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# GLOBAL ITERATION SCHEMES FOR STRONGLY PSEUDO-CONTRACTIVE MAPS

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ABSTRACT. Suppose $E$ is a real *uniformly smooth Banach space*, $K$ is a nonempty closed convex and bounded subset of $E$, and $T : K \to K$ is a strong pseudo-contraction. It is proved that if $T$ has a fixed point in $K$ then both the Mann and the Ishikawa iteration processes, for an arbitrary initial vector in $K$, converge strongly to the unique fixed $T$. No continuity assumption is necessary for this convergence. Moreover, our iteration parameters are independent of the geometry of the underlying Banach space and of any property of the operator.

## 1. INTRODUCTION

Let $E$ be an arbitrary real Banach space. A mapping $T$ with domain $D(T)$ and range $R(T)$ in $E$ is called a *strong pseudo-contraction* if there exists $t > 1$ such that for all $x, y \in D(T)$ and $r > 0$, the following inequality holds:

$$\|x - y\| \leq \|(1 + r)(x - y) - rt(Tx - Ty)\|. \tag{1}$$

If $t = 1$ in inequality (1) then $T$ is called *pseudo-contractive*. The class of pseudo-contractive maps has been studied extensively by various authors (see, for example, [1], [2], [4]–[6], [7], [8], [11]–[14], [15], [16], [20], [21]–[25], [26]–[28], [29]–[30], [34]). Interest in pseudo-contractive mappings stems mainly from their firm connection with the important class of *accretive operators*—a mapping $U$ is called *accretive* if the inequality

$$\|x - y\| \leq \|x - y + s(Ux - Uy)\| \tag{2}$$

holds for every $x, y \in D(U)$ and for all $s > 0$. If $I$ denotes the identity operator on $E$, then for $t = 1$, inequality (1) implies that

$$\|x - y\| \leq \|x - y + r[(I - T)x - (I - T)y]\| \tag{3}$$

holds for all $x, y \in D(T)$ and $r > 0$, so that, from inequalities (2) and (3), it follows that an operator $T$ is pseudo-contractive if and only if $(I - T)$ is accretive. Consequently, the mapping theory for accretive operators is closely related to the fixed point theory for pseudo-contractions.

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Let $E^*$ denote the dual space of $E$ and let $J : E \to 2^{E^*}$ denote the normalized duality mapping of $E$ defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\| \cdot \|x^*\|, \|x^*\| = \|x\|\} \ ,$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known that if $E^*$ is strictly convex then $J$ is single-valued, and if $E^*$ is uniformly convex then $J$ is uniformly continuous on bounded subsets of $E$. In the sequel we shall denote a single-valued normalized duality map by $j$. As a consequence of a result of Kato [17], it follows from inequality (3) that $T$ is pseudo-contractive if and only if for each $x, y \in D(T)$ there exists $j(x-y) \in J(x-y)$ such that

$$\langle (I-T)x - (I-T)y,\ j(x-y) \rangle \geq 0 \ . \tag{4}$$

Furthermore, $T$ is strongly pseudo-contractive if and only if there exists $k > 0$ such that

$$\langle (I-T)x - (I-T)y, j(x-y) \rangle \geq k\|x-y\|^2 \ . \tag{5}$$

If $E = H$, a Hilbert space, then (4) and (5) are equivalent, respectively, to the *monotonicity* and *strong monotonicity* properties of $T$ in the sense of Minty [19].

The accretive operators were introduced independently by Browder [3] and Kato [17] in 1967. Interest in such mappings stems mainly from the fact that many physically significant problems can be modelled in terms of an initial value problem of the form

$$\frac{du}{dt} + Tu = 0,\ u(0) = u_0, \tag{6}$$

where $T$ is either accretive or strongly accretive in an appropriate Banach space. We observe that if $N(T)$ denotes the kernel of $T$, then members of $N(T)$ are the equilibrium points of the system (6). Consequently, considerable effort has been devoted to developing constructive techniques for the determination of the kernels of accretive operators. Moreover, since a continuous accretive operator can be approximated well by a sequence of strongly accretive ones, particular attention has been devoted to the determination of the kernels of strongly accretive maps (see for example, [1], [2], [4]–[6], [7], [8], [11]–[14], [20], [21]–[25], [26]–[28], [29], [30], [31], [32], [34], [35]).

Typical of the results obtained is the following theorem.

**Theorem $*$.** *Let $H$ be a Hilbert space, $T : H \to H$ a bounded strongly accretive map with a nonempty kernel $N(T)$. Then the sequence $\{x_n\}_{n=0}^{\infty}$ defined iteratively by $x_0 \in H$ and*

$$x_{n+1} = x_n - c_n T x_n, \quad n = 0, 1, 2, \ldots, \tag{7}$$

*where $c_n \in \ell^2 \backslash \ell^1$, converges globally to an element of $N(T)$.*

Recall that a Banach space $E$ is called *smooth* if, for every $x \in E$ with $\|x\| = 1$, there exists a unique $f^* \in E^*$ such that $\|f^*\| = f^*(x) = 1$ (see e.g., [9, page 21]. The *modulus of smoothness* of $E$ is the function $\rho_E : [0, \infty) \to [0, \infty)$, defined by

$$\rho_E(\tau) = \sup\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : x, y \in E, \|x\| = 1,\ \|y\| = \tau\}.$$

The Banach space $E$ is called *uniformly smooth* (e.g., [26]) if $\lim_{\tau \to 0} \frac{\rho_E(\tau)}{\tau} = 0$, and for $q > 1$, $E$ is said to be *$q$-uniformly smooth* if there exists a constant $c > 0$ such that

$$\rho_E(\tau) \leq c\tau^q, \ \tau \in [0,\infty). \tag{8}$$

It is well known (see e.g. [26]) that

$$L_p(\text{ or } \ell_p) \ \text{ is } \begin{cases} p\text{–uniformly smooth, if } 1 < p \leq 2, \\ 2\text{–uniformly smooth, if } p \geq 2. \end{cases}$$

Various authors have extended Theorem $*$ to more general Banach spaces. Vainberg [33, pages 276–284] proved the convergence of (7) in $L_p$ spaces ($1 < p < \infty$) when $T$ is Lipschitz continuous and strongly accretive; the author [4] obtained the same result in $L_p$, $p \geq 2$, under less restrictive contidions. Crandall and Pazy [20] proved convergence of (7) for a continuous strongly accretive operator on an arbitrary Banach space. Reich [23], and also Liu [18], proved the convergence of (7) for an arbitrary strongly accretive operator on uniformly smooth Banach spaces. We, however, remark immediately that in these results in general Banach spaces, the conditions imposed on the iteration parameter $c_n$ are not convenient in applications. For instance, Crandall and Pazy [20] required that at each iteration step, $c_k$ be determined by

$$c_k = \delta_{k+1}/(1+\delta_{k+1}),$$

where $\delta_{k+1} = 2^{-n_k}$ and $n_k$ is the least nonnegative integer such that

$$\|A\left(\frac{2^n}{1+2^n}x_k - \frac{1}{1+2^n}Ax_k\right) - Ax_k\| \leq \exp\{-(\delta_1+\delta_2+\cdots+\delta_k+1)\}.$$

It is clear that $c_k$ can hardly be determined in an explicit form from the above conditions. In [23], Reich imposed the additional assumption that $\sum_{n=0}^{\infty} c_n^2\|Tx_n\|^2 < \infty$. Again this condition obviously causes computational difficulties.

Recently, the author, reformulating Theorem $*$ in terms of strong pseudo-contractions proved the following theorem in which the iteration parameter is easily evaluated.

**Theorem C1** ([4])**.** *Suppose $K$ is a nonempty bounded closed convex subset of $L_p$, $p \geq 2$, and $T : K \to K$ is a Lipschitz strong pseudo-contraction. Let $\{c_n\}$ be a real sequence satisfying the following conditions:*

*(i) $0 < c_n < 1$ for all $n \geq 1$;*
*(ii) $\sum_{n=1}^{\infty} c_n = \infty$; and*
*(iii) $\sum_{n=1}^{\infty} c_n^2 < \infty$.*

*Then the sequence $\{x_n\}_{n=1}^{\infty}$ generated from $x_1 \in K$ by*

$$x_{n+1} = (1-c_n)x_n + c_nTx_n, \quad n \geq 1 \ ,$$

*converges strongly to the unique fixed point of $T$.*

Clearly one can choose $c_n = \frac{1}{n+1}$ for all $n$ in Theorem C1.

Several authors have generalized Theorem C1 in various directions. (See, for example, Bethke [2]; Kang [1]; Schur [29], [30]; Weng [34]; Xu et al. [28]; Xu and Roach [36].)

While most of these generalizations extend Theorem C1 either to more general Banach spaces or to larger classes of maps, the iteration parameters involved still depend either on the geometry of the underlying Banach space or on properties of the operator.

For example, in his extension of Theorem C1 to the more general $q$-uniformly smooth Banach spaces and to the more general Ishikawa scheme (see e.g., [6] or [13]), Deng [13] required that the parameters $\alpha_n$, $\beta_n$ in the scheme satisfy the following conditions:

(i) $0 \le \alpha_n^{s-1} \le 2^{-1}q(k - L\beta_n - L^2\beta_n)(bL^q + h)^{-1}, n \ge 0,$

(ii) $0 \le \beta_n^{s-1} \le \min\{\frac{k}{2(L+L^2)},\ \frac{qk}{bL^q+h)}\}$, $n \ge 0$,

where $k$ is the constant appearing in the definition of a strongly pseudo-contractive mapping, $b$ is a constant that appears in an inequality that characterizes $q$-uniformly smooth Banach spaces, $h = \max\{1,\ \frac{q(q-1)}{2}\}$ and $s = \min\{2, q\}$. In [31], Tan and Xu extended Theorem C1 also to $q$-uniformly smooth Banach spaces and required that $c_n$ satisfy the following condition: $0 < c_n < s_q$, $n \ge 1$, where $s_q$ is the (smaller) solution of the equation

$$f(s) := q(q-1)(1-k)s - (1 + d_qL^q)s^{q-1} + \frac{1}{2}qk = 0,\ s > 0$$

(see, e.g., [31] for the meaning of the symbols). Extensions by several authors to even the more general *uniformly smooth* Banach spaces have also been obtained.

The most general results in this class of Banach spaces seem to be the following theorems:

**Theorem R1** ([21], Theorem 1)**.** *Suppose $A$ is a strongly accretive operator with a zero, and $E$ is a uniformly smooth Banach space. Suppose that a sequence $\{x_n\}$ can be defined by*

$$x_{n+1} \in x_n - \lambda_n Ax_n, \quad n \ge 0,$$

*where $x_0 \in E$ and $\{\lambda_n\}$ is a positive sequence such that $\{(x_n - x_{n+1})/\lambda_n\}$ is bounded. If $\lim_{n\to\infty} \lambda_n = 0$ and $\sum_{n=0}^{\infty} \lambda_n = \infty$, then $\{x_n\}$ converges strongly to the zero of $A$.*

**Theorem C2** ([6])**.** *Let $E$ be a real uniformly smooth Banach spae, and let $K$ be a nonempty closed convex and bounded subset of $E$. Let $T : K \to K$ be a* **continuous** *strongly pseudo-contractive mapping of $K$ into itself. Let $\{c_n\}_{n=1}^{\infty}$ be a real sequence satisfying the following conditions:*

*(i) $0 < c_n < 1$ for all $n \ge 1$,*

*(ii) $\sum_{n=1}^{\infty} c_n = \infty$, and*

*(iii) $\sum_{n=1}^{\infty} c_n b(c_n) < \infty$.*

*Then the sequence $\{x_n\}_{n=1}^{\infty}$ generated by $x_1 \in K$,*

$$x_{n+1} = (1 - c_n)x_n + c_nTx_n,\ n \ge 1\ ,$$

*converges strongly to the unique fixed point of $T$.*

While Theorems R1 and C2 extend most of the results that have appeared to either the more general uniformly smooth Banach spaces (e.g. [6], [13] and [31]) or the more general class of *continuous* strong pseudo-contractions, there are still several questions to answer. In particular, condition (iii) of Theorem C2, in some sense, still depends on the geometry of the underlying Banach space, while the requirement in Theorem R1 that $\{(x_n - x_{n+1})/\lambda_n\}$ be bounded depends on a property of the operator $A$. Nevanlinna and Reich [25], however, have shown how to choose the iteration parameters to satisfy condition (iii) of Theorem C2 in the case $E = L_p$ $(1 < p < \infty)$.

It is our purpose in this paper to resolve all these problems by proving that if $E$ is a real uniformly smooth Banach space, $K$ is a nonempty closed convex bounded subset of $E$ and $T : K \to K$ is a strong pseudo-contraction, with a fixed point $x^*$ in $K$, then both the Mann and the Ishikawa iteration schemes converge strongly to $x^*$, for an arbitrary initial point $x_0 \in K$. No continuity assumption is necessary for this convergence. Moreover, our iteration parameters will be totally independent both of the geometry of the underlying Banach space and of any special property of the operator. Consequently, under our setting, our theorem provides a satisfactory solution to the problem of iteratively approximating fixed points of strongly pseudo-contractive maps *in real uniformly smooth Banach spaces.*

## 2. Preliminaries

In the sequel, we shall make use of the following lemma.

**Lemma** ([22])**.** *Let $E$ be a uniformly smooth Banach space. Then there exists a nondecreasing continuous function $b : [0, \infty) \to [0, \infty)$ satisfying the following conditions:*

$$(9)\qquad \begin{cases} (i)\ b(ct) \le cb(t)\ \textit{for all}\ c \ge 1; \\ (ii)\ \lim_{t\to 0^+} b(t) = 0;\ \textit{and} \\ (iii)\ \|x+y|^2 \le \|x\|^2 + 2\ Re\langle y, j(x)\rangle + \max\{\|x\|, 1\}|y\|b(|y\|) \\ \qquad \textit{for all}\ x, y \in E. \end{cases}$$

## 3. Main results

We prove the following theorem.

### 3.1. Convergence theorems for strongly pseudo-contractive maps.

**Theorem.** *Suppose $E$ is a real uniformly smooth Banach space and $K$ is a bounded closed convex and nonempty subset of $E$. Suppose $T : K \to K$ is a strongly pseudo-contractive map such that $Tx^* = x^*$ for some $x^* \in K$. Let $\{\alpha_n\}$, $\{\beta_n\}$ be real sequences satisfying the following conditions:*

*(i) $0 \le \alpha_n$, $\beta_n < 1$ for all $n \ge 0$;*

*(ii) $\lim_{n\to\infty} \alpha_n = 0$; $\lim_{n\to\infty} \beta_n = 0$;*

*(iii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.*

*Then, for arbitrary $x_0 \in K$, the sequence $\{x_n\}$ defined iteratively by*

$$(10)\qquad x_{n+1} = (1-\alpha_n)x_n + \alpha_n Ty_n\ ,$$

$$(11)\qquad y_n = (1-\beta_n)x_n + \beta_n Tx_n,\ n \ge 0\ ,$$

*converges strongly to $x^*$. Moreover, $x^*$ is unique.*

*Proof.* Using (9), we compute as follows:

(12)
$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|(1-\alpha_n)(x_n - x^*) + \alpha_n(Ty_n - x^*)\|^2 \\ &\le (1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n(1-\alpha_n)\langle Ty_n - x^*, j(x_n - x^*)\rangle \\ &\quad + \max\{(1-\alpha_n)\|x_n - x^*\|, 1\}\alpha_n\|Ty_n - x^*\| \max\{\|Ty_n - x^*\|, 1\}b(\alpha_n) \\ &\le (1-\alpha_n)^2\|x_n - x^*\|^2 + M_1\alpha_n b(\alpha_n) + 2\alpha_n(1-\alpha_n)\delta_n\ , \end{aligned}$$

for some constant $M_1 > 0$ (since $K$ is bounded), where

$$\begin{aligned}\delta_n :&= \langle Ty_n - x^*,\ j(x_n - x^*)\rangle \\ &= \langle Ty_n - x^*, j(x_n - x^*) - j(y_n - x^*)\rangle + \langle Ty_n - Tx^*, j(y_n - x^*)\rangle \\ &\le \langle Ty_n - Tx^*, j(x_n - x^*) - j(y_n - x^*)\rangle + k\|y_n - x^*\|^2 \\ &= \Delta_n + k\|y_n - x^*\|^2\end{aligned}$$

where $\Delta_n := \langle Ty_n - Tx^*, j(x_n - x^*) - j(y_n - x^*)\rangle$.

*Claim* 1. $\Delta_n \to 0$ as $n \to \infty$.

*Proof.* Observe that $\{x_n - x^*\}$ and $\{y_n - x^*\}$ are bounded subsets of $E$, and

$$\|(x_n - x^*) - (y_n - x^*)\| = \beta_n\|x_n - Tx_n\| \le (\ \mathrm{diam} K)\beta_n \to 0$$

as $n \to \infty$. Hence, by the uniform continuity of $j$ on bounded subsets of $E$, and since $\{Ty_n - Tx^*\}$ is bounded, the proof of the claim is complete. □

Set $M_2 := 2(1 - \alpha_n)$. From (12) we obtain the following estimates:

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le (1 - \alpha_n)^2\|x_n - x^*\|^2 + M_1\alpha_n b(\alpha_n) \\ &\quad + 2k\alpha_n(1 - \alpha_n)\|y_n - x^*\|^2 + 2\alpha_n(1 - \alpha_n)\Delta_n \\ &\le (1 - \alpha_n)^2\|x_n - x^*\|^2 + 2k\alpha_n(1 - \alpha_n)\|y_n - x^*\|^2 \\ &\quad + \alpha_n[M_2\Delta_n + M_1 b(\alpha_n)].\end{aligned} \tag{13}$$

Now,

$$\|y_n - x^*\|^2 \le (1 - \beta_n)^2\|x_n - x^*\|^2 + 2k\beta_n(1 - \beta_n)\|x_n - x^*\|^2 + M_3\beta_n b(\beta_n)$$

for some constant $M_3 > 0$ (since $K$ is bounded). So,

$$\begin{aligned}\|y_n - x^*\|^2 &\le [1 - (1 - k)\beta_n]\|x_n - x^*\|^2 + M_3\beta_n b(\beta_n) \\ &\le \|x_n - x^*\|^2 + M_3\beta_n b(\beta_n).\end{aligned}$$

Substituting this inequality in (13) and setting $M_4 := 2kM_3(1 - \alpha_n)$ yield the following estimates:

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 \\ &\quad \le (1 - \alpha_n)^2\|x_n - x^*\|^2 + 2k\alpha_n(1 - \alpha_n)[\|x_n - x^*\|^2 + M_3\beta_n b(\beta_n)] \\ &\qquad + \alpha_n[M_2\Delta_n + M_1 b(\alpha_n)] \\ &\quad \le [(1 - \alpha_n) + k\alpha_n]^2\|x_n - x^*\|^2 + M_4\alpha_n\beta_n b(\beta_n) + \alpha_n[M_2\Delta_n + M_1 b(\alpha_n)].\end{aligned} \tag{14}$$

Set $J_n := M_4\beta_n b(\beta_n) + M_2\Delta_n + M_1 b(\alpha_n)$, and observe that condition (ii) and the continuity of (b) imply that $J_n \to 0$ as $n \to \infty$. Inequality (14) now yields that

$$\|x_{n+1} - x^*\|^2 \le [1 - (1 - k)\alpha_n]\|x_n - x^*\|^2 + \alpha_n J_n\ .$$

Set $\lambda_n := \|x_n - x^*\|^2$, $\sigma_n := \alpha_n J_n$, so that the last inequality reduces to

$$\lambda_{n+1} \le [1 - (1 - k)\alpha_n]\lambda_n + \sigma_n. \tag{15}$$

Clearly, $\{\lambda_n\}$ is bounded below. Let $a =\ \inf\{\lambda_n : n \ge 1\}$.

*Claim* 2. $a = 0$.

*Proof.* Assume $a \neq 0$, i.e., assume $a > 0$. Then for all $n \geq 1$, $\lambda_n \geq a > 0$. Observe that $(\sigma_n/\alpha_n) \to 0$ as $n \to \infty$. Hence there exists a positive integer $N_0$ such that for all $n \geq N_0$, $0 < (\sigma_n/\alpha_n) < a \leq \frac{1}{2}(1-k)\lambda_n$. This implies $\sigma_n \leq \frac{1}{2}(1-k)\alpha_n\lambda_n$ for all $n \geq N_0$. Substitute this inequality in (15) to get

$$\begin{aligned} 0 \leq \lambda_{n+1} &\leq [1-(1-k)\alpha_n]\lambda_n + \tfrac{1}{2}(1-k)\alpha_n\lambda_n \\ &= [1-(\tfrac{1-k}{2})\alpha_n]\lambda_n \leq \textstyle\prod_{j=0}^{n}[1-(\tfrac{1-k}{2})\alpha_j]\lambda_j \to 0 \quad \text{as } n \to \infty, \end{aligned}$$

since $\alpha_n \in (0,1)$ for all $n \geq 0$, $\{\lambda_n\}$ is bounded and $\sum_{n=0}^{\infty} \alpha_n = \infty$, a contradiction. Hence $a = 0$. □

*Claim* 3. $\lambda_n \to 0$ as $n \to \infty$.

*Proof.* By Claim 2, there exists a subsequence $\{\lambda_{n_j}\}_{j=0}^{\infty}$ of $\{\lambda_n\}_{n=0}^{\infty}$ such that $\lambda_{n_j} \to 0$ as $j \to \infty$. Now, given any $\epsilon > 0$, there exists an integer $j_*$ sufficiently large that

$$\frac{\sigma_n}{(1-k)\alpha_n} < \epsilon \quad \text{and } \lambda_{n_{j*}} < \epsilon \quad \forall\ n \geq n_{j_*}.$$

Inequality (15) now yields that

$$\lambda_{n_{j*}+1} \leq \left[1-(1-k)\alpha_{n_{j*}}\right]\epsilon + (1-k)\alpha_{n_{j*}}\epsilon = \epsilon\ ,$$

and a simple induction now yields $\lambda_{n_{j*}+p} \leq \epsilon$ for all integers $p \geq 1$, and this implies that $\lambda_n \to 0$ as $n \to \infty$. Uniqueness follows as in [4]. The proof of the theorem is complete. □

**Corollary.** *Suppose $E$ is a real uniformly smooth Banach space and $K$ is a bounded closed convex and nonempty subset of $E$. Suppose $T : K \to K$ is a strongly pseudo-contractive map such that $Tx^* = x^*$ for some $x^* \in K$. Let $\{c_n\}_{n=0}^{\infty}$ be a real sequence satisfying the following conditions:*

*(i)* $0 \leq c_n < 1$ *for all* $n \geq 0$*;*
*(ii)* $\lim_{n\to\infty} c_n = 0$*; and*
*(iii)* $\sum_{n=0}^{\infty} c_n = \infty$
*Then, for arbitrary $x_0 \in K$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined iteratively by*

$$x_{n+1} = (1-c_n)x_n + c_nTx_n, \quad n \geq 0\ ,$$

*converges strongly to $x$. Moreover, $x^*$ is unique.*

*Proof.* Set $c_n \equiv \alpha_n$ and $\beta_n \equiv 0$ for all $n$ in the Theorem, and the result follows immediately. □

*Remark* 1. Our theorem is a significant generalization of most of the results that have appeared on the convergence of either the Mann or the Ishikawa iteration scheme *in uniformly smooth Banach spaces.* In particular, the theorem extends Theorem 2 of [13] to the more general uniformly smooth Banach spaces, to the more general class of strong pseudo-contractions with fixed points and without the dependence of the iteration parameters $\alpha_n$, $\beta_n$ on properties of the operator (conditions (i) and (ii) of Theorem 2 of [13]). Similarly, our theorem extends Theorem 4.2 of [31] to more general Banach spaces, more general Ishikawa schemes, a more general class of operators, and also without the dependence of the iteration parameter $\alpha_n$ on properties of the operator (condition (i)). Also, our theorem extends Theorem C2 first to the more general Ishikawa scheme, then to the more general class of operators and without the dependence of the iteration parameter $c_n$ on the geometry of the underlying Banach space (condition (iii)). Furthermore, our theorem extends Theorem R1 to the case where the iteration parameter is independent of any special property of the operator.

*Remark* 2. In our theorem, the iteration parameters can be chosen at the start of the iteration process. A prototype for the parameters is

$$\alpha_n = \beta_n = c_n = \frac{1}{n+1} \text{ for all } n \geq 0 .$$

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