

AN APPLICATION OF SCHAUDER'S FIXED POINT THEOREM WITH RESPECT TO HIGHER ORDER BVPS

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ABSTRACT. We shall provide conditions on the function $f(t, u_1, \dots, u_{n-1})$.
The higher order boundary value problem

(BVP)

$$\begin{cases} (E) & u^{(n)}(t) + f(t, u(t), u^{(1)}(t), \dots, u^{(n-2)}(t)) = 0 \text{ for } t \in (0, 1) \text{ and } n \geq 2, \\ (BC) & \begin{cases} u^{(i)}(0) = 0, & 0 \leq i \leq n-3, \\ \alpha u^{(n-2)}(0) - \beta u^{(n-1)}(0) = 0, \\ \gamma u^{(n-2)}(1) + \delta u^{(n-1)}(1) = 0 \end{cases} \end{cases}$$

has at least one solution.

1. INTRODUCTION

In this article, we shall attempt to construct some existence criteria for the following n -th order boundary value problem:

(BVP)

$$\begin{cases} (E) & u^{(n)}(t) + f(t, u(t), u^{(1)}(t), \dots, u^{(n-2)}(t)) = 0 \text{ for } t \in (0, 1) \text{ and } n \geq 2, \\ (BC) & \begin{cases} u^{(i)}(0) = 0, & 0 \leq i \leq n-3, \\ \alpha u^{(n-2)}(0) - \beta u^{(n-1)}(0) = 0, \\ \gamma u^{(n-2)}(1) + \delta u^{(n-1)}(1) = 0. \end{cases} \end{cases}$$

The motivation for the present work stems from many recent investigations in [1]–[3], [8], [13], [23]–[24]. In fact, particular cases of the boundary value problem (BVP) occur in various physical phenomena [4]–[7], [9]–[10], [13], especially such as gas diffusion through porous media, thermal self-ignition of a chemically active mixture of gases in a vessel [7], catalysis theory [9], chemically reacting systems, as well as adiabatic tubular reactor processes. For other related works, we refer to recent contributions of Agarwal and Wong [1]–[3], Anuradaha, Hai and Shivaji [4], Bailey, Shampine and Waltman [5], Erbe and Wang [13], Granas, Guenther and Lee [16], Lee and O'Regan [21], Chyan and Henderson [8], Henderson [17], Vasilev and Klovov [23] and Kelevedjiev [18]–[19] and the references therein.

Here, we shall remark that there are four main techniques to treat the existence of (BVP) as follows:

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Method 01. Shooting method ([25]). *This method has been used successfully in the study of some special boundary value problems, if one can guarantee the uniqueness of related initial value problems.*

Method 02. Fixed point index method ([1]–[2], [11], [20]). *This method has many advantages in treating non-singular boundary value problems and relies on the following lemma:*

Let E be a Banach space, and let $C \subseteq E$ be a cone in E . Assume that Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1, \overline{\Omega_1} \subset \Omega_2$, and let

$$T : C \cap (\overline{\Omega_2} \setminus \Omega_1) \longrightarrow C$$

be a completely continuous operator such that either

- (i) $\|Tu\| \leq \|u\|$, $u \in C \cap \partial\Omega_1$, and $\|Tu\| \geq \|u\|$, $u \in C \cap \partial\Omega_2$; or
- (ii) $\|Tu\| \geq \|u\|$, $u \in C \cap \partial\Omega_1$, and $\|Tu\| \leq \|u\|$, $u \in C \cap \partial\Omega_2$.

Then T has a fixed point in $C \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Method 03. Nonlinear alternative or topological method ([6], [15]–[16], [22]). *This method was initiated in Granas, Guenther and Lee [15]–[16]:*

Let X, Z be real vector normed spaces, $L : \text{dom}L \subset X \rightarrow Z$ a linear Fredholm mapping of index zero, $\Omega \subset X$ an open bounded subset, and $N : \overline{\Omega} \rightarrow Z$ an L -compact mapping. If $\ker L = \{0\}$, $0 \in \Omega$ and

$$Lx - \mu Nx \neq 0$$

for every $(x, \mu) \in (\text{dom}L \cap \partial\Omega) \times (0, 1)$, then the equation

$$Lx = Nx$$

has at least one solution in $\text{dom}L \cap \overline{\Omega}$.

Method 04. Schauder's or Barrier's method ([12]). *In the next section, we attempt to establish a general existence principle for (BVP), which relies on Schauder's fixed point theorem:*

Let C be a convex subset of a normed linear space \mathbb{E} . Then every compact continuous function $T : C \rightarrow C$ has at least one fixed point.

2. MAIN RESULTS

Let $\alpha, \gamma, \beta, \delta \geq 0$, $\rho := \gamma\beta + \alpha\gamma + \alpha\delta > 0$ and \mathbb{B} be the Banach space

$$\{u \in C^{(n)}(0, 1) \cap C^{(n-1)}[0, 1] \mid u^{(i)}(0) = 0, 0 \leq i \leq n-3\}$$

with norm $\|u\| \equiv \sup_{t \in [0, 1]} |u^{(n-2)}(t)|$.

In order to abbreviate our discussion, we suppose throughout this paper that the following assumptions hold:

(C_1): $K(t, s)$ is the Green's function of the differential equation

$$-u^{(n)}(t) = 0 \text{ in } (0, 1)$$

subject to the boundary conditions (BC).

(C_2): $k(t, s)$ is the Green's function of the differential equation

$$-u''(t) = 0 \text{ in } (0, 1)$$

subject to the boundary conditions

$$(BC^*) \quad \begin{cases} \alpha u(0) - \beta u'(0) = 0, \\ \gamma u(1) + \delta u'(1) = 0. \end{cases}$$

(C₃): $f \in C([0, 1] \times \mathbb{R}^{n-1}; \mathbb{R})$.

(C₄): $v, w \in \mathbb{B}$ are lower solutions and upper solutions of (BVP) in the sense:

$$\begin{cases} (1^0) & v^{(n)}(t) + f(t, v(t), v^{(1)}(t), \dots, v^{(n-2)}(t)) \geq 0 \text{ for } t \in (0, 1), \\ (2^0) & w^{(n)}(t) + f(t, w(t), w^{(1)}(t), \dots, w^{(n-2)}(t)) \leq 0 \text{ for } t \in (0, 1), \\ (3^0) & \begin{cases} v^{(i)}(0) = 0, \quad 0 \leq i \leq n-3, \\ \alpha v^{(n-2)}(0) - \beta v^{(n-1)}(0) \leq 0, \\ \gamma v^{(n-2)}(1) + \delta v^{(n-1)}(1) \leq 0, \end{cases} \\ (4^0) & \begin{cases} w^{(i)}(0) = 0, \quad 0 \leq i \leq n-3, \\ \alpha w^{(n-2)}(0) - \beta w^{(n-1)}(0) \geq 0, \\ \gamma w^{(n-2)}(1) + \delta w^{(n-1)}(1) \geq 0, \end{cases} \end{cases}$$

respectively.

(C₅): $f(t, u_1, \dots, u_{n-2}, u_{n-1})$, $v(t)$ and $w(t)$ satisfy

$$v^{(n-2)}(t) \leq w^{(n-2)}(t) \text{ on } [0, 1], \text{ and}$$

$$f(t, v(t), \dots, v^{(n-3)}(t), u_{n-1})$$

$$\leq f(t, u_1, \dots, u_{n-2}, u_{n-1})$$

$$\leq f(t, w(t), \dots, w^{(n-3)}(t), u_{n-1})$$

$$\text{for } t \in [0, 1], (v(t), \dots, v^{(n-3)}(t)) \leq (u_1, \dots, u_{n-2}) \leq (w(t), \dots, w^{(n-3)}(t)),$$

in which

$$(x_1, \dots, x_{n-2}) \leq (y_1, \dots, y_{n-2}) \iff x_i \leq y_i \text{ for } i = 1, \dots, n-2.$$

Remark 2.1. It is clear that

(a) if $f(t, u_1, \dots, u_{n-2}, u_{n-1})$ is increasing with respect to (u_1, \dots, u_{n-2}) on \mathbb{R}^{n-2} for each fixed $(t, u_{n-1}) \in [0, 1] \times \mathbb{R}$, then (C₅) holds;

(b) a simple calculation can show that

$$\begin{aligned} \frac{\partial^{n-2}}{\partial t^{n-2}} K(t, s) &= k(t, s) \\ &= \begin{cases} \frac{1}{\rho}(\beta + \alpha s)\{\delta + \gamma(1-t)\}, & 0 \leq s \leq t \leq 1, \\ \frac{1}{\rho}(\beta + \alpha t)\{\delta + \gamma(1-s)\}, & 0 \leq t \leq s \leq 1; \end{cases} \end{aligned}$$

(c) there exists an $M \in (0, 1)$ such that

$$\begin{cases} (R_1) & \frac{k(t, s)}{k(s, s)} \leq 1, \quad \text{for } t \in [0, 1] \text{ and } s \in [0, 1], \\ (R_2) & \frac{k(t, s)}{k(s, s)} \geq M, \quad \text{for } t \in [\frac{1}{4}, \frac{3}{4}] \text{ and } s \in [0, 1]. \end{cases}$$

Now, we can state and prove our main result:

Theorem 2.2 (Main result). *Boundary value problem (BVP) has at least one solution $u \in \mathbb{B}$ such that*

$$v^{(i)}(t) \leq u^{(i)}(t) \leq w^{(i)}(t) \text{ on } [0, 1] \text{ for } i = 0, 1, \dots, n-2.$$

Proof. We separate the proof into the following steps:

Step (1). Consider the modified problem

$$(BVP^*) \begin{cases} (E^*) & u^{(n)}(t) + f^*(t, u(t), u^{(1)}(t), \dots, u^{(n-2)}(t)) = 0 \text{ for } t \in (0, 1), \\ (BC) & \begin{cases} u^{(i)}(0) = 0, \quad 0 \leq i \leq n-3, \\ \alpha u^{(n-2)}(0) - \beta u^{(n-1)}(0) = 0, \\ \gamma u^{(n-2)}(1) + \delta u^{(n-1)}(1) = 0, \end{cases} \end{cases}$$

where

$$f^*(t, u_1, \dots, u_{n-1}) := f(t, \eta_1, \dots, \eta_{n-1}) + \rho(\eta_{n-1} - u_{n-1}),$$

$$\eta_i := \begin{cases} w^{(i-1)}(t) & \text{if } u_i > w^{(i-1)}(t), \\ u_i & \text{if } v^{(i-1)}(t) \leq u_i \leq w^{(i-1)}(t), \\ v^{(i-1)}(t) & \text{if } u_i < v^{(i-1)}(t), \end{cases}$$

for all $i = 1, 2, \dots, n-1$, $t \in [0, 1]$ and $\rho : \mathbb{R} \rightarrow [-1, 1]$ is the radial retraction defined by

$$\rho(r) := \begin{cases} r & \text{for } |r| \leq 1, \\ \frac{r}{|r|} & \text{for } |r| > 1. \end{cases}$$

It is clear that (BVP*) has a solution $u = u(t)$ if, and only if, u is the solution of the operator equation

$$u(t) = \int_0^1 K(t, s) f^*(s, u(s), u^{(1)}(s), \dots, u^{(n-2)}(s)) ds := (Tu)(t), \quad u \in \mathbb{B},$$

or

$$u^{(n-2)}(t) = \int_0^1 k(t, s) f^*(s, u(s), u^{(1)}(s), \dots, u^{(n-2)}(s)) ds := (Tu)^{(n-2)}(t), \quad u \in \mathbb{B}.$$

Since f^* is continuous and bounded on $[0, 1] \times \mathbb{R}^{n-1}$, $T : \mathbb{B} \rightarrow \mathbb{B}$ is continuous and compact. Therefore, it follows from Schauder's fixed point theorem (cf. Method 04) that T has a fixed point $u \in \mathbb{B}$, i.e. (BVP*) has a solution $u \in \mathbb{B}$.

Step (2). Let

$$H(t) := u^{(n-2)}(t) - w^{(n-2)}(t) \text{ on } [0, 1].$$

Then we have

$$\begin{aligned} & H''(\theta) \\ & \geq -f^*(\theta, u(\theta), \dots, u^{(n-3)}(\theta), u^{(n-2)}(\theta)) \\ & \quad + f(\theta, w(\theta), \dots, w^{(n-3)}(\theta), w^{(n-2)}(\theta)) \\ & = -f(\theta, \eta_1, \dots, \eta_{n-2}, w^{(n-2)}(\theta)) - \rho(w^{(n-2)}(\theta) - u^{(n-2)}(\theta)) \\ & \quad + f(\theta, w(\theta), \dots, w^{(n-3)}(\theta), w^{(n-2)}(\theta)) \quad (\eta_i \text{ is defined as in Step (1)}) \\ & \geq -\rho(w^{(n-2)}(\theta) - u^{(n-2)}(\theta)) > 0 \quad (\text{by using } (C_5)) \end{aligned}$$

if $\theta \in (0, 1)$ such that $H(\theta) > 0$.

Therefore, we see that there is no $\theta \in (0, 1)$ such that $H(\theta) > 0$ and $H''(\theta) \leq 0$.

Step (3). Now, we claim that $H(t) \leq 0$ on $[0, 1]$. Suppose to the contrary that there exists a $t_0 \in [0, 1]$ such that $H(t_0) > 0$. Then there is a $\theta \in [0, 1]$ such that

$$H(\theta) := \max_{t \in [0, 1]} H(t) > 0.$$

Case(1). Suppose that $\beta = 0$, which implies $H(0) \leq 0$ and $\theta \in (0, 1]$.

(1⁰) Suppose that $\delta = 0$, which implies $H(1) \leq 0$ and $\theta \in [0, 1)$. Thus, we have $\theta \in (0, 1)$ and $H''(\theta) \leq 0$. This contradicts the conclusion of Step (2).

(2⁰) Suppose that $\gamma = 0$, which implies $H'(1) \leq 0$. It is clear that $\theta = 1$. In fact, if $\theta \in (0, 1)$, then $H''(\theta) \leq 0$. This gives a contradiction.

Suppose that $H'(\theta) = H'(1) < 0$; then $H(t)$ is strictly decreasing near $t = \theta = 1$. This implies $H(\theta) = H(1)$ cannot be the maximum of $H(t)$, thus we obtain $H'(1) = 0$.

Since $H(\theta) = H(1) > 0$, there exists an $\epsilon > 0$ such that $H(t) > 0$ in $(1 - \epsilon, 1]$. Thus $H''(t) > 0$ in $(1 - \epsilon, 1)$, which implies $H'(t)$ is strictly increasing on $[1 - \epsilon, 1]$. It follows from $H'(t) < H'(1) = H'(\theta) = 0$ on $[1 - \epsilon, 1)$ and we see that $H(1) = H(\theta)$ cannot be the maximum of $H(t)$. This gives a contradiction.

(3⁰) Suppose that $\gamma\delta > 0$, which implies

$$w^{(n-1)}(1) \geq \frac{-w^{(n-2)}(1)\gamma}{\delta} \quad \text{and} \quad u^{(n-1)}(1) = \frac{-u^{(n-2)}(1)\gamma}{\delta}.$$

By Case(1)-(1⁰), we see that

$$u^{(n-2)}(1) > w^{(n-2)}(1).$$

Hence, we have

$$w^{(n-1)}(1) \geq \frac{-w^{(n-2)}(1)\gamma}{\delta} > \frac{-u^{(n-2)}(1)\gamma}{\delta} = u^{(n-1)}(1).$$

It follows from Case(1)-(2⁰) that we obtain a contradiction.

Case(2). Suppose that $\alpha = 0$, which implies $H'(0) \geq 0$.

(4⁰) Suppose that $\delta = 0$, which implies $H(1) \leq 0$ and $\theta \in [0, 1)$. It is clear that $\theta = 0$. In fact, if $\theta \in (0, 1)$, then $H''(\theta) \leq 0$. This gives a contradiction.

Suppose that $H'(\theta) = H'(0) > 0$; then $H(t)$ is strictly increasing near $t = \theta = 0$. This implies $H(\theta) = H(0)$ cannot be the maximum of $H(t)$, thus we obtain $H'(0) = 0$.

Since $H(\theta) = H(0) > 0$, there exists an $\epsilon > 0$ such that $H(t) > 0$ in $[0, \epsilon]$. Thus $H''(t) > 0$ in $(0, \epsilon)$, which implies $H'(t)$ is strictly increasing on $[0, \epsilon]$. It follows from $H'(t) > H'(0) = H'(\theta) = 0$ on $(0, \epsilon]$ and we see that $H(0) = H(\theta)$ cannot be the maximum of $H(t)$. This gives a contradiction.

(5⁰) Suppose that $\gamma = 0$, which implies $H'(1) \leq 0$. By Case(1)-(2⁰) and Case(2)-(4⁰), we see that

$$H(0) > 0 \quad \text{and} \quad H(1) > 0.$$

By continuity and Step (2), we see that there exist $t_1, t_2 \in (0, 1)$ such that

$$t_1 < t_2, \quad H(t) > 0, \quad H''(t) > 0 \quad \text{on} \quad (0, t_1) \cup (t_2, 1).$$

It follows from $H'(0) \geq 0$ and $H'(1) \leq 0$ that

$$H'(t) > 0 \quad \text{on} \quad (0, t_1) \quad \text{and} \quad H'(t) < 0 \quad \text{on} \quad (t_2, 1).$$

This implies that there exists a $\theta \in (t_1, t_2)$ such that

$$H(\theta) > 0, H'(\theta) = 0 \text{ and } H''(\theta) \leq 0,$$

which contradicts the conclusion of Step (2).

(6⁰) Suppose that $\gamma\delta > 0$, which implies

$$w^{(n-2)}(1) \geq \frac{-w^{(n-1)}(1)\delta}{\gamma} \text{ and } u^{(n-2)}(1) = \frac{-u^{(n-1)}(1)\delta}{\gamma}.$$

By Case(2)-(5⁰), we see that

$$u^{(n-1)}(1) > w^{(n-1)}(1).$$

Hence, we have

$$w^{(n-2)}(1) \geq \frac{-w^{(n-1)}(1)\delta}{\gamma} > \frac{-u^{(n-1)}(1)\delta}{\gamma} = u^{(n-2)}(1).$$

It follows from Case(2)-(4⁰) that we obtain a contradiction.

Case(3). Suppose that $\delta = 0$, which implies $H(1) \leq 0$. By Case(1)-(1⁰) and Case(2)-(4⁰), we see that

$$\alpha\beta > 0 \text{ and } u^{(n-2)}(0) > w^{(n-2)}(0).$$

Thus, we have

$$w^{(n-1)}(0) \leq \frac{w^{(n-2)}(0)\alpha}{\beta} < \frac{u^{(n-2)}(0)\alpha}{\beta} = u^{(n-1)}(0).$$

It follows from Case(2)-(4⁰) that we obtain a contradiction.

Case(4). Suppose that $\gamma = 0$, which implies $H'(1) \leq 0$. By Case(1)-(2⁰) and Case(2)-(5⁰), we see that

$$\alpha\beta > 0 \text{ and } u^{(n-1)}(0) < w^{(n-1)}(0).$$

Thus, we have

$$w^{(n-2)}(0) \geq \frac{w^{(n-1)}(0)\beta}{\alpha} > \frac{u^{(n-1)}(0)\beta}{\alpha} = u^{(n-2)}(0).$$

It follows from Case(1)-(2⁰) that we obtain a contradiction.

Case(5). Suppose that $\alpha\beta\gamma\delta > 0$. By Case(1)-(3⁰), Case(2)-(6⁰), Case(3) and Case(4), we see that

$$\begin{aligned} u^{(n-2)}(0) &> w^{(n-2)}(0), u^{(n-1)}(0) < w^{(n-1)}(0), \\ u^{(n-2)}(1) &> w^{(n-2)}(1), u^{(n-1)}(1) > w^{(n-1)}(1). \end{aligned}$$

Thus, we have

$$w^{(n-1)}(0) \leq \frac{w^{(n-2)}(0)\alpha}{\beta} < \frac{u^{(n-2)}(0)\alpha}{\beta} = u^{(n-1)}(0) < w^{(n-1)}(0),$$

which gives a contradiction.

From Cases(1)-(5), we see that

$$u^{(n-2)}(t) \leq w^{(n-2)}(t) \text{ on } [0, 1].$$

Similarly, we may show that

$$v^{(n-2)}(t) \leq u^{(n-2)}(t) \text{ on } [0, 1].$$

Since $u, v, w \in \mathbb{B}$ and satisfy

$$v^{(n-2)}(t) \leq u^{(n-2)}(t) \leq w^{(n-2)}(t) \quad \text{on } [0, 1],$$

we obtain

$$v^{(i)}(t) \leq u^{(i)}(t) \leq w^{(i)}(t) \quad \text{on } [0, 1] \quad \text{for } i = 0, 1, \dots, n-2.$$

Therefore,

$$f^*(t, u(t), u^{(1)}(t), \dots, u^{(n-2)}(t)) = f(t, u(t), u^{(1)}(t), \dots, u^{(n-2)}(t)) \quad \text{on } [0, 1].$$

That is, $u(t)$ is a solution of (BVP) and satisfies

$$v^{(i)}(t) \leq u^{(i)}(t) \leq w^{(i)}(t) \quad \text{on } [0, 1] \quad \text{for } i = 0, 1, \dots, n-2. \quad \square$$

Theorem 2.3 (Main result). *Suppose that*

(H) *there exists a function $g \in ([0, 1] \times [0, \infty)^{n-1}; [0, \infty))$ which satisfies*

$$f(t, 0, 0, \dots, 0) \geq 0 \quad \text{on } [0, 1] \quad (f \text{ maybe has negative value for } u_i \neq 0),$$

$$g(t, |u_1|, |u_2|, \dots, |u_{n-1}|) \geq f(t, u_1, u_2, \dots, u_{n-1}) \quad \text{on } [0, 1] \times \mathbb{R}^{n-1}$$

and one of the following:

$$(2.1) \quad \max g_0 = A_1 \in [0, D_1] \quad \text{and} \quad \min g_\infty = A_2 \in (\frac{D_2}{M}, \infty],$$

$$(2.2) \quad \min g_0 = A_3 \in (\frac{D_2}{M}, \infty] \quad \text{and} \quad \max g_\infty = A_4 \in [0, D_1],$$

(2.3) *there exist two non-negative functions $h \in C([0, \infty)^{n-1}; [0, \infty))$, increasing with respect to $u_{n-1} \in [0, \infty)$, and $q \in C([0, 1]; [0, \infty))$ such that*

$$\begin{cases} g(t, u_1, u_2, \dots, u_{n-1}) := q(t)h(u_1, u_2, \dots, u_{n-1}) \quad \text{on } [0, 1] \times [0, \infty)^{n-1}, \\ \sup_{u_{n-1} \in (0, \infty)} \min_{(u_1, \dots, u_{n-2}) \in [0, \infty)} \frac{u_{n-1}}{Qh(u_1, \dots, u_{n-1})} > 1, \end{cases}$$

where

$$\max g_0 := \lim_{u_1, u_2, \dots, u_{n-1} \rightarrow 0^+} \max_{t \in [0, 1]} \frac{g(t, u_1, u_2, \dots, u_{n-1})}{u_{n-1}},$$

$$\min g_0 := \lim_{u_1, u_2, \dots, u_{n-1} \rightarrow 0^+} \min_{t \in [\frac{1}{2}, \frac{3}{4}]} \frac{g(t, u_1, u_2, \dots, u_{n-1})}{u_{n-1}},$$

$$\max g_\infty := \lim_{u_1, u_2, \dots, u_{n-1} \rightarrow \infty} \max_{t \in [0, 1]} \frac{g(t, u_1, u_2, \dots, u_{n-1})}{u_{n-1}},$$

$$\min g_\infty := \lim_{u_1, u_2, \dots, u_{n-1} \rightarrow \infty} \min_{t \in [\frac{1}{2}, \frac{3}{4}]} \frac{g(t, u_1, u_2, \dots, u_{n-1})}{u_{n-1}},$$

$$\left(\int_0^1 k(s, s) ds \right)^{-1} := D_1 = \frac{6\rho}{6\delta\beta + 3\gamma\beta + \alpha\gamma + 3\alpha\delta},$$

$$\left(\int_{\frac{1}{2}}^{\frac{3}{4}} k\left(\frac{1}{2}, s\right) ds \right)^{-1} := D_2 = \frac{64\rho}{16\beta\delta + 6\beta\gamma + 3\alpha\gamma + 8\alpha\delta}$$

and

$$Q := \max_{t \in [0, 1]} \int_0^1 k(t, s) q(s) ds.$$

Then (BVP) has at least one non-negative solution.

Proof. From the results of Agarwal and Wong [1], [2], [3], we can see that

(BVP^{**})

$$\begin{cases} (E^{**}) & w^{(n)}(t) + g(t, u(t), w^{(1)}(t), \dots, w^{(n-2)}(t)) = 0 \text{ for } t \in (0, 1) \text{ and } n \geq 2, \\ (BC^*) & \begin{cases} w^{(i)}(0) = 0, & 0 \leq i \leq n-3, \\ \alpha w^{(n-2)}(0) - \beta w^{(n-1)}(0) = 0, \\ \gamma w^{(n-2)}(1) + \delta w^{(n-1)}(1) = 0 \end{cases} \end{cases}$$

has at least one non-negative solution $w(t)$. It is clear that $w(t)$ and $v(t) := 0$ are the upper-solution and lower-solution of (BVP), respectively. From Theorem 2.2, we obtain the desired results. \square

Remark 2.4. For $n = 2$, there are many functions $g(t, u)$ that do not satisfy

$$\max g_0, \min g_0, \max g_\infty, \min g_\infty \in \{0, \infty\},$$

for example, $g(t, u) := \frac{e^u - 1}{1 + t^2}$ ($\max g_0 = 1$ and $\min g_0 = \frac{16}{25}$), $g(t, u) := (t + 1)\sinh u$ ($\max g_0 = 2$ and $\min g_0 = \frac{3}{2}$), $g(t, u) := u + t^2 e^{-u}$ ($\max g_0 = \infty, \max g_\infty = \min g_\infty = 1$).

Therefore, our main result generalizes all the recent investigations about the existence of non-negative solutions of (BVP).

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