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CAUCHY-SCHWARZ AND MEANS INEQUALITIES FOR ELEMENTARY OPERATORS INTO NORM IDEALS

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Abstract. The Cauchy-Schwarz norm inequality for normal elementary operators

$$\left|\!\left|\!\left|\sum_{n=1}^{\infty} A_n X B_n\right|\!\right|\!\right| \leq \left|\!\left|\!\left|(\sum_{n=1}^{\infty} A_n^* A_n)^{1/2} X (\sum_{n=1}^{\infty} B_n^* B_n)^{1/2}\right|\!\right|\!\right|,$$

implies a means inequality for generalized normal derivations

$$\left|\!\left|\!\left|\frac{AX+XB}{2}\right|\!\right|\!\right| \leq |\!|\!|X|\!|\!|^{1-\frac{1}{r}} \left|\!\left|\!\left|\frac{|A|^r X + X|B|^r}{2}\right|\!\right|\!\right|^{\frac{1}{r}},$$

for all $r \geq 2$, as well as an inequality for normal contractions A and B

$$\left|\!\left|\!\left|(I-A^*A)^{\frac{1}{2}} X (I-B^*B)^{\frac{1}{2}}\right|\!\right|\!\right| \leq |\!|\!|X - AXB|\!|\!|,$$

for all X in $B(H)$ and for all unitarily invariant norms $|\!|\!|\cdot|\!|\!|$.

1. Introduction

Let $B(H)$ and $\mathcal{C}_\infty$ stand respectively for spaces of all bounded and all compact linear operators acting on a separable, infinite-dimensional, complex Hilbert space H. For an $X \in B(H)$ let $\|X\|$ denote its norm, and for an arbitrary $X \in \mathcal{C}_\infty$ let $s_1(X) \geq s_2(X) \geq \cdots$ denote the singular values of X, i.e., the eigenvalues of $|X| = (X^*X)^{1/2}$, arranged in a non-increasing order, with their multiplicities counted. Each "symmetric gauge function" Φ on sequences gives rise to a unitarily invariant (u.i.) norm on operators defined by $\|A\|_\Phi = \Phi(\{s_n(A)\})$. We will denote by the symbol $|\!|\!|\cdot|\!|\!|$ any such norm. Any such norm is defined on a natural subclass $\mathcal{C}_{|\!|\!|\cdot|\!|\!|}$ of $\mathcal{C}_\infty$ called the norm ideal associated with the norm $|\!|\!|\cdot|\!|\!|$, and satisfies the invariance property $|\!|\!|UAV|\!|\!| = |\!|\!|A|\!|\!|$ for all A in this ideal and for all unitary operators U, V. Each norm ideal $\mathcal{C}_{|\!|\!|\cdot|\!|\!|}$ is closed in the topology generated by the norm $|\!|\!|\cdot|\!|\!|$. Particularly well known among unitarily invariant norms are the Schatten p-norms, defined as $\|X\|_p = (\sum_{n=1}^{\infty} s_n^p(A))^{1/p}$ for $1 \leq p < \infty$ and $\|X\|_\infty = \|X\| = s_1(X)$, which represent the norms on the Schatten p-ideals $\mathcal{C}_p$. The Ky Fan norms, defined as $|\!|\!|A|\!|\!|_k = \Phi_k(s_i(A)) = \sum_{i=1}^{k} s_i(A)$ for $k = 1, 2, \cdots$, represent another interesting family of unitarily invariant norms. The associated ideals $\mathcal{C}_\infty^{(k)}$ consist of all compact operators as every Ky Fan k-norm is equivalent to the norm in $\mathcal{C}_\infty$. The property saying that for all $X \in \mathcal{C}_\infty$ and $Y \in \mathcal{C}_{|\!|\!|\cdot|\!|\!|}$ with $|\!|\!|X|\!|\!|_k \leq |\!|\!|Y|\!|\!|_k$ for all $k \geq 1$ we have

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$X \in \mathcal{C}_{|||\cdot|||}$ with $|||X||| \le |||Y|||$ is known as the Ky Fan dominance property ([GK], ch. 3, §4). For a complete account of the theory of norm ideals, the reader is referred to [GK], [Sch] and [Si].

If $\mathcal{A} = (A_1, \dots, A_N)$ and $\mathcal{B} = (B_1, \dots, B_N)$ are N-tuples of bounded Hilbert space operators, then the elementary operator $R = R_{\mathcal{A},\mathcal{B}}$ on $B(H)$ is defined by $R(X) = \sum_{n=1}^{N} A_n X B_n$. Elementary operators were introduced by Lumer and Rosenblum in [LR], who studied their spectral properties. In this setting many authors subsequently studied spectral, algebraic, metric and structural properties of elementary operators (see [F83], [F85], [F87], [FL], [McI], and the references therein).

If both $\{A_n\}_{n=1}^{N}$ and $\{B_n\}_{n=1}^{N}$ are families of commuting normal operators, one can easly show that the associated elementary operator $R_{\mathcal{A},\mathcal{B}}$ is normal when restricted to the Hilbert space $\mathcal{C}_2$. So, in the sequel, by a normal elementary operator we will always mean such an operator, inluding those cases of $N = \infty$ with the convergence guaranted. As was shown by G. Weiss in his paper [W83], a famous Fuglede-Putnam type theorem extends to such operators. Very important examples of normal elementary operators are so called "pinching" operators $R_{\mathcal{P}}(X) = \sum_{n=1}^{\infty} P_n X P_n$, generated by a family of mutually orthogonal self-adjoint projections $\{P_n\}_{n=1}^{\infty}$.

2. Main results

We start with the basic Cauchy-Schwarz norm inequality for normal elementary operators. The following theorem extends "pinching" theorems 2.5.1 of [GK] and 1.19 of [Si].

Theorem 2.1. *If $\sum_{n=1}^{\infty} C_n^* C_n \le 1$, $\sum_{n=1}^{\infty} C_n C_n^* \le 1$, $\sum_{n=1}^{\infty} D_n^* D_n \le 1$ and $\sum_{n=1}^{\infty} D_n D_n^* \le 1$ for some operator families $\{C_n\}_{n=1}^{\infty}$ and $\{D_n\}_{n=1}^{\infty}$, then also $\sum_{n=1}^{\infty} C_n Y D_n \in \mathcal{C}_{|||\cdot|||}$ whenever $Y \in \mathcal{C}_{|||\cdot|||}$ for some unitarily invariant norm $|||\cdot|||$, and moreover*

$$\left|\left|\left| \sum_{n=1}^{\infty} C_n Y D_n \right|\right|\right| \le |||Y||| \,. \tag{2.1}$$

Proof. For arbitrary f and g in H a straightforward calculation gives

$$\begin{aligned} |\langle (\sum_{n=1}^{\infty} C_n Y D_n) f, g\rangle| &\le \sum_{n=1}^{\infty} \|Y\| \, \|D_n f\| \, \|C_n^* g\| \\ &\le \|Y\| \left(\sum_{n=1}^{\infty} \|D_n f\|^2 \right)^{1/2} \left(\sum_{n=1}^{\infty} \|C_n^* g\|^2 \right)^{1/2} \\ &= \|Y\| \left\langle \sum_{n=1}^{\infty} D_n^* D_n f, f \right\rangle^{1/2} \left\langle \sum_{n=1}^{\infty} C_n C_n^* g, g \right\rangle^{1/2} \\ &= \|Y\| \, \|(\sum_{n=1}^{\infty} C_n C_n^*)^{1/2} g\| \, \|(\sum_{n=1}^{\infty} D_n^* D_n)^{1/2} f\| \le \|Y\| \, \|f\| \, \|g\|, \end{aligned}$$

from which we conclude that

$$\| \sum_{n=1}^{\infty} C_n Y D_n \| \le \|Y\|. \tag{2.2}$$

Therefore, for all $N = 1, 2, \cdots$, for $Y \in \mathcal{C}_1$ and for all $W \in B(H)$ we have

$$\begin{aligned} |tr(\sum_{n=1}^{N} C_n Y D_n W^*)| &= |tr(Y(\sum_{n=1}^{N} C_n^* W D_n^*)^*)| \\ &\leq \quad \|Y\|_1 \|\sum_{n=1}^{N} C_n^* W D_n^*\| \leq \|Y\|_1 \|W\|, \end{aligned}$$

according to (2.2), from which we deduce that

$$\|\sum_{n=1}^{N} C_n Y D_n\|_1 \leq \|Y\|_1. \tag{2.3}$$

If $Y \in \mathcal{C}_\infty$, let $Y = \sum_{n=1}^{\infty} s_n(Y) \langle \cdot, e_n \rangle f_n$ be a singular value decompositon for some orthonormal systems $\{e_n\}$ and $\{f_n\}$. For all $k \geq 2$ we introduce operators $Z = \sum_{n=1}^{k-1} (s_n(Y) - s_{n+1}(Y)) \sum_{j=1}^{n} \langle \cdot, e_j \rangle f_j$ and $V = s_k(Y) \sum_{n=1}^{k} \langle \cdot, e_n \rangle f_n + \sum_{n=k+1}^{\infty} s_n(Y) \langle \cdot, e_n \rangle f_n$. We see that

$$\begin{aligned} Z &= \sum_{n=1}^{k-1} \sum_{j=1}^{n} (s_n(Y) - s_{n+1}(Y)) \langle \cdot, e_j \rangle f_j \\ &= \quad \sum_{j=1}^{k} (s_j(Y) - s_k(Y)) \langle \cdot, e_j \rangle f_j \\ &= \quad \sum_{n=1}^{k} s_n(Y) \langle \cdot, e_n \rangle f_n + s_k(Y) \sum_{n=1}^{k} \langle \cdot, e_n \rangle f_n = Y - V. \end{aligned}$$

We can also note that $s_1(V) = \cdots = s_k(V) = s_k(Y)$, due to orthogonality of the systems $\{e_n\}$ and $\{f_n\}$. That will allows us to conclude that for all Ky Fan k-norms we have

$$\left\| \sum_{n=1}^{N} C_n Y D_n \right\|_k \leq \left\| \sum_{n=1}^{N} C_n Z D_n \right\|_k + \left\| \sum_{n=1}^{N} C_n V D_n \right\|_k$$

$$\leq \quad \|Z\|_1 + k \|\sum_{n=1}^{N} C_n Z D_n\|_\infty \tag{2.4}$$

$$\leq \quad \sum_{n=1}^{k-1} (s_n(Y) - s_{n+1}(Y)) \sum_{j=1}^{n} \| \langle \cdot, e_j \rangle f_j \|_\infty + k \|V\|_\infty \tag{2.5}$$

$$\leq \quad \sum_{n=1}^{k-1} n(s_n(Y) - s_{n+1}(Y)) + k s_k(Y) = \sum_{n=1}^{k} s_n(Y) = \|Y\|_k \,, \tag{2.6}$$

with (2.4) following from (2.3) and (2.5) from (2.2).

Moreover, if Y is in $\mathcal{C}_\infty$ then also $\sum_{n=1}^{\infty} C_n Y D_n \in \mathcal{C}_\infty$. Indeed, elementary operators $R_N(Y) = \sum_{n=1}^{N} C_n Y D_n$ acting on $\mathcal{C}_\infty^{(k)}$ represent a bounded family, because $\|R_N(Y)\|_k \leq \|Y\|_k$ for all $Y \in \mathcal{C}_\infty$ by (2.6). Also, for one dimensional operators

$f \otimes g$ and $M > N$ we have

$$\begin{aligned}\|R_M(f\otimes g) - R_N(f\otimes g)\|_k &\le \|\sum_{n=N+1}^{M} D_n^* f \otimes C_n g\|_1 \\ &\le \sum_{n=N+1}^{M} \|D_n^* f\|\,\|C_n g\| \le \|(\sum_{n=N+1}^{M} C_n C_n^*)^{\frac12} g\|\,\|(\sum_{n=N+1}^{M} D_n^* D_n)^{\frac12} f\|,\end{aligned}$$

which $\longrightarrow 0$ as $M, N \longrightarrow \infty$. Therefore $R_N(Y)$ converge in $\mathcal{C}_\infty^{(k)}$ for all finite dimensional Y to a **compact** operator. By the uniform boundedness principle the same is true for all $Y \in \mathcal{C}_\infty^{(k)}$, due to its separability. So (2.1) holds for all Ky Fan k-norms, and we therefore invoke the Ky Fan dominance property to conclude that (2.1) holds for all unitarily invariant norms, as required. □

In the sequel we will refer to a family $\{A_n\}_{n=1}^\infty$ in $B(H)$ as square summable if $\sum_{n=1}^\infty \|A_n f\|^2 < \infty$ for all $f \in H$. Though this means just the weak convergence of $\sum_{n=1}^\infty A_n^* A_n$, an appeal to the resonance principle shows that $\sum_{n=1}^\infty A_n^* A_n$ actually defines a **bounded** Hilbert space operator, and due to the monotonicity of its partial sums, the convergence is moreover strong. For such families the following Cauchy-Schwarz inequality holds:

Theorem 2.2. *For a square summable families $\{A_n\}_{n=1}^\infty$ and $\{B_n\}_{n=1}^\infty$ of commuting normal operators*

$$\left|\!\left|\!\left|\sum_{n=1}^\infty A_n X B_n\right|\!\right|\!\right| \le \left|\!\left|\!\left|(\sum_{n=1}^\infty A_n^* A_n)^{1/2} X (\sum_{n=1}^\infty B_n^* B_n)^{1/2}\right|\!\right|\!\right|, \tag{2.7}$$

for all $X \in B(H)$ and for all u.i. norms $|||\cdot|||$. If $\mathcal{C}_{|||\cdot|||}$ is separable and $X \in \mathcal{C}_{|||\cdot|||}$, then the left-hand side sum converges in the norm of this ideal.

Proof. First, we need a suitable factorization for Hilbert space operators A_n and B_n. Let $A = (\sum_{n=1}^\infty A_n^* A_n)^{\frac12}$ and $B = (\sum_{n=1}^\infty B_n^* B_n)^{\frac12}$, and let P and Q denote respectively the orthogonal projections on $\overline{R(A)}$ and $\overline{R(B)}$. If for a given $f \in H$ we have that $Pf = \lim_{k\longrightarrow\infty} A g_k$ for some sequence $\{g_k\}$ in H, then $\lim_{k\longrightarrow\infty} A_n g_k$ exists for all $n \ge 1$ and does not depend on the chosen sequence. Indeed,

$$\|A_n g_k - A_n g_l\| \le \|A(g_k - g_l)\| \longrightarrow \|Pf - Pf\| = 0$$

as $k, l \longrightarrow \infty$, and also $\|A_n g_k - A_n h_k\| \le \|A(g_k - h_k)\| \longrightarrow 0$ as $k \longrightarrow \infty$ whenever $\lim_{k\longrightarrow\infty} A h_k = Pf$ for some other sequence $\{h_k\}$. Thus we can correctly introduce operators $C_n, n = 1, 2, \cdots$, by $C_n f = \lim_{k\longrightarrow\infty} A_n g_k$, where $\{g_k\}$ is any sequence in H such that $\lim_{k\longrightarrow\infty} A g_k = Pf$. Let us note that due to our definition every C_n vanishes on $N(A)$, i.e., $C_n = C_n P$, and also $C_n A = A C_n = A_n$. Moreover, $\sum_{n=1}^\infty C_n^* C_n = P$. Indeed, $\sum_{n=1}^\infty C_n^* C_n A^2 = \sum_{n=1}^\infty A_n^* A_n = A^2$ implies $\sum_{n=1}^\infty C_n^* C_n P = P$, which together with the fact that $C_n(I - P) = 0$ gives the desired conclusion. For all $m, n = 1, 2, \cdots$, C_m^* and C_n comute on $R(A^2)$ and $N(A^2)$, and so also on all of H. Thus $\{C_n\}_{n=1}^\infty$ is a commuting family of normal contractions which realize the factorizations $C_n A = A C_n = A_n$, with $\sum_{n=1}^\infty C_n^* C_n = P$, and which commute with the family $\{A_n\}_{n=1}^\infty$. Similarly we get a commuting family $\{D_n\}_{n=1}^\infty$ of normal contractions which also commute with $\{B_n\}_{n=1}^\infty$ and satisfy $D_n B = B D_n = B_n$ and $\sum_{n=1}^\infty D_n^* D_n = Q$. One could easily derive the next explicit formula: $C_n = \overline{A_n A^\dagger} = \overline{A^\dagger A_n}$, where $A^\dagger$ denotes a (densely defined) Moore-Penrose (generalized) inverse for A.

For $Y = AXB \in \mathcal{C}_{|||\cdot|||}$ (there is nothing to prove in the opposite case), an application of Theorem 2.1 gives

$$\left|\left|\left|\sum_{n=1}^{\infty} A_n X B_n\right|\right|\right| = \left|\left|\left|\sum_{n=1}^{\infty} C_n Y D_n\right|\right|\right|$$

$$\leq \quad |||Y||| = \left|\left|\left|(\sum_{n=1}^{\infty} A_n^* A_n)^{1/2} X (\sum_{n=1}^{\infty} B_n^* B_n)^{1/2}\right|\right|\right|, \tag{2.8}$$

which proves the first part of theorem.

Finally, if $\mathcal{C}_{|||\cdot|||}$ is separable, then for all $N = 1, 2, \cdots$, an application of the just proven part of theorem combined with the arithmetic-geometric means inequality in [BhD] gives

$$\left|\left|\left|\sum_{n=N}^{\infty} A_n X B_n\right|\right|\right| \leq \left|\left|\left|(\sum_{n=N}^{\infty} A_n^* A_n)^{\frac{1}{2}} X (\sum_{n=N}^{\infty} B_n^* B_n)^{\frac{1}{2}}\right|\right|\right|$$

$$= \quad \left|\left|\left|(\sum_{n=N}^{\infty} C_n^* C_n)^{\frac{1}{2}} AXB (\sum_{n=N}^{\infty} D_n^* D_n)^{\frac{1}{2}}\right|\right|\right|$$

$$\leq \quad \frac{1}{2}\left|\left|\left|(\sum_{n=N}^{\infty} C_n^* C_n) AXB + AXB (\sum_{n=N}^{\infty} D_n^* D_n)\right|\right|\right|. \tag{2.9}$$

We see by (2.8) that $\{\sum_{n=N}^{\infty} C_n^* C_n\}_{N=1}^{\infty}$ and $\{\sum_{n=N}^{\infty} D_n^* D_n\}_{N=1}^{\infty}$ represent bounded sequences of selfadjoint operators which strongly converge to 0 as $N \longrightarrow \infty$. As $AXB \in \mathcal{C}_{|||\cdot|||}$, which is separable, then the right-hand side of (2.9) tends to 0 as $N \longrightarrow \infty$ by Theorem 3.6.3. of [GK]. The conclusion follows. □

Corollary 2.1. *For normal A and B in $B(H)$ and for all real $r \geq 2$,*

$$\left|\left|\left|\frac{AX + XB}{2}\right|\right|\right| \leq \left|\left|\left|\left(\frac{1 + |A|^r}{2}\right)^{\frac{1}{r}} X \left(\frac{1 + |B|^r}{2}\right)^{\frac{1}{r}}\right|\right|\right| \tag{2.10}$$

as well as

$$\left|\left|\left|\frac{X + AXB}{2}\right|\right|\right| \leq \left|\left|\left|\left(\frac{1 + |A|^r}{2}\right)^{\frac{1}{r}} X \left(\frac{1 + |B|^r}{2}\right)^{\frac{1}{r}}\right|\right|\right| \tag{2.11}$$

for all $X \in B(H)$ and for all u.i. norms $|||\cdot|||$.

Proof. $\{A, I\}$ and $\{I, B\}$ are families of normal commuting operators, and so for $r = 2$ the desired conclusion follows by Theorem 2.2. For $r > 2$ the mapping $t \longrightarrow t^{\frac{2}{r}}$ is operator monotone by a well known Heinz therem, and therefore this is an operator concave mapping (see [BSh]). Specifically, $\frac{1+|A|^2}{2} \leq \left(\frac{1+|A|^r}{2}\right)^{\frac{2}{r}}$, from which we obtain $\left|\left|\left(\frac{1+|A|^2}{2}\right)^{\frac{1}{2}} \left(\frac{1+|A|^r}{2}\right)^{-\frac{1}{r}}\right|\right| \leq 1$ and similarly $\left|\left|\left(\frac{1+|B|^2}{2}\right)^{\frac{1}{2}} \left(\frac{1+|B|^r}{2}\right)^{-\frac{1}{r}}\right|\right| \leq 1$. Therefore

$$\left|\left|\left|\left(\frac{1 + |A|^2}{2}\right)^{\frac{1}{2}} X \left(\frac{1 + |B|^2}{2}\right)^{\frac{1}{2}}\right|\right|\right| \leq \left|\left|\left|\left(\frac{1 + |A|^r}{2}\right)^{\frac{1}{r}} X \left(\frac{1 + |B|^r}{2}\right)^{\frac{1}{r}}\right|\right|\right|,$$

which completes the proof. □

Corollary 2.2. *For normal A and B in $B(H)$ the inequality*

$$\left|\!\left|\!\left| \frac{AX+XB}{2} \right|\!\right|\!\right| \le |\!|\!| X |\!|\!|^{1-\frac{1}{r}} \left|\!\left|\!\left| \frac{|A|^r X + X|B|^r}{2} \right|\!\right|\!\right|^{\frac{1}{r}} \tag{2.12}$$

holds for all real $r \ge 2$, for all u.i. norms $|\!|\!|\cdot|\!|\!|$ and for all $X \in \mathcal{C}_{|\!|\!|\cdot|\!|\!|}$.

Proof. By Corollary 2.1, for all $t > 0$,

$$\begin{aligned} \left|\!\left|\!\left| \frac{AX+XB}{2} \right|\!\right|\!\right| &= t^{-1} \left|\!\left|\!\left| \frac{tAX+XtB}{2} \right|\!\right|\!\right| \\ &\le t^{-1} \left|\!\left|\!\left| \left(\frac{1+|tA|^r}{2}\right)^{\frac{1}{r}} X \left(\frac{1+|tB|^r}{2}\right)^{\frac{1}{r}} \right|\!\right|\!\right|, \end{aligned}$$

and therefore

$$\left|\!\left|\!\left| \frac{AX+XB}{2} \right|\!\right|\!\right| \le t^{-1} |\!|\!| X |\!|\!|^{1-\frac{2}{r}} \left|\!\left|\!\left| \left(\frac{1+|tA|^r}{2}\right)^{\frac{1}{2}} X \left(\frac{1+|tB|^r}{2}\right)^{\frac{1}{2}} \right|\!\right|\!\right|^{\frac{2}{r}},$$

by [Ki], because $\frac{2}{r} < 1$. Therefore, the arithmetic-geometric mean inequality implies

$$\begin{aligned} \left|\!\left|\!\left| \frac{AX+XB}{2} \right|\!\right|\!\right| &\le \frac{1}{2t} |\!|\!| X |\!|\!|^{1-\frac{2}{r}} \left|\!\left|\!\left| \frac{1+|tA|^r}{2} X + X \frac{1+|tB|^r}{2} \right|\!\right|\!\right|^{\frac{2}{r}} \\ &\le \frac{1}{2} |\!|\!| X |\!|\!|^{1-\frac{2}{r}} \left(t^{-\frac{r}{2}} |\!|\!| X |\!|\!| + t^{\frac{r}{2}} \left|\!\left|\!\left| \frac{|A|^r X + X|B|^r}{2} \right|\!\right|\!\right| \right)^{\frac{2}{r}}. \end{aligned} \tag{2.13}$$

As the right-hand side equals $|\!|\!| X |\!|\!|^{1-\frac{1}{r}} \left|\!\left|\!\left| \frac{|A|^r X + X|B|^r}{2} \right|\!\right|\!\right|^{\frac{1}{r}}$, which attains its minimum for $t = |\!|\!| X |\!|\!|^{\frac{1}{r}} \left|\!\left|\!\left| \frac{|A|^r X + X|B|^r}{2} \right|\!\right|\!\right|^{-\frac{1}{r}}$, the conclusion follows. □

Theorem 2.3. *For normal contractions A and B the inequality*

$$\left|\!\left|\!\left| (I - A^*A)^{\frac{1}{2}} X (I - B^*B)^{\frac{1}{2}} \right|\!\right|\!\right| \le |\!|\!| X - AXB |\!|\!|, \tag{2.14}$$

holds for all $X \in B(H)$ and for all unitarily invariant norms $|\!|\!|\cdot|\!|\!|$.

Proof. First, we note that $\text{s-}\lim_{n\longrightarrow\infty} A^n (I - A^*A)^{\frac{1}{2}} = 0$. Indeed, by a spectral theorem, for every $f \in H$ there is a positive, finite Borel measure μ concentrated on $D = \{z \in \mathbf{C} : |z| \le 1\}$ such that $\|A^n (I - A^*A)^{\frac{1}{2}} f\|^2 = \int_D |z|^{2n} (1 - |z|^2) d\mu_f(z)$, whence the desired conclusion follows by Lebesgue's dominating convergence theorem. Therefore

$$\text{w-}\lim_{n\longrightarrow\infty} (I - A^*A)^{\frac{1}{2}} (X - A^n X B^n)(I - B^*B)^{\frac{1}{2}} = (I - A^*A)^{\frac{1}{2}} X (I - B^*B)^{\frac{1}{2}}.$$

So by Theorem 2.2 we get

$$\begin{aligned}
&\left\|\!\left|(I-A^*A)^{\frac{1}{2}}X(I-B^*B)^{\frac{1}{2}}\right|\!\right\| \\
&\quad = \left\|\!\left|\lim_{n\longrightarrow\infty}(I-A^*A)^{\frac{1}{2}}(X-A^nXB^n)(I-B^*B)^{\frac{1}{2}}\right|\!\right\| \\
&\quad = \left\|\!\left|\sum_{k=0}^{\infty}(I-A^*A)^{\frac{1}{2}}A^k(X-AXB)B^k(I-B^*B)^{\frac{1}{2}}\right|\!\right\| \\
&\quad \le \left\|\!\left|\left(\sum_{k=0}^{\infty}(I-|A|^2)|A|^{2k}\right)^{\frac{1}{2}}(X-AXB)\left(\sum_{k=0}^{\infty}|B|^{2k}(I-|B|^2)\right)^{\frac{1}{2}}\right|\!\right\| \\
&\quad = |||(I-P)(X-AXB)(I-Q)||| \le |||X-AXB|||\,, \qquad (2.15)
\end{aligned}$$

where P and Q are the orthogonal projections on $Ker(I-A^*A)$ and $Ker(I-B^*B)$ respectively. This concludes the proof. □

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