $e' = 1$ and $f = f_\lambda$, we may assume every automorphism appearing in f is E-linear.

According to [K2, p.59], we induce a certain *"topology"* on the K-module M. Let Γ be a directed set. We call $m \in M$ a *limit* of a family $\{m_\alpha \in M | \alpha \in \Gamma\}$ if there exists a family $\{e_\alpha \in E | \alpha \in \Gamma\}$ satisfying

(i) $e_\alpha \leq e_\beta$ if $\alpha \leq \beta$,

(ii) $\sup\{e_\alpha | \alpha \in \Gamma\} = 1$, and

(iii) $me_\alpha = m_\alpha e_\alpha$ for all $\alpha \in \Gamma$.

A subset $S \subseteq M$ is said to be *closed* if S contains any limit of the family in S. The least closed set containing $T \subseteq M$ is called the *closure* of T and denoted by $\widehat{T}$. For K-modules M_1, M_2, a mapping $\varphi : M_1 \to M_2$ is called *completely continuous* if $\varphi(m) = \lim_{\alpha\in\Gamma} \varphi(m_\alpha)$ for any $m, m_\alpha \in M_1$ $(\alpha \in \Gamma)$ with $m = \lim_{\alpha\in\Gamma} m_\alpha$. All automorphisms and E-linear skew-derivations are completely continuous.

Lemma 10 ([K2, Lemma 1.6.6, 1.6.10, 1.6.11, 1.6.26])**.** (1) *If $f(T) = 0$ for an automorphic-skew-differential endomorphism f and $T \subseteq R_{\mathcal{F}}$, then $f(\widehat{T}) = 0$.*

(2) *If T is a subring of $R_{\mathcal{F}}$, so is $\widehat{T}$.*

(3) *If I is a right ideal of a subring T, $\widehat{I}$ is a right ideal of $\widehat{T}$.*

(4) *Let T be a closed additive subset of $R_{\mathcal{F}}$ with $TE \subseteq T$ and $f : T \to R_{\mathcal{F}}$ an E-linear mapping. There is an element $t \in T$ satisfying $e(f(t)) = e(f(T))$.*

By Lemma 10(2), $\widehat{RE}$ is a subring of Q, where RE is the set of all linear combinations $\sum a_i e_i$, $a_i \in R, e_i \in E$. Let $\wp$ be a maximal ideal of K. The ideal of $R_{\mathcal{F}}$ generated by $\wp$ is denoted by $\wp R_{\mathcal{F}}$. Let ϕ be a canonical homomorphism $R_{\mathcal{F}} \to R_{\mathcal{F}}/\wp R_{\mathcal{F}}$. Then $\phi(\widehat{RE})$ is a prime ring ([K2, Lemma 1.12.1]). Moreover, $\phi(R_{\mathcal{F}})$ is contained in the left Martindale quotient ring of $\phi(\widehat{RE})$ ([K2, Proposition 1.12.3]). We denote the left Martindale quotient ring by $\phi(\widehat{RE})_{\mathcal{F}}$.

Lemma 11. *For $a \in R_{\mathcal{F}}$, $a \in \ker\phi \Leftrightarrow e(a) \in \wp$.*

Proof. See the proof of [K2, Lemma 1.9.18]. □

Lemma 12. *Let $\mathfrak{A}$ be a right ideal of R with $\ell(\mathfrak{A}) = 0$, and a an element of $R_{\mathcal{F}}$. Then $a\widehat{\mathfrak{A}E} \subseteq \ker\phi \Leftrightarrow a \in \ker\phi$.*

Proof. Assume that $a\widehat{\mathfrak{A}E} \subseteq \ker\phi$. By Lemma 10(4), there exists $t \in \widehat{\mathfrak{A}E}$ with $e(at) = e(a\widehat{\mathfrak{A}E})$. As $(1 - e(at))a\widehat{\mathfrak{A}E} = 0$ and $\ell_{\widehat{RE}}(\widehat{\mathfrak{A}E}) = 0$ for $\ell(\mathfrak{A}) = 0$, we have $(1-e(at))a = 0$, and so $e(a) = e(a)e(at)$. Since $at \in \ker\phi$, by Lemma 11, $e(at) \in \wp$. Thus, $e(a) \in \wp$ and $a \in \ker\phi$. The other direction is easy. □

From Lemma 11, it is clear that $\ker\phi \cap Q = E(\wp)Q$, where $E(\wp) = \wp \cap E$. So this ideal is invariant under any E-linear mapping $\beta : Q \to Q$. Thus, we can define a mapping $\widehat{\beta}$ of $\phi(Q)$ by $\widehat{\beta}(\phi(q)) = \phi(\beta(q))$.

Moreover, if β is an automorphism or skew-derivation, by restricting $\widehat{\beta}$ to $\phi(\widehat{RE})$, we have an automorphism or skew-derivation of $\phi(\widehat{RE})_{\mathcal{F}}$ from Proposition 4. We denote it by the same $\widehat{\beta}$.

Now, as all automorphisms and skew-derivations appearing in f are E-linear, we can define $\widehat{f}$, which is an automorphic-skew-differential endomorphism of a prime ring $\phi(\widehat{RE})$, by changing each automorphism g to $\widehat{g}$, each skew-derivation d to $\widehat{d}$ and coefficient a to $\phi(a) \in \phi(\widehat{RE})_{\mathcal{F}}$.

Proof of Theorem 6. Let $\wp$ and ϕ be as above. For $\mathfrak{A} \lhd_r R$ with $f(\mathfrak{A}) = 0$ and $\ell(\mathfrak{A}) = 0$, $\phi(\widehat{\mathfrak{A}E})$ is a right ideal of $\phi(\widehat{RE})$ and $\widehat{f}(\phi(\widehat{\mathfrak{A}E})) = \phi(f(\widehat{\mathfrak{A}E})) = 0$ by the fact that f is E-linear and Lemma 10(1),(3). By Lemma 12, $\ell_{\phi(\widehat{RE})}(\phi(\widehat{\mathfrak{A}E})) = 0$, so we can apply Theorem 5 to a prime ring $\phi(\widehat{RE})$ and $\widehat{f}$; we have $\widehat{f}(\phi(\widehat{RE})) = 0$. Thus, $f(R) \subseteq f(\widehat{RE}) \subseteq \wp R_{\mathcal{F}}$ for each maximal ideal $\wp$ of K.

Assume that a is contained in $\wp R_{\mathcal{F}}$ for any maximal ideal $\wp \lhd K$. By Lemma 11, $e(a)$ is contained in $\bigcap_{\wp} E(\wp) \subset \bigcap_{\wp} \wp$, where $\wp$ runs over all maximal ideals of K. Since K is a regular ring, the intersection of all maximal ideals is zero. Hence $a = e(a)a = 0$.

Thus, we have $f(R) = 0$, and the proof is completed. □

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References

[B] J. Bergen, *Automorphic-differential identities in rings*, Proc. Amer. Math. Soc. **106** (1989), 297–305. MR **89k:**16064

[K1] V. K. Kharchenko, *Skew derivations of semiprime rings*, Siberian Math. J. **32** (1991), 1045–1051. MR **93b:**16032

[K2] V. K. Kharchenko, *Automorphisms and derivations of associative rings*, Kluwer, Dordrecht, 1991. MR **93i:**16048

[KP] V. K. Kharchenko and A. Z. Popov, *Skew derivations of prime rings*, Comm. Algebra **20** (1992), 3321–3345. MR **93k:**16071

[Ma] T. Marlowe, *The diagonal of a pointed coalgebra and incidence-like structure*, J. Pure and Appl. Algebra **35** (1985), 157–169. MR **86j:**16002

[M1] S. Montgomery, *Bi-invertible actions of Hopf algebras*, Isr. J. Math. **83** (1993), 45–71. MR **94g:**16047

[M2] S. Montgomery, *Hopf algebras and their actions on rings*, CBMS Regional Conference Series in Mathematics 82, AMS, Providence, R.I., 1993. MR **94i:**16019

[O] A. Ouarit, *Identités auto-différentielles d'anneaux semi-primiers*, C. R. Acad. Sci. Paris **314** (1992), 173–176. MR **93c:**16026

[TW] E. Taft and R. Wilson, *On antipodes in pointed Hopf algebras*, J. Algebra **29** (1974), 27–32. MR **49:**2820

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