

THE MAXIMUM CONDITION ON ANNIHILATORS FOR POLYNOMIAL RINGS

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(Communicated by Ken Goodearl)

ABSTRACT. For each positive integer n , we construct a commutative ring \mathcal{R} such that the polynomial ring $\mathcal{R}[x_1, \dots, x_n]$ satisfies the maximum condition on annihilators and $\mathcal{R}[x_1, \dots, x_{n+1}]$ does not. In particular, there exists a commutative Kerr ring \mathcal{R} such that $\mathcal{R}[x]$ is not Kerr. This answers in the negative a question of Faith's.

INTRODUCTION

All rings in this paper are commutative. We say that a ring R has ascending chain condition on annihilators ($\text{acc } \perp$) if it satisfies the maximum condition on annihilators (of subsets of R). Recall that a ring R is Goldie if it has $\text{acc } \perp$ and has no direct sum of infinitely many non-zero ideals. In [3] Kerr constructs a Goldie ring R such that $R[x]$ does not satisfy $\text{acc } \perp$. Following [2], we say that a ring R is Kerr if $R[x]$ has $\text{acc } \perp$.

In [1], Camillo and Guralnick show that if R is an algebra over an uncountable field K satisfying $\text{acc } \perp$ then $R[x]$ satisfies $\text{acc } \perp$; thus, if R is a Kerr algebra over an uncountable field K then $R[x]$ is Kerr. On the other hand, in [4] Roitman proves that for any countable field K there exists a K -algebra R satisfying $\text{acc } \perp$ such that $R[x]$ does not satisfy $\text{acc } \perp$. In [2], Faith asks: If R is a Kerr ring, is $R[x]$? We use Roitman's techniques [4] to construct for each non-negative integer n and for each countable field K a K -algebra \mathcal{R} such that $\mathcal{R}[x_1, \dots, x_n]$ satisfies $\text{acc } \perp$ and $\mathcal{R}[x_1, \dots, x_{n+1}]$ does not. This answers Faith's question in the negative. For $n = 0$, \mathcal{R} is the example constructed by Roitman in [4].

Our ring \mathcal{R} is semiprimary, with index of nilpotency 3, and has infinite Goldie dimension. We do not know whether for any countable field K , there exists a Goldie K -algebra R such that $R[x]$ is not Goldie. We remark that Kerr's example is a $\mathbb{Z}/2\mathbb{Z}$ -algebra.

1. PRELIMINARY LEMMAS

Let K be a field. Let $u_{\alpha_1, \dots, \alpha_{n+1}}, v_{\beta_1, \dots, \beta_{n+1}}, x_1, \dots, x_{n+1}$ be indeterminates over K , where α_i, β_i, n are non-negative integers, $0 \leq \alpha_1 + \dots + \alpha_{n+1} \leq m$,

Received by the editors May 10, 1996 and, in revised form, January 30, 1997.

1991 *Mathematics Subject Classification*. Primary 16P60, 13B25.

Both authors are partially supported by the DGICYT (Spain), through the grant PB95-0626, and by the Comissionat per Universitats i Recerca de la Generalitat de Catalunya.

$0 \leq \beta_1 + \dots + \beta_{n+1} \leq p$ and $m, p \geq 1$. Set

$$U = \{u_{\alpha_1, \dots, \alpha_{n+1}} \mid 0 \leq \alpha_1 + \dots + \alpha_{n+1} \leq m\},$$

$$V = \{v_{\beta_1, \dots, \beta_{n+1}} \mid 0 \leq \beta_1 + \dots + \beta_{n+1} \leq p\}.$$

Let $B = K[U, V]$. Let

$$f(x_1, \dots, x_{n+1}) = \sum_{\alpha_1 + \dots + \alpha_{n+1} = 0}^m u_{\alpha_1, \dots, \alpha_{n+1}} x_1^{\alpha_1} \cdot \dots \cdot x_{n+1}^{\alpha_{n+1}},$$

$$g(x_1, \dots, x_{n+1}) = \sum_{\beta_1 + \dots + \beta_{n+1} = 0}^p v_{\beta_1, \dots, \beta_{n+1}} x_1^{\beta_1} \cdot \dots \cdot x_{n+1}^{\beta_{n+1}}$$

be polynomials in $B[x_1, \dots, x_{n+1}]$. Let

$$h(x_1, \dots, x_{n+1}) = f(x_1, \dots, x_{n+1})g(x_1, \dots, x_{n+1}).$$

Thus $h(x_1, \dots, x_{n+1})$ has the form $\sum_{\gamma_1 + \dots + \gamma_{n+1} = 0}^{m+p} s_{\gamma_1, \dots, \gamma_{n+1}} x_1^{\gamma_1} \cdot \dots \cdot x_{n+1}^{\gamma_{n+1}}$, where $s_{\gamma_1, \dots, \gamma_{n+1}}$ are the indicated coefficients in $K[U, V]$.

Lemma 1.1. *Let z, t be non-zero elements in the $K[x_1, \dots, x_n]$ -submodule of $B[x_1, \dots, x_{n+1}]$ generated by $U \cup V$. If zt is in the $K[x_1, \dots, x_n]$ -submodule generated by $s_{\gamma_1, \dots, \gamma_{n+1}}$ ($0 \leq \gamma_1 + \dots + \gamma_{n+1} \leq m + p$), then either $z \in K[U, x_1, \dots, x_n]$ and $t \in K[V, x_1, \dots, x_n]$, or $z \in K[V, x_1, \dots, x_n]$ and $t \in K[U, x_1, \dots, x_n]$.*

Proof. This follows from the fact that the $s_{\gamma_1, \dots, \gamma_{n+1}}$ are linear both in the indeterminates U and in the indeterminates V . □

Lemma 1.2. *Let z be in the $K[x_1, \dots, x_n]$ -submodule of $B[x_1, \dots, x_{n+1}]$ generated by U and let t be in the $K[x_1, \dots, x_n]$ -submodule generated by V . If z and t are non-zero then the following conditions are equivalent:*

- (i) zt is in the $K[x_1, \dots, x_n]$ -submodule generated by $s_{\gamma_1, \dots, \gamma_{n+1}}$ ($0 \leq \gamma_1 + \dots + \gamma_{n+1} \leq m + p$).
- (ii) Either zt is in the $K[x_1, \dots, x_n]$ -submodule generated by $s_{\gamma_1, \dots, \gamma_{n+1}}$, where $\gamma_1 + \dots + \gamma_{n+1} = m + p$, or there exist $c_0, c_1, \dots, c_{n+1}, d_0 \in K[x_1, \dots, x_n]$ such that $z = c_0 f\left(\frac{c_1}{c_0}, \dots, \frac{c_{n+1}}{c_0}\right)$ and $t = d_0 g\left(\frac{c_1}{c_0}, \dots, \frac{c_{n+1}}{c_0}\right)$.
- (iii) Either zt is in the $K[x_1, \dots, x_n]$ -submodule generated by $s_{\gamma_1, \dots, \gamma_{n+1}}$, where $\gamma_1 + \dots + \gamma_{n+1} = m + p$, or there exist $c_0, c_1, \dots, c_{n+1}, d_0 \in K[x_1, \dots, x_n]$ such that $zt = c_0 d_0 h\left(\frac{c_1}{c_0}, \dots, \frac{c_{n+1}}{c_0}\right)$.

Proof. (ii) \Rightarrow (iii) and (iii) \Rightarrow (i) are obvious.

(i) \Rightarrow (ii). Let

$$z = \sum_{\alpha_1 + \dots + \alpha_{n+1} = 0}^m \lambda_{\alpha_1, \dots, \alpha_{n+1}} u_{\alpha_1, \dots, \alpha_{n+1}},$$

$$t = \sum_{\beta_1, \dots, \beta_{n+1} = 0}^p \mu_{\beta_1, \dots, \beta_{n+1}} v_{\beta_1, \dots, \beta_{n+1}},$$

and

$$zt = \sum_{\gamma_1 + \dots + \gamma_{n+1} = 0}^{m+p} \nu_{\gamma_1, \dots, \gamma_{n+1}} s_{\gamma_1, \dots, \gamma_{n+1}},$$

where $\lambda, \mu, \nu \in K[x_1, \dots, x_n]$.

Let $\alpha_1^0, \dots, \alpha_{n+1}^0$ be such that $\lambda_{\alpha_1^0, \dots, \alpha_{n+1}^0} \neq 0$ and $\alpha_1^0 + \dots + \alpha_{n+1}^0$ is the least possible. Let $\beta_1^0, \dots, \beta_{n+1}^0$ be such that $\mu_{\beta_1^0, \dots, \beta_{n+1}^0} \neq 0$ and $\beta_1^0 + \dots + \beta_{n+1}^0$ is the least possible. If $\alpha_1^0 + \dots + \alpha_{n+1}^0 + \beta_1^0 + \dots + \beta_{n+1}^0 = m + p$ then $zt = \sum_{\gamma_1 + \dots + \gamma_{n+1} = m+p} \nu_{\gamma_1, \dots, \gamma_{n+1}} s_{\gamma_1, \dots, \gamma_{n+1}}$. Suppose that $\alpha_1^0 + \dots + \alpha_{n+1}^0 + \beta_1^0 + \dots + \beta_{n+1}^0 < m + p$. In this case we shall see that $\alpha_1^0 = \dots = \alpha_{n+1}^0 = \beta_1^0 = \dots = \beta_{n+1}^0 = 0$. Suppose that $\alpha_i^0 > 0$ for some i . If $\beta_1^0 + \dots + \beta_{n+1}^0 < p$, then

$$u_{\alpha_1^0, \dots, \alpha_i^0 - 1, \dots, \alpha_{n+1}^0} v_{\beta_1^0, \dots, \beta_i^0 + 1, \dots, \beta_{n+1}^0}$$

appears in $s_{\alpha_1^0 + \beta_1^0, \dots, \alpha_{n+1}^0 + \beta_{n+1}^0}$. Thus $\lambda_{\alpha_1^0, \dots, \alpha_i^0 - 1, \dots, \alpha_{n+1}^0} \neq 0$, in contradiction with the selection of $\alpha_1^0, \dots, \alpha_{n+1}^0$. Hence $\beta_1^0 + \dots + \beta_{n+1}^0 = p$. Now $\beta_j^0 > 0$ for some j and $\alpha_1^0 + \dots + \alpha_{n+1}^0 < m$. As above, we have a contradiction. Thus $\alpha_i^0 = 0$ for all i , and by symmetry $\beta_j^0 = 0$ for all j . Note that then $\lambda_{0, \dots, 0} \neq 0$ and $\mu_{0, \dots, 0} \neq 0$.

In this case, let $c_0 = \lambda_{0, \dots, 0}$, $c_1 = \lambda_{1, 0, \dots, 0}, \dots, c_{n+1} = \lambda_{0, \dots, 0, 1}$, $d_0 = \mu_{0, \dots, 0}$. Now

$$\begin{aligned} \nu_{1, 0, \dots, 0} &= c_0 \mu_{1, 0, \dots, 0} = c_1 d_0, \\ &\dots \\ \nu_{0, \dots, 0, 1} &= c_0 \mu_{0, \dots, 0, 1} = c_{n+1} d_0. \end{aligned}$$

If $\beta_1 + \dots + \beta_{n+1} \geq 2$, then there exists i such that $\beta_i \geq 1$ and

$$\nu_{\beta_1, \dots, \beta_{n+1}} = \lambda_{0, \dots, 0} \mu_{\beta_1, \dots, \beta_{n+1}} = \lambda_{0, \dots, 1, \dots, 0} \mu_{\beta_1, \dots, \beta_i - 1, \dots, \beta_{n+1}};$$

hence

$$\mu_{\beta_1, \dots, \beta_{n+1}} = \frac{c_i}{c_0} \mu_{\beta_1, \dots, \beta_i - 1, \dots, \beta_{n+1}}.$$

Now, by induction on $\beta_1 + \dots + \beta_{n+1}$, we see that

$$\mu_{\beta_1, \dots, \beta_{n+1}} = \frac{c_1^{\beta_1} \dots c_{n+1}^{\beta_{n+1}}}{c_0^{\beta_1 + \dots + \beta_{n+1}}} d_0.$$

In a similar way it is easy to see that

$$\lambda_{\alpha_1, \dots, \alpha_{n+1}} = \frac{c_1^{\alpha_1} \dots c_{n+1}^{\alpha_{n+1}}}{c_0^{\alpha_1 + \dots + \alpha_{n+1}}} c_0.$$

Thus $z = c_0 f\left(\frac{c_1}{c_0}, \dots, \frac{c_{n+1}}{c_0}\right)$ and $t = d_0 g\left(\frac{c_1}{c_0}, \dots, \frac{c_{n+1}}{c_0}\right)$. □

Let $u_{\alpha_1, \dots, \alpha_{n+1}}^{(m)}, v_{\beta_1, \dots, \beta_{n+1}}^{(m)}$ be indeterminates over K for $m \geq 1, 0 \leq \alpha_1 + \dots + \alpha_{n+1} \leq d_m, 0 \leq \beta_1 + \dots + \beta_{n+1} \leq e_m$, where $d_m \geq 1, e_m \geq 1$. Let

$$\begin{aligned} \mathcal{U} &= \{u_{\alpha_1, \dots, \alpha_{n+1}}^{(m)} \mid m \geq 1, 0 \leq \alpha_1 + \dots + \alpha_{n+1} \leq d_m\}, \\ \mathcal{V} &= \{v_{\beta_1, \dots, \beta_{n+1}}^{(m)} \mid m \geq 1, 0 \leq \beta_1 + \dots + \beta_{n+1} \leq e_m\}. \end{aligned}$$

Let $\mathcal{B} = K[\mathcal{U}, \mathcal{V}]$ and

$$f_m = f_m(x_1, \dots, x_{n+1}) = \sum_{\alpha_1 + \dots + \alpha_{n+1} = 0}^{d_m} u_{\alpha_1, \dots, \alpha_{n+1}}^{(m)} x_1^{\alpha_1} \cdots x_{n+1}^{\alpha_{n+1}},$$

$$g_m = g_m(x_1, \dots, x_{n+1}) = \sum_{\beta_1 + \dots + \beta_{n+1} = 0}^{e_m} v_{\beta_1, \dots, \beta_{n+1}}^{(m)} x_1^{\beta_1} \cdots x_{n+1}^{\beta_{n+1}},$$

$$h_{m,p}(x_1, \dots, x_{n+1}) = f_m g_p = \sum_{\gamma_1 + \dots + \gamma_{n+1} = 0}^{d_m + e_p} s_{\gamma_1, \dots, \gamma_{n+1}}^{(m,p)} x_1^{\gamma_1} \cdots x_{n+1}^{\gamma_{n+1}}.$$

Let \mathcal{C} be the $K[x_1, \dots, x_n]$ -submodule of $\mathcal{B}[x_1, \dots, x_{n+1}]$ generated by $s_{\gamma_1, \dots, \gamma_{n+1}}^{(m,p)}$ for $0 \leq \gamma_1 + \dots + \gamma_{n+1} \leq d_m + e_p$ and for all $m, p \geq 1$.

Lemma 1.3. *Let $p_1 = \sum \lambda_m f_m + \sum \lambda'_m g_m$ and $p_2 = \sum \mu_m f_m + \sum \mu'_m g_m$ be in $\mathcal{B}[x_1, \dots, x_{n+1}] \setminus \{0\}$, where $\lambda, \lambda', \mu, \mu' \in K[x_1, \dots, x_n]$.*

Let $c_1, \dots, c_{n+1}, c'_1, \dots, c'_{n+1} \in K(x_1, \dots, x_n)$. Suppose that

$$\begin{aligned} & \sum \lambda_m f_m(c_1, \dots, c_{n+1}) + \sum \lambda'_m g_m(c_1, \dots, c_{n+1}) \\ &= \sum \mu_m f_m(c'_1, \dots, c'_{n+1}) + \sum \mu'_m g_m(c'_1, \dots, c'_{n+1}). \end{aligned}$$

Then $p_1 = p_2$ and $c_i = c'_i$ for all i .

Proof. Since the coefficients of $f_m(x_1, \dots, x_{n+1})$ and $g_m(x_1, \dots, x_{n+1})$ are independent over $K[x_1, \dots, x_{n+1}]$, we have

$$\begin{aligned} \lambda_m f_m(c_1, \dots, c_{n+1}) &= \mu_m f_m(c'_1, \dots, c'_{n+1}), \\ \lambda'_m g_m(c_1, \dots, c_m) &= \mu'_m g_m(c'_1, \dots, c'_{n+1}) \end{aligned}$$

for all $m \geq 1$.

Now $\lambda_m u_{0, \dots, 0}^{(m)} = \mu_m u_{0, \dots, 0}^{(m)}$ and $\lambda'_m v_{0, \dots, 0}^{(m)} = \mu'_m v_{0, \dots, 0}^{(m)}$. Thus $\lambda_m = \mu_m$ and $\lambda'_m = \mu'_m$. Hence $p_1 = p_2$.

Without lost of generality we may assume that $\lambda_m \neq 0$. Then

$$\lambda_m c_i u_{0, \dots, 1, \dots, 0}^{(m)} = \mu_m c'_i u_{0, \dots, 1, \dots, 0}^{(m)}.$$

Thus $c_i = c'_i$ for all i . □

The next lemma is an easy generalization of Lemma 1.1.

Lemma 1.4. *Let z, t be non-zero elements in the $K[x_1, \dots, x_n]$ -submodule of $\mathcal{B}[x_1, \dots, x_{n+1}]$ generated by $\mathcal{U} \cup \mathcal{V}$. If $zt \in \mathcal{C}$, then either $z \in K[\mathcal{U}, x_1, \dots, x_n]$ and $t \in K[\mathcal{V}, x_1, \dots, x_n]$, or $z \in K[\mathcal{V}, x_1, \dots, x_n]$ and $t \in K[\mathcal{U}, x_1, \dots, x_n]$.*

Lemma 1.5. *Let z be in the $K[x_1, \dots, x_n]$ -submodule of $\mathcal{B}[x_1, \dots, x_{n+1}]$ generated by \mathcal{U} and let t be in the $K[x_1, \dots, x_n]$ -submodule generated by \mathcal{V} . Then the following conditions are equivalent:*

- (i) $zt \in \mathcal{C}$.
- (ii) *Either zt is in the $K[x_1, \dots, x_n]$ -submodule generated by $s_{\gamma_1, \dots, \gamma_{n+1}}^{(m,p)}$, where $\gamma_1, \dots, \gamma_{n+1} = d_m + e_p$, or there exist $\lambda_m, \mu_m \in K[x_1, \dots, x_n]$ and $c_1, \dots, c_{n+1} \in K(x_1, \dots, x_n)$ such that*

$$z = \sum \lambda_m f_m(c_1, \dots, c_{n+1}) \text{ and } t = \sum \mu_m g_m(c_1, \dots, c_{n+1}).$$

In fact, in (ii) $\lambda_m f_m(c_1, \dots, c_{n+1})$ and $\mu_m g_m(c_1, \dots, c_{n+1}) \in \mathcal{B}[x_1, \dots, x_n]$.

Proof. (ii) \Rightarrow (i) is obvious.

(i) \Rightarrow (ii) The element z has a unique representation as a sum $z = \sum_m z^{(m)}$, where $z^{(m)}$ is in the $K[x_1, \dots, x_n]$ -submodule generated by $u_{\alpha_1, \dots, \alpha_{n+1}}^{(m)}$ ($0 \leq \alpha_1 + \dots + \alpha_{n+1} \leq d_m$). Similarly $t = \sum_m t^{(m)}$, where $t^{(m)}$ is in the $K[x_1, \dots, x_n]$ -submodule generated by $v_{\beta_1, \dots, \beta_{n+1}}^{(m)}$ ($0 \leq \beta_1 + \dots + \beta_{n+1} \leq e_m$).

Now it is easy to see that since $zt \in \mathcal{C}$, $z^{(m)}t^{(p)}$ is in the $K[x_1, \dots, x_n]$ -submodule generated by $s_{\gamma_1, \dots, \gamma_{n+1}}^{(m,p)}$ ($0 \leq \gamma_1 + \dots + \gamma_{n+1} \leq d_m + e_p$). Suppose that $z^{(m)}$ and $t^{(p)}$ are non-zero. By Lemma 1.2, either $z^{(m)}t^{(p)}$ is in the $K[x_1, \dots, x_n]$ -submodule generated by $s_{\gamma_1, \dots, \gamma_{n+1}}^{(m,p)}$, where $\gamma_1 + \dots + \gamma_{n+1} = d_m + e_p$, or there exist $\lambda_{m,p}, \mu_{m,p}, c_1^{(m,p)}, \dots, c_{n+1}^{(m,p)} \in K[x_1, \dots, x_n]$ such that

$$z^{(m)} = \lambda_{m,p} f_m \left(\frac{c_1^{(m,p)}}{\lambda_{m,p}}, \dots, \frac{c_{n+1}^{(m,p)}}{\lambda_{m,p}} \right),$$

$$t^{(p)} = \mu_{m,p} g_p \left(\frac{c_1^{(m,p)}}{\lambda_{m,p}}, \dots, \frac{c_{n+1}^{(m,p)}}{\lambda_{m,p}} \right).$$

Furthermore, in the latter case, if $z^{(m')} \neq 0$ then there exist $\lambda_{m',p}, \mu_{m',p}, c_1^{(m',p)}, \dots, c_{n+1}^{(m',p)} \in K[x_1, \dots, x_n]$ such that

$$z^{(m')} = \lambda_{m',p} f_{m'} \left(\frac{c_1^{(m',p)}}{\lambda_{m',p}}, \dots, \frac{c_{n+1}^{(m',p)}}{\lambda_{m',p}} \right),$$

$$t^{(p)} = \mu_{m',p} g_p \left(\frac{c_1^{(m',p)}}{\lambda_{m',p}}, \dots, \frac{c_{n+1}^{(m',p)}}{\lambda_{m',p}} \right).$$

By Lemma 1.3, $\mu_{m,p} = \mu_{m',p}$ and $\frac{c_i^{(m,p)}}{\lambda_{m,p}} = \frac{c_i^{(m',p)}}{\lambda_{m',p}}$. Thus $\mu_{m,p}$ depends just on p and not on m , so we set $\mu_{m,p} = \mu_p$ for all m such that $t^{(p)} \neq 0$. Similarly, $\lambda_{m,p}$ depends just on m and not on p , so we set $\lambda_{m,p} = \lambda_m$. We have seen that $\frac{c_i^{(m,p)}}{\lambda_{m,p}}$ does not depend on m and similarly for p , so we set $\frac{c_i^{(m,p)}}{\lambda_{m,p}} = c_i$. Hence

$$z = \sum_m \lambda_m f_m(c_1, \dots, c_{n+1}) \text{ and } t = \sum_p \mu_p g_p(c_1, \dots, c_{n+1}).$$

□

2. THE RESULT

In this section we assume that K is a field with $|K| \leq \aleph_0$. Thus

$$|K(x_1, \dots, x_n)^{n+1}| \leq \aleph_0.$$

Given $c \in K(x_1, \dots, x_n)$, we denote by p_c and q_c two coprime elements of $K[x_1, \dots, x_n]$ such that $c = p_c/q_c$. Let $(c_{1,1}, \dots, c_{1,n+1}), (c_{2,1}, \dots, c_{2,n+1}), \dots, (c_{m,1}, \dots, c_{m,n+1}), \dots$ be the sequence of all elements in $K(x_1, \dots, x_n)^{n+1}$. As in the above section we define $\mathcal{U}, \mathcal{V}, \mathcal{B}, f_m, g_m$ and \mathcal{C} . Let $c_{m,0}$ be the least common

multiple of $q_{c_{m,1}}, \dots, q_{c_{m,n+1}}$, so $c_{m,0}c_{m,j} \in K[x_1, \dots, x_n]$ for $1 \leq j \leq n + 1$. For $m \geq 1$ and $i \geq 1$, let a_{mi} be a non-zero polynomial of least degree in $K[x_1, \dots, x_n]$ such that $a_{mi}f_m(c_{i,1}, \dots, c_{i,n+1})g_m(c_{i,1}, \dots, c_{i,n+1}) \in \mathcal{B}[x_1, \dots, x_n]$. Note that the degree of a_{mi} is $\leq \deg(c_{i,0}) \cdot (d_m + e_m)$.

Let I be the ideal of \mathcal{B} generated by $s_{\gamma_1, \dots, \gamma_{n+1}}^{(m,p)}$ for all $m \neq p$, the coefficients of the polynomials

$$a_{mi}f_m(c_{i,1}, \dots, c_{i,n+1})g_m(c_{i,1}, \dots, c_{i,n+1})$$

for all $1 \leq i < m$, the elements $s_{\gamma_1, \dots, \gamma_{n+1}}^{(m,m)}$ for $\gamma_1 + \dots + \gamma_{n+1} = d_m + e_m$ and all monomials in the indeterminates $\mathcal{U} \cup \mathcal{V}$ of degree 3. Let \mathcal{R} be the ring \mathcal{B}/I . Thus \mathcal{R} has a natural grading inherited from \mathcal{B} , $\mathcal{R} = \mathcal{R}_0 \oplus \mathcal{R}_1 \oplus \mathcal{R}_2$, and $\mathcal{R}_0 = K$. We denote by \bar{r} the canonical image of $r \in \mathcal{B}[x_1, \dots, x_{n+1}]$ in $\mathcal{R}[x_1, \dots, x_{n+1}]$.

Lemma 2.1. *The ring $\mathcal{R}[x_1, \dots, x_n]$ satisfies $\text{acc } \perp$.*

Proof. To prove that $\mathcal{R}[x_1, \dots, x_n]$ satisfies $\text{acc } \perp$ it is enough to show that for any pair of sequences $\{z_m\}_{m \geq 1}$ and $\{t_m\}_{m \geq 1}$ of nonzero elements in $\mathcal{B}_1[x_1, \dots, x_n]$ such that $\bar{z}_m \bar{t}_p = 0$ for $1 \leq p < m$, we have $\bar{z}_m \bar{t}_m = 0$ for almost all m .

For $m > p$ the elements $z_m t_p$ are in $I_2[x_1, \dots, x_n] \subseteq \mathcal{C}$. By Lemma 1.4 we may assume that $z_m \in K[\mathcal{U}, x_1, \dots, x_n]$ and $t_p \in K[\mathcal{V}, x_1, \dots, x_n]$ for all $p, m \geq 1$.

By Lemma 1.5 there exist $\lambda_{k,m,p}, \mu_{k,m,p} \in K[x_1, \dots, x_n] \setminus \{0\}$, and $c_{1,m,p}, \dots, c_{n+1,m,p} \in K(x_1, \dots, x_n)$ such that

$$z_m = \sum_k \lambda_{k,m,p} f_k(c_{1,m,p}, \dots, c_{n+1,m,p}) \text{ and}$$

$$t_p = \sum_k \mu_{k,m,p} g_k(c_{1,m,p}, \dots, c_{n+1,m,p}),$$

with $\lambda_{k,m,p} f_k(c_{1,m,p}, \dots, c_{n+1,m,p}), \mu_{k,m,p} g_k(c_{1,m,p}, \dots, c_{n+1,m,p}) \in \mathcal{B}[x_1, \dots, x_n]$.

By Lemma 1.3, $\lambda_{k,m,p}$ does not depend on p , $\mu_{k,m,p}$ does not depend on m and $c_{i,m,p}$ depends on neither m nor p . So we set $\lambda_{k,m,p} = \lambda_{k,m}, \mu_{k,m,p} = \mu_{k,p}$ and $c_{i,m,p} = c_i$. Thus

$$z_m = \sum_k \lambda_{k,m} f_k(c_1, \dots, c_{n+1}) \text{ and}$$

$$t_p = \sum_k \mu_{k,p} g_k(c_1, \dots, c_{n+1}).$$

Choose j such that $(c_1, \dots, c_{n+1}) = (c_{j,1}, \dots, c_{j,n+1})$. Let

$$z'_m = \sum_{k=1}^j \lambda_{k,m} f_k(c_1, \dots, c_{n+1}) \text{ and}$$

$$t'_p = \sum_{k=1}^j \mu_{k,p} g_k(c_1, \dots, c_{n+1}).$$

Let $k > j$. Since a_{kj} is the polynomial of minimal degree such that

$$a_{kj} f_k(c_{j,1}, \dots, c_{j,n+1}) g_k(c_{j,1}, \dots, c_{j,n+1}) \in \mathcal{B}[x_1, \dots, x_n]$$

and $f_k g_{k'} \in I[x_1, \dots, x_{n+1}]$ for all $k \neq k'$, we have that $\bar{z}_m \bar{t}_p$ is the canonical image of $z'_m t'_p$.

Now there exists p_0 such that for any $p \geq p_0$, t'_p is a $K(x_1, \dots, x_n)$ -linear combination of t'_1, \dots, t'_{p_0-1} . Thus for each $p \geq p_0$ there exist $b_{1,p}, \dots, b_{p_0-1,p} \in K(x_1, \dots, x_n)$ such that

$$t'_p = \sum_{l=1}^{p_0-1} b_{l,p} t'_l.$$

Hence for any $p \geq p_0$

$$z'_p t'_p = \sum_{l=1}^{p_0-1} b_{l,p} t'_l z'_p \in I(x_1, \dots, x_n) \cap \mathcal{B}[x_1, \dots, x_n] = I[x_1, \dots, x_n].$$

Therefore $\bar{z}_p \bar{t}_p = 0$ for any $p \geq p_0$. □

Assume $m > 1$ and let l_m be the maximum of the degrees of c_{j0} , for $1 \leq j < m$. Let l'_m be the maximum of the degrees of $c_{j0} c_{ji}$, for $1 \leq j < m$ and $1 \leq i \leq n + 1$. Hence, the degree of $a_{mj} f_m(c_{j,1}, \dots, c_{j,n+1}) g_m(c_{j,1}, \dots, c_{j,n+1})$ is $\leq l_m(d_m + e_m) + l'_m(d_m + e_m) = (l_m + l'_m)(d_m + e_m)$. Thus the number of non-zero coefficients of this polynomial, as an element of $\mathcal{B}[x_1, \dots, x_n]$, is $\leq \binom{n + (l_m + l'_m)(d_m + e_m)}{(l_m + l'_m)(d_m + e_m)}$.

Note that the number of coefficients of $f_m g_m \in \mathcal{B}[x_1, \dots, x_{n+1}]$ is

$$\binom{n + 1 + d_m + e_m}{d_m + e_m}.$$

Observe that $\binom{n + 1 + d_m + e_m}{d_m + e_m}$ is a polynomial of degree $(n + 1)!$ in $d_m + e_m$ while $\binom{n + (l_m + l'_m)(d_m + e_m)}{(l_m + l'_m)(d_m + e_m)}$ is a polynomial of degree $n!$ in $d_m + e_m$. Now it is clear that for each $m > 1$ there exists a positive integer N_m such that if $d_m + e_m \geq N_m$ then

$$\binom{n + 1 + d_m + e_m}{d_m + e_m} > m \binom{n + (l_m + l'_m)(d_m + e_m)}{(l_m + l'_m)(d_m + e_m)}.$$

Proposition 2.2. *Suppose that $d_m + e_m \geq N_m$ for all $m > 1$. Then the ring $\mathcal{R}[x_1, \dots, x_{n+1}]$ does not satisfy $\text{acc } \perp$.*

Proof. Clearly, $\bar{f}_m \bar{g}_p = 0$ for $m \neq p$. On the other hand, for any m , the K -subspace W_m generated by the coefficients of $f_m g_m$ has dimension $\binom{n + 1 + d_m + e_m}{d_m + e_m}$ and $W_m \cap I$ is generated by the coefficients of $a_{mi} f_m(c_{i,1}, \dots, c_{i,n+1}) g(c_{i,1}, \dots, c_{i,n+1})$ for all $1 \leq i < m$. So

$$\dim_K(W_m \cap I) \leq m \binom{n + (l_m + l'_m)(d_m + e_m)}{(l_m + l'_m)(d_m + e_m)}.$$

Since $d_m + e_m \geq N_m$ for all $m > 1$, we have

$$\dim_K(W_m \cap I) < \dim_K W_m$$

for all $m > 1$. Thus $f_m g_m \notin I[x_1, \dots, x_{n+1}]$ and $\bar{f}_m \bar{g}_m \neq 0$ for all $m > 1$. Hence $\mathcal{R}[x_1, \dots, x_{n+1}]$ does not satisfy $\text{acc } \perp$. □

REFERENCES

1. V. Camillo and R. Guralnick, *Polynomial rings over Goldie rings are often Goldie*, Proc. A.M.S. **98** (1986), 567-568. MR **87k**:16018
2. C. Faith, *Polynomial rings over Goldie-Kerr commutative rings*, Proc. A.M.S. **120** (1994), 989-993. MR **94k**:13024
3. J. W. Kerr, *The polynomial ring over a Goldie ring need not be a Goldie ring*, J. Alg. **134** (1990), 344-352. MR **91h**:16042
4. M. Roitman, *On polynomial extensions of Mori domains over countable fields*, J. of Pure and Appl. Algebra **64** (1990), 315-328. MR **91i**:13021

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