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A SHORT PROOF OF A CHARACTERIZATION OF REFLEXIVITY OF JAMES

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Abstract. A short direct proof is given to a well-known intrinsic characterization of reflexivity due to R. C. James.

The following famous intrinsic geometric characterization of reflexivity is due to R. C. James [4] (cf. also e.g. [1, p. 51], [2, p. 58] or [3]).

Theorem. *A Banach space X is reflexive if and only if there is a $\theta \in (0,1)$ such that if $(x_n)_{n=1}^{\infty}$ is a sequence of elements of the unit sphere of X, S_X, with $\|u\| > \theta$ for all $u \in \operatorname{conv}\{x_1, x_2, \dots\}$, then there are $n_0 \in \mathbb{N}$, $u \in \operatorname{conv}\{x_1, x_2, \dots, x_{n_0}\}$ and $v \in \operatorname{conv}\{x_{n_0+1}, x_{n_0+2}, \dots\}$ such that $\|u - v\| \le \theta$.*

This note provides a short and easy direct proof of the James theorem. It does not rely on Helly's theorem (like the proof in [4]) nor the Šmulian-Eberlein theorem (like the proof in [5, pp. 95–99]).

Proof of the Theorem. Necessity. The following proof is traditional. We present it here for the sake of completeness. Let X be reflexive. Fix any $\theta > 0$, consider any $(x_n)_{n=1}^{\infty} \subset S_X$, and denote $K_n = \overline{\operatorname{conv}}\{x_{n+1}, x_{n+2}, \dots\}$. Since $(K_n)_{n=0}^{\infty}$ is a nested sequence of weakly compact sets, there is $x \in \bigcap_{n=0}^{\infty} K_n$. Since $x \in K_0$, there are $n_0 \in \mathbb{N}$ and $u \in \operatorname{conv}\{x_1, \dots, x_{n_0}\}$ such that $\|x - u\| < \theta/2$. Since $x \in K_{n_0}$, there is $v \in \operatorname{conv}\{x_{n_0+1}, x_{n_0+2}, \dots\}$ such that $\|x - v\| < \theta/2$. Hence $\|v - u\| < \theta$.

Sufficiency. Denote $B_\theta = \{F \in X^{**} : \|F\| \le \theta\}$. This is a weak* closed set. If $X \ne X^{**}$, then by Riesz's lemma there is $F_\theta \in S_{X^{**}} \backslash (B_\theta + \{x\})$ for all $x \in X$. Note that F_θ is in the weak* closure of S_X (by Goldstine's theorem and weak* lower-semicontinuity of the norm). Pick any $x_0 \in S_X$. Since the weak* open set $X^{**} \backslash (B_\theta + \{x_0\})$ contains F_θ, it also contains a convex weak* neighbourhood V_1 of F_θ, which means

$$\|v - x_0\| > \theta \qquad \forall v \in V_1.$$

Since $F_\theta \in X^{**} \backslash B_\theta$, we can assume that $V_1 \subset X^{**} \backslash B_\theta$. Pick any $x_1 \in V_1 \cap S_X$. Since $F_\theta \in X^{**} \backslash (B_\theta + \operatorname{conv}\{x_0, x_1\})$, there is a convex weak* neighbourhood $V_2 \subset V_1$ of F_θ such that

$$\|v - u\| > \theta \qquad \forall u \in \operatorname{conv}\{x_0, x_1\}, \quad \forall v \in V_2.$$

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Pick any $x_2 \in V_2 \cap S_X$ and continue as above. The sequences of convex sets $V_1 \supset V_2 \supset \cdots$ and elements $(x_n)_{n=1}^{\infty} \subset S_X$ satisfy $x_n \in V_n$ and

$$\|v - u\| > \theta \qquad \forall u \in \operatorname{conv}\{x_1, \dots, x_n\}, \quad \forall v \in V_{n+1}.$$

Since $\operatorname{conv}\{x_n, x_{n+1}, \dots\} \subset V_n \subset X^{**} \backslash B_\theta$, this contradicts the assumption. □

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