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MERGELYAN PAIRS FOR HARMONIC FUNCTIONS

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ABSTRACT. Let $\Omega \subseteq \mathbb{R}^n$ be open and $E \subseteq \Omega$ be a bounded set which is closed relative to Ω. We characterize those pairs (Ω, E) such that, for each harmonic function h on Ω which is uniformly continuous on E, there is a sequence of harmonic polynomials which converges to h uniformly on E. As an immediate corollary we obtain a characterization of Mergelyan pairs for harmonic functions.

1. RESULTS

Let Ω be an open set in Euclidean space $\mathbb{R}^n$ $(n \geq 2)$ and let $E \subseteq \Omega$ be a bounded set which is closed relative to Ω. Also, let $f|_E$ denote the restriction of a function f to E. We call (Ω, E) a *Mergelyan pair for harmonic functions* if each harmonic function h on Ω for which $h|_E$ is uniformly continuous on E can be uniformly approximated by harmonic polynomials on $K \cup E$ for every compact subset K of Ω. This paper presents a complete characterization of Mergelyan pairs for harmonic functions. The corresponding problem for holomorphic functions was solved by Stray [12] in the particular case where Ω is the unit disc, and then by Brown and Shields [2, p. 79 and Theorem 3] for general plane domains.

We will need some notation. If K is a compact subset of $\mathbb{R}^n$, then $K^\wedge$denotes the union of K with the bounded (connected) components of $\mathbb{R}^n \backslash K$. Let $\mathcal{A}$ denote the Alexandroff (ideal) point for Ω. If V is a connected open subset of Ω and there is a continuous function $p : [0, +\infty) \to V$ such that $p(t) \to \mathcal{A}$ as $t \to +\infty$, then we say that $\mathcal{A}$ is *accessible* from V. We define $E^\sim$ to be the union of E with the connected components of $\Omega \backslash E$ from which $\mathcal{A}$ is not accessible. Note that the definition of $E^\sim$ involves Ω, whereas the definition of $K^\wedge$does not. Also, the notation $\overline{E}^\wedge$ means $(\overline{E})^\wedge$. We refer to Doob [6, 1.XI] for an account of thin sets.

Theorem 1. *Let Ω be an open set in $\mathbb{R}^n$ and $E \subseteq \Omega$ be a bounded set which is closed relative to Ω. The following are equivalent*:

(a) *each harmonic function h on Ω for which $h|_E$ is uniformly continuous on E can be uniformly approximated on E by harmonic polynomials*;

(b) *$\mathbb{R}^n \backslash \overline{E}^\wedge$ and $\mathbb{R}^n \backslash E^\sim$ are thin at the same points of $\overline{E}$.*

Corollary 1. *Let Ω be an open set in $\mathbb{R}^n$ and $E \subseteq \Omega$ be a bounded set which is closed relative to Ω. The following are equivalent*:

(a) *(Ω, E) is a Mergelyan pair for harmonic functions*;

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(b) *for each compact subset K of Ω, the sets $\mathbb{R}^n\backslash(\overline{E}\cup K)^{\frown}$ and $\mathbb{R}^n\backslash(E\cup K)^{\sim}$ are thin at the same points of $\overline{E}\cup K$.*

Corollary 1 is an immediate consequence of Theorem 1 since $\overline{E\cup K}=\overline{E}\cup K$ for each compact $K\subseteq\Omega$. To see that conditions (b) of Corollary 1 and Theorem 1 are distinct, let B_1 and B_2 be open balls such that B_1 is internally tangent to B_2 at precisely one point, let $\Omega=B_2$ and $E=\Omega\cap\partial B_1$. Then $\overline{E}^{\frown}=\overline{B_1}$ and $E^{\sim}=E$, so condition (b) of Theorem 1 holds. However, if $K=\{x\}$, where x is the centre of B_1, then $(E\cup K)^{\sim}=E\cup K$ and x is in the interior of $(\overline{E}\cup K)^{\frown}$, so condition (b) of Corollary 1 fails. Similar considerations show that condition (b) of Corollary 1 implies that $\overline{E}^{\frown}\cap\Omega=E^{\sim}$, and even that $(\overline{E}\cup K)^{\frown}\cap\Omega=(E\cup K)^{\sim}$ for every compact $K\subseteq\Omega$.

We note from Brown and Shields [2, Theorem 3] that a Mergelyan pair for holomorphic functions must satisfy $\overline{E^{\sim}}=\overline{E}^{\frown}$. To see that this is not the case for harmonic functions, let $\Omega=B_2\backslash\overline{B_1}$, where B_1 and B_2 are open balls as above. Further, let E be a countable subset of Ω whose set of accumulation points is ∂B_1. Then (Ω,E) satisfies condition (b) of Corollary 1, but $\overline{E^{\sim}}=E\cup\partial B_1\neq E\cup\overline{B_1}=\overline{E}^{\frown}$.

Let $B(x,r)$ denote the open ball in $\mathbb{R}^n$ with centre x and radius r. For any set F in $\mathbb{R}^n$ we define

$$F^{\vee}=F\cup\{x\in\mathbb{R}^n\colon B(x,r)\backslash F \text{ is polar for some } r>0\}.$$

Clearly $F^{\vee}\backslash F$ is a polar set. We will show that, when $n=2$, condition (b) of Theorem 1 simplifies to the condition that $\partial(\overline{E}^{\frown})=\partial((E^{\sim})^{\vee})$. Thus we have the following.

Corollary 2. *Let Ω be an open set in $\mathbb{R}^2$ and $E\subseteq\Omega$ be a bounded set which is closed relative to Ω. The following are equivalent*:

(a) (Ω,E) *is a Mergelyan pair for harmonic functions*;

(b) $\partial((\overline{E}\cup K)^{\frown})=\partial(((E\cup K)^{\sim})^{\vee})$ *for each compact subset K of Ω.*

Corollary 2 extends recent work of Bonilla, Perez-Gonzalez and Trujillo-Gonzalez [1, Theorem 3.3 and Correction] which characterizes Mergelyan pairs for harmonic functions in the plane in the special case where Ω is a bounded open set satisfying $\partial\Omega=\partial(\overline{\Omega}^{\frown})$.

2. Proofs

2.1. Suppose that condition (b) of Theorem 1 holds and let h be a harmonic function on Ω such that $h|_E$ is uniformly continuous on E. Then h can be extended to $\Omega\cup\overline{E}$ in such a way that the restriction of h to $\overline{E}$ is continuous.

We claim that the restriction of h to $\overline{E^{\sim}}$ is continuous. To see this, let $W=E^{\sim}\backslash E$. Then each point y of $\partial W\cap\partial\Omega$ is regular for the Dirichlet problem on W; for otherwise, $\mathbb{R}^n\backslash W$ would be thin at y, and a result of Deny [5] then yields the contradictory conclusion that there are (many) paths emanating from y which lie initially in W. Also, it is clear that $\partial W\subseteq\overline{E}$. Let $u=H_h^W-h$, where H_h^W denotes the Perron-Wiener-Brelot solution of the Dirichlet problem on W with boundary function h. Then $u(x)\to 0$ as $x\to y$ for each $y\in\partial W\cap\Omega$ which is regular for W, and $\limsup_{x\to y}|u(x)|<+\infty$ for each y in the polar set of irregular boundary points of W. Let w be a positive superharmonic function on an open set containing

$\overline{E^{\sim}}$ such that $w = +\infty$ on this polar set. If $\delta > 0$, then

$$\limsup_{x\to y, x\in W} \{|u(x)| - \delta w(x)\} \le 0 \qquad (y \in \partial W \cap \Omega).$$

Since $\mathcal{A}$ is not accessible from any component of W, it follows that $|u| - \delta w \le 0$ on W (see [3], for example), and since δ can be arbitrarily small, we can conclude that $u \equiv 0$. Thus, by the regularity of points of $\partial W \cap \partial\Omega$ for W, the restriction of h to $\overline{E^{\sim}}$ is continuous.

Let $F = \overline{E^{\sim}}$ and let A denote the fine interior of F. Then h is finely harmonic on $A \cap \Omega$ (see [7] for an account of finely harmonic functions). Noting that $F \subseteq \overline{E}^{\frown}$, we see from condition (b) that

$$\begin{aligned} A\backslash\Omega &= \{x \in \overline{E} \cap \partial\Omega\colon \mathbb{R}^n\backslash F \text{ is thin at } x\} \\ &\subseteq \{x \in \overline{E} \cap \partial\Omega\colon \mathbb{R}^n\backslash \overline{E}^{\frown} \text{ is thin at } x\} \\ &= \{x \in \overline{E} \cap \partial\Omega\colon \mathbb{R}^n\backslash E^{\sim} \text{ is thin at } x\} \\ &\subseteq \{x \in \partial\Omega\colon \mathbb{R}^n\backslash \Omega \text{ is thin at } x\}, \end{aligned}$$

and this latter set is polar. Since polar sets are removable for bounded finely harmonic functions (see [7, Theorem 9.15]), h is finely harmonic on A.

Let $\varepsilon > 0$. In view of the properties of h established in the preceding two paragraphs, we can apply a result of Debiard and Gaveau [4] to see that there is a harmonic function ν on a neighbourhood of F such that $|h - \nu| < \varepsilon/2$ on F. Since $E^{\sim} \subseteq F \subseteq F^{\frown} = \overline{E}^{\frown}$, we see from condition (b) that $\mathbb{R}^n\backslash F^{\frown}$ and $\mathbb{R}^n\backslash F$ are thin at the same points of $\overline{E}$, and hence at the same points of F. By [8, Theorem 1.10], there is a harmonic function h_0 on $\mathbb{R}^n$ such that $|\nu - h_0| < \varepsilon/2$ on F, and hence $|h - h_0| < \varepsilon$ on $\overline{E}$. By suitably truncating the expansion of h_0 in terms of homogeneous harmonic polynomials, we obtain condition (a) of Theorem 1.

2.2. Conversely, suppose that condition (a) of Theorem 1 holds. Since $E^{\sim} \subseteq \overline{E}^{\frown}$, it is enough to show that, if $\mathbb{R}^n\backslash E^{\sim}$ is non-thin at a point y in $\overline{E}$, then so also is $\mathbb{R}^n\backslash\overline{E}^{\frown}$. If $n = 2$, then let ω denote an open disc which contains $\overline{E}$; otherwise let $\omega = \mathbb{R}^n$. Thus, in either case, ω possesses a Green function. Let $u^{\#}$ denote a continuous potential on ω which determines thinness (see [6, 1.XI.10]), and let $\widehat{R}_u^A$ denote the regularized reduced function (balayage) of a positive superharmonic function u relative to a set A in ω. Also, let B be an open ball such that $\overline{E} \subset B$ and $\overline{B} \subset \omega$.

Suppose that $\mathbb{R}^n\backslash E^{\sim}$ is non-thin at a fixed point y of $\overline{E}$ and let $\varepsilon > 0$. For each m in $\mathbb{N}$ let

$$A_m = B\backslash[B(y, 1/m) \cup \{x \in \Omega\colon \operatorname{dist}(x, E^{\sim}) < 1/m\}].$$

Since $A_m \uparrow B\backslash(E^{\sim} \cup \{y\})$ and $\mathbb{R}^n\backslash E^{\sim}$ is non-thin at y, we see that

$$\widehat{R}_{u^{\#}}^{A_m}(y) \uparrow \widehat{R}_{u^{\#}}^{B\backslash E^{\sim}}(y) = u^{\#}(y) \qquad (m \to \infty).$$

We choose k large enough so that

$$\widehat{R}_{u^{\#}}^{A_k}(y) \ge u^{\#}(y) - \varepsilon.$$

The measure μ associated with the potential $\widehat{R}_{u^{\#}}^{A_k}$ has support $\overline{A}_k$. It follows from a theorem of Choquet (see [11, Theorem 6.21]) and the fact that $y \notin \overline{A}_k$, that we can restrict μ to a smaller set in such a way that the corresponding potential u is continuous on ω and satisfies

$$u(y) > u^{\#}(y) - 2\varepsilon. \tag{1}$$

We now appeal to a recent result of the author [10, Theorem 2(b)]. This asserts that, if u is uniformly continuous on $E^{\sim}$ and finely harmonic on the fine interior of $E^{\sim}$, then u can be uniformly approximated on $E^{\sim}$ by harmonic functions on Ω whose restrictions to $E^{\sim}$ are uniformly continuous. The hypotheses of this result are certainly satisfied in the case of the function u constructed in the previous paragraph, since u is continuous on ω and harmonic on the open set $\{x \in \Omega\colon \operatorname{dist}(x, E^{\sim}) < 1/k\}$, which contains $E^{\sim}$. Hence there is a harmonic function h on Ω such that $h|_{E^{\sim}}$ is uniformly continuous on $E^{\sim}$ and such that

$$|u(x) - h(x)| < \varepsilon \qquad (x \in E^{\sim}). \tag{2}$$

Let $L = \overline{E}^{\frown}$. We next apply condition (a) to obtain a harmonic polynomial h_0 such that $|h - h_0| < \varepsilon$ on E. Thus, in view of (2), $|u - h_0| \le 2\varepsilon$ on $\overline{E}$, which contains ∂L. If we define

$$V_m = \{x \in \omega\colon \operatorname{dist}(x, L) < 1/m\} \qquad (m \in \mathbb{N}),$$

then for large values of m,

$$|H_u^{V_m} - h_0| = |H_{u-h_0}^{V_m}| < 3\varepsilon \quad \text{on } \overline{E}.$$

Since $H_u^{V_m} = \widehat{R}_u^{\omega\setminus V_m}$ on V_m, we see that

$$|\widehat{R}_u^{\omega\setminus V_m} - u| \le |\widehat{R}_u^{\omega\setminus V_m} - h_0| + |h_0 - u| < 5\varepsilon \quad \text{ on } \overline{E}. \tag{3}$$

From (3), (1) and the fact that $u^{\#} \ge u$ we obtain

$$\widehat{R}_{u^{\#}}^{\omega\setminus L}(y) \ge \widehat{R}_u^{\omega\setminus L}(y) \ge u(y) - 5\varepsilon > u^{\#}(y) - 7\varepsilon.$$

Since ε can be arbitrarily small, we see that

$$\widehat{R}_{u^{\#}}^{\omega\setminus L}(y) = u^{\#}(y).$$

It follows that $\omega\setminus L$, and hence $\mathbb{R}^n\setminus L$, is non-thin at y, as required.

2.3. In order to prove Corollary 2, it is enough to show that, when $n = 2$, condition (b) of Theorem 1 can be replaced by:

(b′) $\partial(\overline{E}^{\frown}) = \partial((E^{\sim})^{\vee})$.

First suppose that condition (b′) holds and suppose further that $\mathbb{R}^2\setminus\overline{E}^{\frown}$ is thin at a point y of $\overline{E}$. It follows that there are (many) circles centred at y which are contained in $\overline{E}^{\frown}$. Hence y lies in the interior of $\overline{E}^{\frown}$. By condition (b′), $y \notin \partial((E^{\sim})^{\vee})$. Clearly $y \in \overline{(E^{\sim})^{\vee}}$, so $\mathbb{R}^2\setminus(E^{\sim})^{\vee}$ is thin at y. Since $(E^{\sim})^{\vee}\setminus E^{\sim}$ is polar, it follows that $\mathbb{R}^2\setminus E^{\sim}$ is thin at y. Thus condition (b) of Theorem 1 holds.

Conversely, suppose that condition (b′) fails to hold. We will show that condition (a) of Theorem 1 then fails. Let $F = \overline{E^{\sim}}$ and let ω be an open disc that contains F. Since $F^{\frown} = \overline{E}^{\frown}$, it is clear that $\partial(\overline{E}^{\frown}) \subseteq \partial F \subseteq \partial((E^{\sim})^{\vee})$. Thus there exists a point y in $\partial((E^{\sim})^{\vee})\setminus\partial(\overline{E}^{\frown})$. We note that $y \in F \subseteq \overline{E}^{\frown}$, so $y \in (\overline{E}^{\frown})^0$. Let $r > 0$ be such that $B(y, 2r) \subseteq \overline{E}^{\frown}$. Since $y \in \partial((E^{\sim})^{\vee})$, it follows that $B(y, r)\setminus E^{\sim}$ is non-polar. Thus we can choose a non-zero measure on $B(y, r)\setminus E^{\sim}$ such that the corresponding potential u on ω is continuous (in addition to being harmonic on $(E^{\sim})^0$) and satisfies

$$u(y) > \sup\{u(x) \colon x \in \partial(\overline{E^{\frown}})\}.$$

By [9, Corollary 1] (or [10, Corollary 2]) there is a function h which is harmonic on Ω such that $h|_E$ is uniformly continuous on E, and such that

$$h(y) > \sup\{h(x) : x \in \partial(\overline{E^{\frown}})\}. \tag{4}$$

It is now clear that condition (a) of Theorem 1 fails, for otherwise (4) would lead us to contradict the maximum principle for harmonic functions in view of the fact that $y \in (\overline{E^{\frown}})^0$.

The proof of Corollary 2 is now complete.

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