

## VECTOR BUNDLES ON LOG TERMINAL VARIETIES

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ABSTRACT. Let  $X$  be an  $n$ -dimensional variety and  $E$  an ample vector bundle on  $X$  of rank  $e$ . We give a complete classification of pairs  $(X, E)$ , with  $X$  log terminal and  $e \geq n$  such that  $K_X + \det E$  is not ample. The results we obtain were conjectured by Fujita, and recently by Zhang.

### INTRODUCTION

Let  $X$  be a log terminal variety and  $E$  an ample vector bundle of rank  $e$ . We are interested in studying those pairs  $(X, E)$  for which the adjoint  $\mathbf{Q}$ -Cartier divisor  $K_X + \det E$  is not ample—this is what is usually called generalized adjunction.

In the attempt to prove the Mukai conjectures, [Mu], such pairs with  $X$  smooth have been studied, mainly with the help of Mori theory, in a series of papers [YZ], [Fu3], [Pe], and then further investigations took place in [ABW], [PSW], [Zh1], [Ma], [AM]. On the contrary, only recently the full power of minimal model theory was used to extend this study to the case of mildly singular varieties [Zh2], [Zh3], [AM]. In this context pairs  $(X, E)$  with  $e \geq n + 1$  and  $X$  log terminal were classified independently in [Zh2] and [AM].

The following theorem characterizes the pairs  $(X, E)$ , with  $e \geq n$ ; this result was conjectured first by Fujita [Fu3] and then by Zhang [Zh3].

**Theorem 1.** *Let  $X$  be a log terminal variety of dimension  $n$  and  $E$  an ample vector bundle of rank  $e \geq \dim X$ . Assume that  $K_X + \det E$  is nef but not ample.*

*If  $e \geq n + 1$  then  $(X, E) \simeq (\mathbf{P}^n, \mathcal{O}(1)^{\oplus n+1})$ , [Zh1], [AM].*

*If  $e = n$  then  $(X, E)$  is one of the following:*

- $(\mathbf{P}^n, \mathcal{O}(1)^{\oplus n-1} \oplus \mathcal{O}(2))$ ,
- $(\mathbf{Q}^n, \mathcal{O}(1)^{\oplus n})$ , where  $\mathbf{Q}^n$  is a, maybe singular, hyperquadric,
- $(\mathbf{P}^n, T_{\mathbf{P}^n})$ ,
- $X$  is a smooth scroll over a smooth curve and  $E|_F \simeq \mathcal{O}(1)^{\oplus n}$ , where  $F \simeq \mathbf{P}^{n-1}$ .

The idea of the proof is to use the variety  $Y = \mathbf{P}(E)$  and the extremal contraction naturally associated to it, to study the possible singularities of  $X$ . To do this we mainly use the slicing technique of [AW] and some byproducts of it [An]. We complete the study of the positivity of  $K_X + \det E$  for a pair  $(X, E)$  with  $rk E \geq$

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$\dim X$ , giving a different proof of the main theorem in [Zh3], and start the case  $\operatorname{rk} E = \dim X - 1$ , with the following.

**Theorem 2.** *Let  $X$  be a log terminal variety of dimension  $n$  and  $E$  an ample vector bundle of rank  $e \geq \dim X - 1$ . Then  $K_X + \det E$  is nef unless*

- $e \geq n$  and  $(X, E) \simeq (\mathbf{P}^n, \mathcal{O}(1)^{\oplus n})$ , [Zh3], or*
- $e = n - 1$  and  $(X, E)$  is one of the following:*
  - *$(\mathbf{P}^n, \mathcal{O}(1)^{\oplus n-1})$ ,*
  - *$(\mathbf{P}^n, \mathcal{O}(1)^{\oplus n-2} \oplus \mathcal{O}(2))$ ,*
  - *$(\mathbf{Q}^n, \mathcal{O}(1)^{\oplus n-1})$ , where  $\mathbf{Q}^n$  is a (possibly singular) hyperquadric,*
  - *$X$  is a smooth scroll over a smooth curve and  $E|_F \simeq \mathcal{O}(1)^{\oplus n-1}$ , where  $F \simeq \mathbf{P}^{n-1}$ .*

In this case the main part is to prove that  $X$  is Gorenstein, so as to be able to apply a trick, borrowed from [AM], that reduces the pair  $(X, E)$  to a pair  $(X, E_1)$  in the list of Theorem 1. We believe that, with similar arguments, it is possible to describe those pairs  $(X, E)$  with nef but not ample adjoint  $\mathbf{Q}$ -divisor and  $\operatorname{rk} E = n - 1$ ; this should be the borderline of our techniques.

## 1. PRELIMINARIES

We use the standard notation from algebraic geometry. In particular it is compatible with that of [KMM], to which we refer constantly. Everything is defined over  $\mathbf{C}$ . A  $\mathbf{Q}$ -Cartier divisor is an element in  $\operatorname{Div}(X)_{\mathbf{Q}} := \operatorname{Div}(X) \otimes \mathbf{Q}$ , where  $\operatorname{Div}(X)$  is the group of Cartier divisors. A contraction  $f : Y \rightarrow X$  is a birational morphism of normal varieties with connected fibers. Given a contraction  $f : X \rightarrow Y$  and  $A, B \in \operatorname{Div}(X)_{\mathbf{Q}}$ , we say that  $A$  is  $f$ -numerically equivalent to  $B$  ( $A \equiv_f B$ ) if  $A \cdot C = B \cdot C$  for any curve contracted by  $f$ ; and  $A$  is  $f$ -linearly equivalent to  $B$  ( $A \sim_f B$ ) if  $A - B \sim f^*(M)$ , for some Cartier divisor  $M \in \operatorname{Div}(Y)$ .

A contraction  $f : X \rightarrow Y$  is extremal if  $-K_X$  is  $f$ -ample; if  $\dim X > \dim Y$  then  $f$  is said to be of fiber type, otherwise it is birational. We will say that  $K_X + rL$  supports the contraction  $f$ , for some  $L \in \operatorname{Div}(X)$ , if  $K_X + rL \equiv_f \mathcal{O}_X$ . In many occasions we are interested only in a neighborhood of a special fiber; to study this situation we will use a local set-up as described in [AW]; that is, we will shrink  $Y$  to an affine scheme in such a way that any line bundle on  $Y$  is a unit.

1.1. Let us recall various properties of extremal contractions. Let  $\varphi : X \rightarrow Y$  be an extremal contraction supported by  $K_X + rL$  and  $F$  an irreducible component of a positive dimensional fiber of  $\varphi$ ; then

- i)  $\dim F \geq r - 1$ , and if  $\varphi$  is birational then  $\dim F \geq r$ , [Fu1],[AW];
- ii) if  $\dim F < r + 1$  (if  $\varphi$  is birational  $\dim F \leq r + 1$ ), then  $L$  is  $\varphi$ -spanned, [AW];
- iii) if  $\varphi$  is equidimensional of dimension  $r - 1$  and there are no fibers contained in  $\operatorname{Sing}(X)$ , then  $(X, L) \simeq (\mathbf{P}(\mathcal{F}), \mathcal{O}(1))$ , for some ample vector bundle  $\mathcal{F}$  on a smooth variety, [BS], [An];
- iv) if  $\varphi$  is birational and  $\dim F = r$ , then  $(F, L) \simeq (\mathbf{P}^r, \mathcal{O}(1))$ , [An].

The following lemma is just a restatement of [Me, Lemma 1.9].

**Lemma 1.2.** *Let  $\varphi : X \rightarrow Y$  be a local extremal contraction and  $T$  an integral Weil divisor with  $T \equiv_{\varphi} \mathcal{O}_X$ . Let  $H \in |\varphi^* M|$ , for some spanned line bundle  $M$  on  $Y$ , a generic section. Assume that  $T|_H \sim_{\varphi} \mathcal{O}_H$ . Then  $T \sim_{\varphi} \mathcal{O}_X$ .*

*Proof.* Since  $T$  is a Weil divisor and  $f^*M$  is Cartier, the following sequence is exact:

$$0 \rightarrow \mathcal{O}(T - H) \rightarrow \mathcal{O}_X(T) \rightarrow \mathcal{O}_H(T) \rightarrow 0.$$

By hypothesis  $H^0(H, T)$  has a nonvanishing section  $\sigma$ ; therefore, using the above sequences and the Kawamata–Viehweg vanishing theorem, we can lift  $\sigma$  to a nonvanishing section of  $H^0(X, T)$ .  $\square$

2. PROOF OF THE THEOREMS

*Proof of Theorem 1.* If  $X$  is smooth then the theorem follows by [Fu3], [Pe]. By hypothesis there exists an extremal contraction  $\pi : X \rightarrow W$  supported by  $K_X + \det E$ . Let  $Y = \mathbf{P}(E)$ ; then  $\dim Y = n + e - 1$ ,  $Y$  is log terminal, in particular CM, and  $\text{Sing}(Y) = p^{-1}\text{Sing}(X)$ ; furthermore  $K_Y + e\xi_E \sim p^*(K_X + \det E)$ , where  $\xi_E$  is the tautological line bundle on  $Y$ . Therefore  $-K_Y$  is  $(\pi \circ p)$ -ample and we have the following commutative diagram:

$$\begin{array}{ccc} \mathbf{P}(E) = Y & \xrightarrow{\varphi} & V \\ \downarrow p & & \downarrow \psi \\ X & \xrightarrow{\pi} & W, \end{array}$$

where  $\varphi$  is an extremal contraction supported by  $K_Y + e\xi_E$ . Since  $p$  is an elementary contraction then  $\dim F_\varphi \leq \dim F_\pi$ , where  $F_f$  is an irreducible component of a fiber of the contraction  $f$ . If  $e \geq n + 1$  then, by (1.1 i)),  $e - 1 \leq \dim F_\varphi \leq n$ ; therefore  $e = n + 1$  and  $\varphi$  is equidimensional of dimension  $e - 1$ . Since  $\varphi|_{F_\varphi} : F_\varphi \rightarrow X$  is surjective,  $F_\varphi$  is not contained in  $\text{Sing}(Y)$ . Therefore by (1.1 iii))  $Y$  is smooth and consequently  $X$  is smooth.

Assume now that  $e = n$ . Then  $\varphi$  is supported by  $K_Y + nH$ ; hence  $\dim F_\varphi \geq n - 1$ .

*Case 1.* All fibers of  $\varphi$  have dimension  $n - 1$ . Thus  $\varphi$  is of fiber type and  $Y$  and  $X$  are smooth as above; see also [AM, Theorem 4.2].

*Case 2.* The generic fiber has dimension  $n - 1$  but there are special fibers of dimension  $n$ . Let  $S$  the locus of fibers of dimension  $n$  and  $G = \varphi(S)$ ; then  $\dim S < \dim Y = 2n - 1$ , that is,  $\text{cod} G \geq 2$ . Slice  $V$  with  $(n - 1)$  generic hyperplane sections  $A_i$ , in such a way that  $C = \bigcap A_i$  is outside of  $G$  and  $C \not\subset \varphi(F_p)$ , for any fiber  $F_p$ . Let  $T := \varphi^*(C)$  and  $f_T := \varphi|_T$ ; then  $T$  is lt, and  $f_T$  is an equidimensional extremal contraction, supported by  $K_T + n\xi_{E|T}$ . Since  $p|_T : T \rightarrow X$  is surjective and  $X$  is normal, there are no fibers of  $f_T$  contained in  $\text{Sing}(T)$  and  $T$  is smooth by (1.1 iii)). Since  $T$  is obtained by  $Y$  slicing with Cartier divisors, then  $Y$  is smooth along  $T$ . Furthermore  $p|_T$  is surjective and  $\text{Sing}(Y) = p^{-1}\text{Sing}(X)$ ; thus  $X$  is smooth as well.

*Case 3.* The generic fiber  $F$  has dimension  $n$  and  $\varphi$  is of fiber type. By the Kobayashi–Ochiai criterion the generic fiber  $F \simeq \mathbf{Q}^n$ , maybe singular, and  $p|_F : F \rightarrow X$  is finite and surjective. Let  $C \subset F$  be a rational curve such that  $p|_C : C \rightarrow B \subset X$  is birational onto the image. Such a curve always exists, for instance  $H_1 \cap \dots \cap H_{n-1}$ , where  $H_i \in |\mathcal{O}_{\mathbf{Q}^n}(1)|$  are generic elements. Let  $\nu : \mathbf{P}^1 \rightarrow B$  be the normalization and  $Y_C = \mathbf{P}(\nu^*\mathcal{E})$ ; then we have a morphism  $h : Y_C \rightarrow V$ , induced by the normalization and  $\varphi$ . Let

$$Y_C \xrightarrow{g} U \xrightarrow{i} V$$

be the Stein factorization of  $h$ . Since  $n = \dim Y_C = \dim V + 1$ , then  $g$  is not birational, that is,  $\nu^* \mathcal{E} \simeq \mathcal{O}(a)^{\oplus n}$  and  $U \simeq \mathbf{P}^{n-1}$ . On the other hand,  $i$  is birational and  $V$  is normal; therefore  $V \simeq \mathbf{P}^{n-1}$ . By the universal property of a product and our construction we have a morphism  $\alpha$

$$Y \xrightarrow{\alpha} X \times \mathbf{P}^{n-1} \begin{array}{l} \xrightarrow{p_1} X \\ \xrightarrow{p_2} \mathbf{P}^{n-1} \end{array},$$

compatible with the projections, that is,  $p_1 \circ \alpha = p$  and  $p_2 \circ \alpha = \varphi$ . In other words,  $Y \simeq \mathbf{Q}^n \times \mathbf{P}^{n-1}$  and  $E \simeq \mathcal{O}(1)^{\oplus n}$ .

*Case 4.*  $\varphi$  is birational. In this case all fibers have dimension  $n$ ,  $(F, \xi_{E|F}) \simeq (\mathbf{P}^n, \mathcal{O}(1))$  and  $\xi_E$  is  $\varphi$ -spanned (1.1 iv), ii); let us consider a local set-up and slice with  $(n-1)$  sections  $A_i \in |\xi_E|$ , to get a log terminal  $n$ -fold  $T$  with a birational contraction  $f := \varphi|_T : T \rightarrow Z$ , supported by  $K_X + L$ , where  $L = \xi_{E|T}$ , with one-dimensional fibers.  $X$  is normal; therefore the generic fiber  $F \simeq \mathbf{P}^1$  is out of  $\text{Sing}(T)$ , and so, by the Ionescu–Wisniewski inequality, [Wi], the exceptional locus of  $f$ , hence of  $\varphi$ , is a divisor. In particular we can slice vertically on  $Y$  with  $(n-2)$  generic hyperplane sections  $\varphi^* H_i$ , to get  $S = Y \cap (\bigcap_i \varphi^* H_i)$  and a birational extremal contraction,  $f_S : S \rightarrow U$  supported by  $K_Y + n\xi_{E|S}$ , whose general fiber  $F$  has dimension  $n$ . By [An] we know that under this hypothesis either  $\dim(F \cap \text{Sing}(S)) = n-1$  or  $F$  is on the smooth locus. The former is impossible since  $X$  is normal; therefore the latter is the case and  $X$  is smooth.  $\square$

**Lemma 2.1.** *Let  $X$  be a log terminal variety of dimension  $n$  and  $E$  an ample vector bundle on  $X$  of rank  $e \geq n-1$ . Assume that  $K_X + \det E$  is not nef. Then  $X$  is Gorenstein.*

*Proof.* Let  $Y = \mathbf{P}(E)$ , and let  $\varphi : Y \rightarrow V$ , as in the above commutative diagram, be the extremal contraction supported by  $K_Y + r\xi_E$  for some  $r > e$ . Let  $F$  be a generic fiber of  $\varphi$ . Since  $\text{Sing}(Y) = \text{Sing}(X) \times \mathbf{P}^{e-1}$ , it is enough to prove that for the generic point  $y \in Y$  of any fiber  $F_p$ ,  $K_Y$  is Cartier at  $y$ .

*Case 1.*  $\dim F = n$  and  $\varphi$  of fiber type. Consider a local set-up and slice on  $V$  with  $(e-1)$  general hyperplane sections  $A_i$ . Let  $F = \varphi^*(\bigcap A_i)$ , with  $(K_X + r\xi_E)|_F \equiv \mathcal{O}_F$ , where  $r > e \geq n-1$ . Since  $p|_F : F \rightarrow X$  is surjective,  $X$  is Gorenstein if  $Y$  is Gorenstein along  $F$ . On the other hand  $F$  is obtained from  $Y$  by vertical slicing; therefore by Lemma 1.2,  $Y$  is Gorenstein along  $F$  if  $F$  is Gorenstein and  $r$  is an integer.

*Claim.*  $F$  is either a hyperquadric or the projective space, and  $r$  is an integer.

*Proof of the claim.* This is a standard application of the Kawamata–Viehweg vanishing theorem.  $F$  is log terminal. Let  $p(t) = \chi(F, t\xi_{E|F}) = \sum a_i t^i$ . Since  $K_F + r\xi_{E|F} \sim \mathcal{O}_F$  with  $r > n-1$ , then  $p(t) = 0$  for  $0 > t \geq n-1$  and  $p(0) = 1$ . Furthermore, by the Riemann–Roch formula we know that  $-K_F \cdot \xi_{E|F} = 2(n-1)!a_{n-1}$ , and we can now immediately write down the possible polynomials of degree  $n$  satisfying those prescriptions and check that they are the Hilbert polynomials of projective spaces and hyperquadrics.  $\square$

*Case 2.*  $\dim F \leq n-1$ . Since  $r > e \geq n-1$ , then  $\dim F = n-1$  and  $\varphi$  is of fiber type. Slice on  $V$  with  $(e-1)$  general hyperplane sections  $A_i$ . Let  $T = \varphi^*(\bigcap A_i)$ , with  $\dim T = n$  and  $\varphi_T = \varphi|_T$  an extremal contraction supported by  $K_T + r\xi_{E|T}$ ,

with  $r > n - 1$ ; by an easy dimension count, as in the proof of Theorem 1,  $\varphi_T$  is equidimensional of dimension  $n - 1$ . As in the above claim the general fiber of  $f_T$  is  $\mathbf{P}^{n-1}$  and  $r = n$ ; therefore by (1.1 iii))  $T$  is smooth. Since  $p|_T : T \rightarrow X$  is surjective and  $T$  is obtained from  $Y$  by vertical slicing, then  $X$  is smooth.

*Case 3.*  $\dim F = n$  and  $\varphi$  is birational. With the same argument as in Case 4 of Theorem 1, we can prove that the exceptional locus of  $\varphi$  is a divisor. Slice vertically on  $Y$  with  $(e - 2)$  generic hyperplane sections, to get a birational extremal contraction  $f_S : S \rightarrow U$ , from an  $n + 1$  log terminal variety  $S$  with fibers  $F$  of dimension  $n$ . Again we conclude that  $F \simeq \mathbf{P}^n$  is on the smooth locus and  $r = n$ . Therefore  $X$  is smooth.  $\square$

*Proof of Theorem 2.* Let  $\pi : X \rightarrow W$  be an extremal contraction such that the divisor  $-(K_X + \det E)$  is  $\pi$ -ample. By Lemma 2.1 we know that  $X$  is Gorenstein. Let  $E_1 = E \oplus (-(K_X + \det E))$  then  $rk E_1 = e + 1$  and  $K_X + \det E_1$  is  $\pi$ -nef. Therefore the relative version of Theorem 1 and easy computations to recover  $E$  from  $E_1$  allow us to draw the desired conclusion.  $\square$

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