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C^*-ALGEBRAS THAT ARE ONLY WEAKLY SEMIPROJECTIVE

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ABSTRACT. We show that the C^*-algebra of continuous functions on the Cantor set is a weakly semiprojective C^*-algebra that is not semiprojective.

The two most basic forms of semiprojectivity, both equivalent to useful notions of stability for C^*-algebra relations, are called semiprojectivity and weak semiprojectivity. A C^*-algebra A is *weakly semiprojective* if we can always solve the $*$-homomorphism lifting problem

$$\begin{array}{ccc} & & \prod_N^\infty B_n \\ & \overset{\bar{\varphi}}{\nearrow} & \downarrow \rho_N \\ A & \xrightarrow{\varphi} & \prod_1^\infty B_n \Big/ \bigoplus_1^\infty B_n \end{array} \qquad \begin{array}{c} (b_N, b_{N+1}, \dots) \\ \downarrow \\ [(0, \dots, 0, b_N, b_{N+1}, \dots)] \end{array}$$

and it is *semiprojective* if we can always solve the lifting problem

$$\begin{array}{ccc} & & B / J_N \\ & \overset{\bar{\varphi}}{\nearrow} & \downarrow \\ A & \xrightarrow{\varphi} & B \Big/ \overline{\bigcup_1^\infty J_n} \end{array} \qquad (J_1 \lhd J_2 \lhd \cdots \lhd B).$$

If A is finitely generated, we have an advantage proving weak semiprojectivity, as it suffices to prove that the lifting can be done only to produce a $*$-homomorphism $\bar{\varphi}$ so that

$$\|\rho_N \circ \bar{\varphi}(g) - \varphi(g)\| \leq \epsilon \quad (\forall g \in G)$$

for preordained positive ϵ and finite subset G. (See [9] or [3].)

What has been lacking is an example of a C^*-algebra that is weakly semiprojective but not semiprojective. With the Cantor set

$$X = \prod_1^\infty \{0,1\}, \quad X = \varprojlim X_j, \quad X_j = \prod_1^j \{0,1\},$$

we get such an example, $C(X)$. It is not semiprojective because its spectrum X is not an ANR, as is necessary for a commutative C^*-algebra to be semiprojective. It

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is weakly semiprojective by the following theorem, the well-known semiprojectivity of finite-dimensional C^*-algebras, and the existence of the $*$-homomorphisms

$$\theta_j : C(X) \longrightarrow C(X_j) \quad \gamma_j : C(X_j) \longrightarrow C(X)$$

of truncation and zero-padding, whose composition $\gamma_j \circ \theta_j$ converges pointwise to the identity. Although $C(X)$ is the direct limit of the $C(X_j)$, this is not very relevant. The point is that $C(X)$ is an "approximate retract" of $C(X_j)$. A retract of a semiprojective is semiprojective ([1]), so it is not surprising that we can conclude that $C(X)$ is weakly semiprojective.

The proof of Theorem 0.1 was inspired by similar arguments by Lin. See any of the references for interesting results regarding semiprojectivity and stable relations.

Theorem 0.1. *Suppose that $A_1, A_2, \ldots$ are semiprojective C^*-algebras and that A is a finitely generated C^*-algebra. If there are $*$-homomorphisms $\theta_j : A \to A_j$ and $\gamma_j : A_j \to A$ such that*

$$\|\gamma_j \circ \theta_j(a) - a\| \to 0$$

for all a in A, then A is weakly semiprojective.

Proof. Let $\epsilon > 0$ and a finite set $G \subset A$ be given, as well as a $*$-homomorphism of the form

$$\varphi : A \longrightarrow \prod_1^\infty B_n \Big/ \bigoplus_1^\infty B_n \ ,$$

which we now must lift, approximately. Fix j such that $\|\gamma_j \circ \theta_j(g) - g\| \leq \epsilon$ for all g in G. Since A_j is semiprojective, there exist N and a $*$-homomorphism

$$\psi : A_j \longrightarrow \prod_N^\infty B_n$$

such that $\rho_N \circ \psi = \varphi \circ \gamma_j$. Let $\bar{\varphi} = \psi \circ \theta_j$, so that, for any g in G,

$$\begin{aligned} \|\rho_K \circ \bar{\varphi}(g) - \varphi(g)\| &= \|\varphi \circ \gamma_j \circ \theta_j(g) - \varphi(g)\| \\ &\leq \|\gamma_j \circ \theta_j(g) - g\| \\ &\leq \epsilon. \end{aligned} \tag{0.1}$$

□

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