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A STRUCTURAL RESULT OF IRREDUCIBLE INCLUSIONS OF TYPE III_λ FACTORS, $\lambda \in (0,1)$

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Abstract. Given an irreducible inclusion of factors with finite index $N \subset M$, where M is of type $\mathrm{III}_{\lambda^{1/m}}$, N of type $\mathrm{III}_{\lambda^{1/n}}$, $0 < \lambda < 1$, and m, n are relatively prime positive integers, we will prove that if $N \subset M$ satisfies a commuting square condition, then its structure can be characterized by using fixed point algebras and crossed products of automorphisms acting on the middle inclusion of factors associated with $N \subset M$. Relations between $N \subset M$ and a certain G-kernel on subfactors are also discussed.

1. Introduction

Let $N \overset{E_N^M}{\subset} M$ be an irreducible inclusion of type III factors such that M is of type $\mathrm{III}_{\lambda^{1/m}}$ and N of type $\mathrm{III}_{\lambda^{1/n}}$, where $\lambda \in (0,1), m, n$ are positive relatively prime integers, and $E_N^M\colon M \to N$ is a normal faithful conditional expectation with finite index. We are interested in studying the structure of such an inclusion and its relevance to the classification problem. By the results in [17], such an inclusion can be decomposed into separate sub-inclusions, each of which admits a simple description using automorphisms; more specifically, there exist type III_λ subfactors P and Q with $N \overset{E_N^Q}{\subset} Q \overset{E_Q^P}{\subset} P \overset{E_P^M}{\subset} M$ and such that $\mathrm{Ind}(E_P^M) = m$, $\mathrm{Ind}(E_N^Q) = n$. Moreover, P is the fixed point algebra of the restriction of a modular automorphism of order m on M, whereas Q is the crossed product of N with a modular automorphism of order n on N. As for $Q \subset P$, there exists a joint discrete decomposition in the sense that there exist type II_∞ factors $Q^\infty \subset P^\infty$ and an automorphism θ which acts simultaneously on $Q^\infty \subset P^\infty$ with $\mathrm{mod}(\theta) = \lambda$ and $Q \subset P$ is isomorphic to $Q^\infty \times_\theta \mathbf{Z} \subset P^\infty \times_\theta \mathbf{Z}$. The classification of these sub-inclusions is now well understood. For instance, the top and bottom inclusions $P \subset M$ and $N \subset Q$ are each uniquely determined (cf. [17]) and the middle inclusion $Q \subset P$ is classified by the type II core $Q^\infty \subset P^\infty$, the module of θ and the standard invariant of θ on the tower of higher relative commutants of $Q^\infty \subset P^\infty$ by [23].

Despite the success in classifying these separate sub-inclusions, it remains an open problem to classify the original inclusion $N \subset M$. In this paper, we continue to investigate the feasibility of characterizing the structure of a general inclusion of factors such as $N \subset M$ in terms of automorphisms. First we observe that using Takesaki duality, we can find properly outer, periodic automorphisms α and β

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with non-trivial modules, acting on P and Q respectively, so that M is the crossed product $P \times_\alpha \mathbf{Z}_m$ and N is the fixed point algebra $Q^{\mathbf{Z}_n}$ via β. However, it is not possible, in general, to have α and β act on the inclusion $Q \subset P$. In fact we will prove that this is the case if and only if the inclusions $N \subset Q \subset P \subset M$ satisfy certain commuting square conditions which are equivalent to some restriction and extension conditions of the Longo canonical endomorphism of $Q \subset P$ and this property is also equivalent to the existence of two trace-scaling automorphisms θ_1 and θ_2 on $Q^\infty \subset P^\infty$ such that: $\theta = \theta_1^m = \theta_2^n$, $M = P^\infty \times_{\theta_1} \mathbf{Z}$ and $N_1 = Q^\infty \times_{\theta_2} \mathbf{Z}$, where N_1 is the basic construction of $N \subset Q$. We can then associate to $N \subset M$ a G-kernel on $Q \subset P$, i.e., a homomorphism of the group G into $\mathrm{Aut}(P, Q)/\operatorname{Int}(Q)$, arising from the subgroup generated by α and β modulo $\operatorname{Int}(Q)$. We can then show that the isomorphism class of $N \subset M$ is determined by the conjugacy class of the corresponding G-kernel.

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2. Main results

We begin by recalling the definition of commuting squares and some basic results about them that we will need.

An inclusion of factors

$$\begin{array}{ccc} P & \subset & M \\ \cup & & \cup \\ Q & \subset & N \end{array}$$

is said to be a *commuting square* with respect to the expectations of minimal indices E_N^M, E_P^M, E_Q^N and E_Q^P if $E_N^M|P = E_Q^P$ or equivalently, if $E_P^M|N = E_Q^N$ (cf. [8]).

The commuting square

$$\begin{array}{ccc} P & \subset & M \\ \cup & & \cup \\ Q & \subset & N \end{array}$$

is called *co-commuting* (cf. [24]) or *non-degenerate* (cf. [22]) if

$$\begin{array}{ccc} N' & \subset & Q' \\ \cup & & \cup \\ M' & \subset & P' \end{array}$$

is also a commuting square with respect to the expectations with minimal indices. We refer the reader to [13, 22, 24] for additional properties of commuting and co-commuting squares. We only mention the following result in [24, 13]: the commuting square

$$\begin{array}{ccc} P & \subset & M \\ \cup & & \cup \\ Q & \subset & N \end{array}$$

is co-commuting if and only if $\operatorname{Ind} E_N^M = \operatorname{Ind} E_Q^P$ or $\operatorname{Ind} E_P^M = \operatorname{Ind} E_Q^N$.

For convenience, unless otherwise stated, E_B^A will denote the conditional expectation of minimal index from A onto B and $[A\colon B]_0$ the minimal index value.

Lemma 1. *Let*

$$\begin{array}{ccc} P & \subset & M \\ \cup & & \cup \\ Q & \subset & N \end{array}$$

be a commuting square (with respect to the expectations with minimal indices). Let $S \subset R$ *be intermediate subfactors such that* $P \subset R \subset M$, $Q \subset S \subset N$, $E_P^M = E_P^R E_R^M$ *and* $E_Q^N = E_Q^S E_S^N$. *Then*

$$\begin{array}{ccc} P & \subset & R \\ \cup & & \cup \\ Q & \subset & S \end{array}$$

is also a commuting square.

Proof. For any $s \in S, E_P^R(s) = E_P^M(s)$, which belongs to Q because

$$\begin{array}{ccc} P & \subset & M \\ \cup & & \cup \\ Q & \subset & N \end{array}$$

is a commuting square by assumption. And so by [8],

$$\begin{array}{ccc} P & \subset & R \\ \cup & & \cup \\ Q & \subset & S \end{array}$$

is a commuting square. Q.E.D.

Lemma 2. *Let*

$$\begin{array}{ccc} P & \subset & M \\ \cup & & \cup \\ Q & \subset & N \end{array}$$

be a commuting square of factors with respect to the expectations with minimal indices. Then $[M\colon N]_0 \geq [P\colon Q]_0$ *and* $[M\colon P]_0 \geq [N\colon Q]_0$.

Proof. By [21], for any $x \in P_+$, $E_N^M(x) \geq [M\colon N]_0^{-1}x$. Since $E_Q^P = E_N^M|P, E_Q^P(x) \geq [M\colon N]_0^{-1}x$. Using [21] again, we have $[P\colon Q]_0^{-1} \geq [M\colon N]_0^{-1}$ and so $[M\colon N]_0 \geq [P\colon Q]_0$. Similarly, $[M\colon P]_0 \geq [N\colon Q]_0$. Q.E.D.

In the following, all factors under consideration are of type III and thus without loss of generality, we may assume that they have a common cyclic and separating vector and are acting standardly on the same Hilbert space by the results of [5]. Let us also recall that for an inclusion of factors $N \subset M$ which act standardly on the same Hilbert space, the canonical endomorphism of $N \subset M$ is defined by $\gamma_{M,N} = \mathrm{Ad}\, J_N J_M|M$. For additional properties on the canonical endomorphism in relation to the index theory of $N \subset M$, we refer the reader to [16].

Proposition 1. *Let* $N \overset{E_N^Q}{\subset} Q \overset{E_Q^P}{\subset} P$ *be an irreducible inclusion of type* III *factors with finite index. The following are equivalent.*

(i) *There is a subfactor* N_0 *such that*

$$\begin{array}{ccc} Q & \subset & P \\ \cup & & \cup \\ N & \subset & N_0 \end{array}$$

is a commuting and co-commuting square.

(ii) *There is a choice of the canonical endomorphisms* $\gamma_{Q,N}$ *and* $\gamma_{P,Q}$ *such that* $\gamma_{Q,N}(P) \subset P$, $\gamma_{P,Q}(N) \subset N$ *and*

$$\begin{array}{ccc} \gamma_{P,Q}(Q) & \subset & Q \\ \cup & & \cup \\ \gamma_{P,Q}(N) & \subset & N \end{array}$$

is a commuting and co-commuting square.

Proof. (i) $\Rightarrow$ (ii) By [9], there is a choice of $\gamma_{Q,N}$ and $\gamma_{P,Q}$ of $Q \subset P$ such that $\gamma_{Q,N}(P) \subset P$ and $\gamma_{P,Q}(N) \subset N$. It follows that $\gamma_{P,Q}(Q)$ and $\gamma_{P,Q}(N)$ are the second factors in the downward basic constructions of $Q \subset P$ and $N \subset N_0$, respectively. Since

$$\begin{array}{ccc} Q & \subset & P \\ \cup & & \cup \\ N & \subset & N_0 \end{array}$$

is a co-commuting and commuting square, it is easy to check that

$$\begin{array}{ccc} \gamma_{P,Q}(Q) & \subset & Q \\ \cup & & \cup \\ \gamma_{P,Q}(N) & \subset & N \end{array}$$

is a commuting square which is also co-commuting because $[Q\colon N]_0 = [\gamma_{P,Q}(Q)\colon \gamma_{P,Q}(N)]_0$.

(ii) $\Rightarrow$ (i) Let J_P, J_Q and J_N be the modular conjugate operators on their respective factor such that $\gamma_{P,Q} = \mathrm{Ad}(J_Q J_P)$ and $\gamma_{Q,N} = \mathrm{Ad}(J_N J_Q)$. Since $\gamma_{P,N} = \gamma_{Q,N} \cdot \gamma_{P,Q}$, we have the following inclusions of factors:

$$\begin{array}{ccccc} \gamma_{P,Q}(Q) & \subset & \gamma_{P,Q}(P) & \subset & Q \\ \cup & & & & \cup \\ \gamma_{P,Q}(N) & \subset & & & N \\ \cup & & & & \cup \\ \gamma_{P,N}(Q) & \subset & \gamma_{P,N}(P) & \subset & \gamma_{Q,N}(Q) \\ \cup & & & & \\ \gamma_{P,N}(N) & & & & \end{array}$$

Since

$$\begin{array}{ccc} \gamma_{P,Q}(Q) & \subset & Q \\ \cup & & \cup \\ \gamma_{P,Q}(N) & \subset & N \end{array}$$

is a commuting and co-commuting square by assumption, Proposition 2.3 in [9] implies that $\gamma_{Q,N}$ restricts to a canonical endomorphism of $\gamma_{P,Q}(N) \subset \gamma_{P,Q}(Q)$ and thus $\gamma_{P,Q}(N)$ is the basic construction of $\gamma_{P,N}(N) \subset \gamma_{P,N}(Q)$. Let $\widetilde{N}$ be the von Neumann algebra generated by $\gamma_{P,Q}(N)$ and $\gamma_{P,N}(P)$. As $N' \cap P = \mathbf{C}$, $\gamma_{P,N}(P)$ is irreducible in N and hence $\widetilde{N}$ is a factor. Also since $\gamma_{Q,N}(P) \subset P$, $\gamma_{P,N}(P) \subset \gamma_{P,Q}(P)$ and so $\widetilde{N} \subset \gamma_{P,Q}(P)$ as well. By repeated applications of Takesaki's criterion in [26], we see that there exist conditional expectations on each of the following inclusions: $\gamma_{P,Q}(N) \subset \widetilde{N}$, $\gamma_{P,N}(P) \subset \widetilde{N}$, $N_0 \subset \gamma_{P,Q}(P)$ and $\widetilde{N} \subset N$. Moreover, as these expectations all have finite indices, we may just assume that they have minimal indices.

From the assumption that

$$\begin{array}{ccc} \gamma_{P,Q}(Q) & \subset & Q \\ \cup & & \cup \\ \gamma_{P,Q}(N) & \subset & N \end{array}$$

is a commuting and co-commuting square, it follows that

$$\begin{array}{ccc} \gamma_{P,Q}(N) & \subset & N \\ \cup & & \cup \\ \gamma_{P,N}(Q) & \subset & \gamma_{Q,N}(Q) \end{array}$$

is also a commuting and co-commuting square. Hence by Lemma 1,

$$\begin{array}{ccc} \gamma_{P,Q}(Q) & \subset & \gamma_{P,Q}(P) \\ \cup & & \cup \\ \gamma_{P,Q}(N) & \subset & \widetilde{N} \end{array} \quad \text{and} \quad \begin{array}{ccc} \gamma_{P,Q}(N) & \subset & \widetilde{N} \\ \cup & & \cup \\ \gamma_{P,N}(Q) & \subset & \gamma_{P,N}(P) \end{array}$$

are commuting squares. Thus by Lemma 2,

$$[P\colon Q]_0 = [\gamma_{P,Q}(P)\colon \gamma_{P,Q}(Q)]_0 \geq [\widetilde{N}\colon \gamma_{P,Q}(N)]_0 \geq [\gamma_{P,N}(P)\colon \gamma_{P,N}(Q)]_0 = [P\colon Q]_0$$

and so $[\widetilde{N}\colon \gamma_{P,Q}(N)]_0 = [P\colon Q]_0$. Therefore

$$\begin{array}{ccc} \gamma_{P,Q}(Q) & \subset & \gamma_{P,Q}(P) \\ \cup & & \cup \\ \gamma_{P,Q}(N) & \subset & \widetilde{N} \end{array}$$

is a commuting and co-commuting square by [24]. Now let $N_0 = \gamma_{P,Q}^{-1}(\widetilde{N})$; then

$$\begin{array}{ccc} Q & \subset & P \\ \cup & & \cup \\ N & \subset & N_0 \end{array}$$

is a commuting and co-commuting square. Q.E.D.

Let us also state and prove the following dual version of Proposition 1.

Proposition 2. *Let* $Q \overset{E_Q^P}{\subset} P \overset{E_P^M}{\subset} M$ *be an irreducible inclusion of type* III *factors with finite indices. The following are equivalent.*

(i) *There is a factor* M_0 *such that*

$$\begin{array}{ccc} P & \subset & M \\ \cup & & \cup \\ Q & \subset & M_0 \end{array}$$

is a commuting and co-commuting square.

(ii) *There is a choice of the canonical endomorphisms* $\gamma_{M,P}$ *and* $\gamma_{P,Q}$ *such that* $\gamma_{M,P}(Q) \subset Q$, $\gamma_{P,Q}(M) \subset M$ *and*

$$\begin{array}{ccc} \gamma_{P,Q}(M) & \subset & M \\ \cup & & \cup \\ \gamma_{P,Q}(P) & \subset & P \end{array}$$

is a commuting and co-commuting square.

Proof. (i) $\Rightarrow$ (ii) By Proposition 2.3 of [9], there exist canonical endomorphisms $\gamma_{M,P}$ and $\gamma_{P,Q}$ that satisfy the stated extension and restriction conditions. Moreover, $\gamma_{P,Q}(P)$ and $\gamma_{P,Q}(M)$ are the basic constructions of $Q \subset P$ and $M_0 \subset M$, respectively, and hence

$$\begin{array}{ccc} \gamma_{P,Q}(M) & \subset & M \\ \cup & & \cup \\ \gamma_{P,Q}(P) & \subset & P \end{array}$$

is a commuting and co-commuting square because

$$\begin{array}{ccc} P & \subset & M \\ \cup & & \cup \\ Q & \subset & M_0 \end{array}$$

is commuting and co-commuting by assumption.

(ii) $\Rightarrow$ (i) By taking the commutants of the factors, the assumptions in (ii) imply that the inclusion $M' \subset P' \subset Q'$ satisfies the hypotheses of (ii) of Proposition 1. Therefore there is a factor M_0 such that

$$\begin{array}{ccc} P' & \subset & Q' \\ \cup & & \cup \\ M' & \subset & M_0' \end{array}$$

is commuting and co-commuting. Passing to the commutants of these factors will yield the desired commuting and co-commuting square. Q.E.D.

In order to describe the structure of an inclusion of type III factors that satisfies both the commuting and co-commuting square conditions of Propositions 1 and 2, we need the information about the Connes-Takesaki modules (cf. [4]) of the associated automorphisms provided by the next lemma.

Lemma 3. (i) *Let P be a factor of type* III_λ*,* $\lambda \in (0,1)$*, and* $m \in \mathbf{N}$*. Let* φ *be a generalized trace on* P *and* $T = [2\pi/\ln\lambda|$*. Then* $P^{\sigma^\varphi_{T/m}}$ *and* $P \times_{\sigma^\varphi_{T/m}} \mathbf{Z_m}$ *are both of type* III_{λ^m} *and if* α *is either the dual or the pre-dual automorphism of* $\sigma^\varphi_{T/m}$ *on* $P \times_{\sigma^\varphi_T} \mathbf{Z_m}$ *or* $P^{\sigma^\varphi_T}$*, then* $\mathrm{mod}(\alpha) \equiv \lambda$*.*

(ii) *Let* $\lambda \in (0,1)$ *and* $Q \overset{E}{\subset} P$ *be an inclusion of type* III_λ *factors with a common discrete decomposition. Then for any* $\alpha \in \mathrm{Aut}(P,Q)$ *which commutes with* E*,* $\mathrm{mod}(\alpha|P) = \mathrm{mod}(\alpha|Q)$*.*

Proof. (i) We will only prove the statement about the module of α as it is well known that $P^{\sigma^\varphi_{T/m}}$ and $P \times_{\sigma^\varphi_{T/m}} \mathbf{Z_m}$ are both of type III_{λ^m}. Suppose that α is the automorphism that is pre-dual to $\sigma^\varphi_{T/m}$ on $P^{\sigma^\varphi_{T/m}}$. Then $P = P^{\sigma^\varphi_{T/m}} \times_\alpha \mathbf{Z_m}$ and $\sigma^\varphi_{T/m}(U) = e^{-2\pi i/m}U$, where U is the unitary in P that implements α. On the other hand, it follows from [17] that the pair $P^{\sigma^\varphi_T} \subset P$ is isomorphic to $P^\infty \times_{\theta^m} \mathbf{Z} \subset P^\infty \times_\theta \mathbf{Z}$, where $\{P^\infty, \theta\}$ is a discrete decomposition of P. Let V be the implementing unitary of θ; then as φ is a generalized trace on P (cf. [3]), $\sigma^\varphi_t(V) = \lambda^{it}V$, for all $t \in \mathbf{R}$. In particular, $\sigma^\varphi_{T/m}(V) = \lambda^{iT/m}V = e^{-2\pi i/m}V$. Hence $W = UV^* \in P^{\sigma^\varphi_{T/m}}$ and $\theta(x) = \mathrm{Ad}\, W^* \cdot \alpha(x)$ for every $x \in P^\varphi$. Since $\mathrm{mod}(\theta) = \lambda$, we deduce that $\mathrm{mod}(\alpha) \equiv \lambda$ (cf. [4]).

If α is the dual automorphism to $\sigma^\varphi_{T/m}$, then using the just established result for the predual of $\sigma^\varphi_{T/m}$ and the Takesaki Duality Theorem, we infer that $\mathrm{mod}(\alpha) \equiv \lambda$.

(ii) Let φ be a normal faithful semi-finite weight on Q. Then $\varphi\cdot\alpha \sim \mu^{-1}\varphi$. Since α and E_Q^P commute, $\varphi \cdot E_Q^P \cdot \alpha \sim \mu^{-1}\varphi \cdot E_Q^P$ as well, i.e., $\mathrm{mod}(\alpha) = \mathrm{mod}(\alpha|Q)$. Q.E.D.

We can now prove the following characterization based on automorphisms of an inclusion of the form $N \subset Q \subset P \subset M$ that satisfies the commuting and co-commuting square condition.

Theorem 1. *Let $\lambda \in (0,1)$, and $m, n \in \mathbf{N}$ be relatively prime. Suppose that $N \overset{E_N^Q}{\subset} Q \overset{E_Q^P}{\subset} P \overset{E_P^M}{\subset} M$ is an irreducible inclusion of factors such that Q and P are both of type III_λ, M of type $\mathrm{III}_{\lambda^{1/m}}$ and N of type $\mathrm{III}_{\lambda^{1/n}}$, $[M\colon P]_0 = m$ and $[Q\colon N]_0 = n$. The following are equivalent.*

(i) *There exist canonical endomorphisms $\gamma_{Q,N}$, $\gamma_{P,Q}$ and $\gamma_{M,P}$ which satisfy: $\gamma_{P,Q}(N) \subset N$, $\gamma_{P,Q}(M) \subset M$, $\gamma_{M,P}(Q) \subset Q$, $\gamma_{Q,N}(N) \subset N$ and such that*

$$\begin{array}{ccc} \gamma_{P,Q}(Q) & \subset & Q \\ \cup & & \cup \\ \gamma_{P,Q}(N) & \subset & N \end{array} \quad \text{and} \quad \begin{array}{ccc} \gamma_{P,Q}(M) & \subset & M \\ \cup & & \cup \\ \gamma_{P,Q}(P) & \subset & P \end{array}$$

are commuting and co-commuting squares.

(ii) *There exist properly outer and periodic automorphisms α, β acting on $Q \subset P$ such that α has order m, β has order n, $\mathrm{mod}(\alpha) \equiv \lambda^{1/m}$, $\mathrm{mod}(\beta) \equiv \lambda^{1/n}$ and such that $N = Q^\beta$ and $M = P \times_\alpha \mathbf{Z}_m$.*

(iii) *Let $\{Q^\infty \subset P^\infty, \theta\}$ be a common discrete decomposition of $Q \subset P$. Then θ is both an m and an n power, i.e., there exist trace-scaling automorphisms θ_1 and θ_2 on $Q^\infty \subset P^\infty$ such that $\theta = \theta_1^m = \theta_2^n$, $M = P^\infty \times_{\theta_1} \mathbf{Z}$ and $N_1 = Q^\infty \times_{\theta_2} \mathbf{Z}$, where N_1 is the basic construction of $N \subset Q$.*

Proof. The equivalence between (ii) and (iii) was established in [19] and so we only need to prove that (i) and (ii) are equivalent.

(ii) ⇒ (i) Assuming that α, β exist, then it is easy to check that

$$\begin{array}{ccc} P & \subset & P \times_\alpha \mathbf{Z}_m = M \\ \cup & & \cup \\ Q & \subset & Q \times_\alpha \mathbf{Z}_m \end{array} \quad \text{and} \quad \begin{array}{ccc} P^\beta & \subset & P \\ \cup & & \cup \\ N = Q^\beta & \subset & Q \end{array}$$

are commuting co-commuting squares and (ii) follows from Propositions 1 and 2.

(ii) ⇒ (i) By Propositions 1 and 2, there exist factors M_0 and N_0 such that

$$\begin{array}{ccc} P & \subset & M \\ \cup & & \cup \\ Q & \subset & M_0 \end{array} \quad \text{and} \quad \begin{array}{ccc} N_0 & \subset & P \\ \cup & & \cup \\ N & \subset & Q \end{array}$$

are commuting and co-commuting squares.

First let us consider the diagram

$$\begin{array}{ccc} P & \subset & M \\ \cup & & \cup \\ Q & \subset & M_0. \end{array}$$

Let ψ be a generalized trace on Q so that $\sigma_T^\psi = \mathrm{Id}$, where $T = |2\pi/\ln\lambda|$. Then

$\sigma_T^{\psi\cdot E_Q^P} = \mathrm{Id}$ on P and, as proved in [17], $P = M^{\sigma_T^{\psi\cdot E_Q^M}}$. It follows that $Q \subset M_0^{\sigma_T^{\psi\cdot E_Q^{M_0}}} \subset M_0$. But because

$$\begin{array}{ccc} P = M^{\sigma^{\psi\cdot E_Q^M}} & \subset & M \\ \cup & & \cup \\ M_0^{\sigma_T^{\psi\cdot E_Q^{M_0}}} & \subset & M_0 \end{array}$$

is a commuting and co-commuting square, the subfactor $M_0^{\sigma_T^{\psi\cdot E_Q^{M_0}}}$ has index value m in M_0, hence $Q = M_0^{\sigma_T^{\psi\cdot E_Q^{M_0}}}$. Now if α is the predual automorphism of $\sigma_T^{\psi\cdot E_Q^M}$ on Q, then $M_0 = Q \times_\alpha \mathbf{Z}_m$ and similarly $M = P \times_\alpha \mathbf{Z}_m$. By Lemma 3, we have $\mathrm{mod}(\alpha) = \mathrm{mod}(\alpha|Q) \equiv \lambda^{1/m}$.

As for the commuting and co-commuting square

$$\begin{array}{ccc} N_0 & \subset & P \\ \cup & & \cup \\ N & \subset & Q, \end{array}$$

since N is of type $\mathrm{III}_{\lambda^{1/n}}$, there exists a generalized trace φ on N such that $\sigma_{nT}^{\varphi} = \mathrm{Id}$ on N, where T is as before. Now $\sigma_T^{\varphi\cdot E_N^Q}$ is inner on Q, say $\sigma_T^{\varphi\cdot E_N^Q} = \mathrm{Ad}\, u$ for some unitary u in Q; then we may assume that $u^n = 1$ as $N' \cap Q = \mathbf{C}$ and hence $\sigma_{nT}^{\varphi\cdot E_N^Q} = \mathrm{Id}$ on N. As $Q' \cap P = \mathbf{C}$ and P is of type III_λ, we also have $\sigma_T^{\varphi\cdot E_N^Q\cdot E_Q^P} = \mathrm{Ad}\, u$ on P and because $E_N^Q \cdot E_Q^P E_N^{N_0} \cdot E_{N_0}^P$, $\sigma_T^{\varphi\cdot E_N^{N_0}\cdot E_{N_0}^P} = \mathrm{Ad}\, u$ on P as well. Hence for $x \in N_0$, $\sigma_T^{\varphi\cdot E_N^{N_0}}(x) = \sigma_T^{\varphi\cdot E_N^{N_0}\cdot E_{N_0}^P}(x) = uxu^*$ so that $uN_0u^* = N_0$ and $\mathrm{Ad}\, u$ defines a properly outer action of $\mathbf{Z}_n$ on N_0. Moreover, $E_{N_0}^P(u^j) = E_N^Q(u^j) = 0$ for $0 \le j \le n-1$. Thus $\{N_0, u\}''$ is the crossed product of N_0 by $\sigma_T^{\varphi\cdot E_N^{N_0}}$ and so it contains P as a subfactor with index value n. Thus $\{N_0, u\}'' = P$. Similarly, Q is the crossed product N by σ_T^{φ}. Now if we let β be the dual action of $\sigma_T^{\varphi\cdot E_N^{N_0}}$, then $N = Q^\beta$ and $N_0 = P^\beta$. By Lemma 3, $\mathrm{mod}(\beta) = \mathrm{mod}(\beta|_Q) \equiv \lambda^{1/n}$. Q.E.D.

Remarks. 1) In [2] inclusions of the form $R^H \subset R \times K$, where H and K are finite groups of outer automorphisms on a type II_1 factor R, are studied and thus inclusions satisfying any one of the equivalent conditions of Theorem 1 may be viewed as a subfactor analogue of these group-like inclusions.

2) In view of condition (ii) in Theorem 1, we can define a G-kernel on $Q \subset P$, i.e., a homomorphism of G into $\mathrm{Aut}(P,Q)/\mathrm{Int}(Q)$, where G is the group generated by α and β in $\mathrm{Aut}(P,Q)/\mathrm{Int}(Q)$ and the homomorphism is given by the quotient map. Such a G-kernel may be viewed as a subfactor analogue to those for single factors that were studied in [11, 20, 25].

3) Let $N \overset{E_N^Q}{\subset} Q \overset{E_Q^P}{\subset} P \overset{E_P^M}{\subset} M$ be an irreducible inclusion of factors that satisfy the hypotheses of the equivalent conditions in Theorem 1. It is easy to see that the isomorphism class of $N \subset M$ determines the isomorphism class of the associated

commuting and co-commuting square

$$\begin{array}{ccccc} N_0 & \subset & P & \subset & M \\ \cup & & \cup & & \cup \\ N & \subset & Q & \subset & M_0 \end{array}$$

and vice versa. Indeed, let φ be a generalized trace on N, put $\psi = \varphi \cdot E_N^M$ and let $T = |2\pi/\ln\lambda|$; then by [17] $Q = N \times_{\sigma_T^\varphi} \mathbf{Z_n}$ and $P = M^{\sigma_{nT}^\psi}$. By the uniqueness of the generalized trace φ, we see that $Q \subset P$ is invariant under isomorphisms of $N \subset M$. Using the spatial uniqueness of the standard form as proved in [10], the extension and restriction properties of the canonical endomorphisms: $\gamma_{Q,N}$, $\gamma_{P,Q}$ and $\gamma_{M,P}$ and the commuting and co-commuting square conditions of

$$\begin{array}{ccc} \gamma_{P,Q}(Q) & \subset & Q \\ \cup & & \cup \\ \gamma_{P,Q}(N) & \subset & N \end{array} \quad \text{and} \quad \begin{array}{ccc} \gamma_{P,Q}(M) & \subset & M \\ \cup & & \cup \\ \gamma_{P,Q}(P) & \subset & P \end{array}$$

are also preserved under isomorphisms of $N \subset M$. Finally, from the proof of Propositions 1 and 2, the constructions of the subfactors M_0 and N_0 are also preserved under isomorphisms of $N \subset M$.

We now turn to the study of inclusions that satisfy the commuting square conditions explained above by means of the discrete decomposition. We recall the following result proved in [19].

Proposition 3. *Let $\lambda \in (0,1)$ and $m \in \mathbf{N}$. Suppose that*

$$\begin{array}{ccc} P & \subset & M \\ \cup & & \cup \\ Q & \subset & M_0 \end{array}$$

is a commuting square of factors with finite indices such that: $Q \subset P$ are both of type III_λ and $M_0 \subset M$ are of type $\mathrm{III}_{\lambda^{1/m}}$, $[M_0 : Q]_0 = [M : P]_0 = m$. Then there exist type II_∞ factors $Q^\infty \subset P^\infty$ and a trace-scaling automorphism $\theta \in \mathrm{Aut}(P^\infty, Q^\infty)$ such that $\mathrm{mod}(\theta) = \lambda$ and

$$\begin{array}{ccc} P & \subset & M \\ \cup & & \cup \\ Q & \subset & M_0 \end{array}$$

is isomorphic to

$$\begin{array}{ccc} P^\infty \times_{\theta^m} \mathbf{Z} & \subset & P^\infty \times_\theta \mathbf{Z} \\ \cup & & \cup \\ Q^\infty \times_{\theta^m} \mathbf{Z} & \subset & Q^\infty \times_\theta \mathbf{Z}. \end{array}$$

Using the classification result in [23] for trace-scaling automorphisms on strongly amenable type II_∞ inclusions, we obtain the following classification result of inclusions of type III factors satisfying the hypotheses of Proposition 1.

Corollary 1. *Let*

$$\begin{array}{ccc} P & \subset & M \\ \cup & & \cup \\ Q & \subset & M_0 \end{array}$$

be as in Proposition 1 and assume further that $Q \subset P$ is strongly amenable, i.e., its type II *core is strongly amenable in the sense defined in* [22]. *Then the commuting* *square is classified by its type* II *core* $Q^\infty \subset P^\infty$ *and the standard invariant of* θ *on* $Q^\infty \subset P^\infty$.

Similarly, let

$$\begin{array}{ccc} N_0 & \subset & P \\ \cup & & \cup \\ N & \subset & Q \end{array}$$

be a commuting square of factors with finite indices such that: $Q \subset P$ are both of type III_λ and $N \subset N_0$ are of type $\mathrm{III}_{\lambda^{1/n}}$, $[N_0\colon N]_0 = [P\colon Q]_0 = n$, and let M_0 be the basic construction of $N \subset Q$ and M the basic construction of $N_0 \subset P$. We then obtain a commuting square satisfying the same assumptions as in Proposition 1 and thus

$$\begin{array}{ccc} N_0 & \subset & P \\ \cup & & \cup \\ N & \subset & Q \end{array}$$

is also classified by the type II invariants associated with $Q^\infty \subset P^\infty$ as in Corollary 1 when $Q \subset P$ is strongly amenable.

As an application of Theorem 1, we are going to show that the middle inclusion $Q \subset P$ and the associated G-kernel can be used to study the structure of $N \subset M$. The following proposition is an easy extension to the subfactor case of the results proved in [2] for group-like inclusions.

Proposition 4. *Let* $N \overset{E_N^Q}{\subset} Q \overset{E_Q^P}{\subset} P \overset{E_P^M}{\subset} M$ *be an irreducible inclusion of type* III *factors satisfying the hypotheses of the equivalent conditions in Theorem 1. Then* $N \subset M$ *has finite depth if and only if* $Q \subset P$ *has finite depth and* G *is finite.*

Proof. Suppose that $N \subset M$ has finite depth. Then by [1], $Q \subset P$ and $P^\beta \subset P \times_\alpha \mathbf{Z_m}$ both have finite depth, and so $\langle\alpha,\beta\rangle/\langle\alpha,\beta\rangle \cap \mathrm{Int}(P)$ is finite by [2]. On the other hand, as $Q \subset P$ has finite index, $\langle\alpha,\beta\rangle \cap \mathrm{Int}(P)/\langle\alpha,\beta\rangle \cap \mathrm{Int}(Q)$ is finite by [21] and hence $G = \langle\alpha,\beta\rangle/\langle\alpha,\beta\rangle \cap \mathrm{Int}(Q)$ is also finite

Conversely, suppose that $Q \subset P$ has finite depth and G is finite. Then $N = Q^\beta \subset Q \times_\alpha \mathbf{Z_m}$ and $P^\beta \subset M = P \times_\alpha \mathbf{Z_m}$ both have finite depth by [2]. Since

$$\begin{array}{ccc} P^\beta & \subset & P \\ \cup & & \cup \\ N = Q^\beta & \subset & Q \end{array} \quad \text{and} \quad \begin{array}{ccc} P & \subset & P \times_\alpha \mathbf{Z_m} \\ \cup & & \cup \\ Q & \subset & Q \times_\alpha \mathbf{Z_m} \end{array}$$

are commuting squares,

$$\begin{array}{ccc} P^\beta & \subset & M \\ \cup & & \cup \\ N = Q^\beta & \subset & Q \times_\alpha \mathbf{Z_m} \end{array}$$

is also a commuting square, and hence $N \subset M$ has finite depth by [27]. Q.E.D.

Let G be a countable discrete group and let Θ_1 and Θ_2 be two G-kernels on an inclusion of factors $Q \subset P$. As in the single factor case that was studied in [11, 25] we say that Θ_1 and Θ_2 are *conjugate* if there exists $\Phi \in \mathrm{Aut}(P,Q)/\mathrm{Int}(Q)$ such that $\Phi \cdot \Theta_1 \cdot \Phi^{-1} = \Theta_2$. According to [22] an inclusion of factors of the form $R^\beta \subset R \times_\alpha \mathbf{Z_m}$, where α and β are outer automorphisms with order m and n, respectively, on a type II_1 factor R, is classified by the conjugacy class of the G-kernel coming from $\langle\alpha,\beta\rangle/\mathrm{Int}(R)$. It is thus not surprising that a similar result holds for an inclusion of AFD factors that satisfies any one of the equivalent conditions in Theorem 1.

Theorem 2. *Let* $N \overset{E_N^Q}{\subset} Q \overset{E_Q^P}{\subset} P \overset{E_P^M}{\subset} M$ *be an irreducible inclusion of type* III *factors that satisfies any one of the equivalent conditions in Theorem 1. Let* α *and* β *be the associated automorphisms and* $\{G, \Theta\}$ *be the kernel on* $Q \subset P$ *arising from the subgroup generated by* α *and* β *modulo* $\mathrm{Int}(Q)$*. Then the isomorphism class of* $N \subset M$ *is determined by the conjugacy class of* G*.*

Proof. Let $N_1 \subset M_1$ be another irreducible inclusion satisfying the same properties as $N \subset M$. Let $Q_1 \subset P_1, \alpha_1, \beta_1$ and Θ_1 be defined accordingly from $N_1 \subset M_1$.

Suppose first that Φ is an isomorphism of $N \subset M$ onto $N_1 \subset M_1$. Then as noted in the Remarks after Theorem 1 above, Φ actually maps $N \subset Q \subset P \subset M$ onto $N_1 \subset Q_1 \subset P_1 \subset M_1$ and so we can identify all the respective factors: $N = N_1$, $Q = Q_1$, $P = P_1$, $M = M_1$. As a result we see that Φ can be extended to an isomorphism mapping

$$\begin{matrix} P & \subset & P \times_\alpha \mathbf{Z_m} \\ \cup & & \cup \\ Q & \subset & Q \times_\alpha \mathbf{Z_m} \end{matrix} \quad \text{and} \quad \begin{matrix} P & \subset & P \times_\beta \mathbf{Z_n} \\ \cup & & \cup \\ Q & \subset & Q \times_\beta \mathbf{Z_n} \end{matrix}$$

onto

$$\begin{matrix} P & \subset & P \times_{\alpha_1} \mathbf{Z_m} \\ \cup & & \cup \\ Q & \subset & Q \times_{\alpha_1} \mathbf{Z_m} \end{matrix} \quad \text{and} \quad \begin{matrix} P & \subset & P \times_{\beta_1} \mathbf{Z_n} \\ \cup & & \cup \\ Q & \subset & Q \times_{\beta_1} \mathbf{Z_n}. \end{matrix}$$

It is then straightforward to prove that there exist unitaries u and v in Q such that $\Phi \cdot \alpha \cdot \Phi^{-1} = \mathrm{Ad}\, u \cdot \alpha_1$ and $\Phi \cdot \beta \cdot \Phi^{-1} = \mathrm{Ad}\, v \cdot \beta_1$. Hence Θ and Θ_1 are conjugate.

Conversely, suppose that Θ and Θ_1 are two G-kernels on $Q \subset P$ that are conjugate via an isomorphism Φ of $Q \subset P$ onto $Q_1 \subset P_1$. Then there exist unitaries u and v in Q_1 such that $\Phi \cdot \alpha \cdot \Phi^{-1} = \mathrm{Ad}\, u \cdot \alpha_1$ and $\Phi \cdot \beta \cdot \Phi^{-1} = \mathrm{Ad}\, v \cdot \beta_1^l$, where l and n are relatively prime. Then by standard arguments, Φ can be extended to isomorphisms between

$$\begin{matrix} P & \subset & P \times_\alpha \mathbf{Z_m} \\ \cup & & \cup \\ Q & \subset & Q \times_\alpha \mathbf{Z_m} \end{matrix} \quad \text{and} \quad \begin{matrix} P & \subset & P \times_\beta \mathbf{Z_n} \\ \cup & & \cup \\ Q & \subset & Q \times_\beta \mathbf{Z_n} \end{matrix}$$

and

$$\begin{matrix} P_1 & \subset & P_1 \times_{\alpha_1} \mathbf{Z_m} \\ \cup & & \cup \\ Q_1 & \subset & Q_1 \times_{\alpha_1} \mathbf{Z_m} \end{matrix} \quad \text{and} \quad \begin{matrix} P_1 & \subset & P_1 \times_{\beta_1} \mathbf{Z_n} \\ \cup & & \cup \\ Q_1 & \subset & Q_1 \times_{\beta_1} \mathbf{Z_n}. \end{matrix}$$

As $\Phi(N)$ is the downward construction of $Q_1 \subset Q_1 \times_{\beta_1} \mathbf{Z_n}$, we can find a unitary w in Q_1 such that $\mathrm{Ad}\, w \cdot \Phi(N) = N_1$ and therefore $\mathrm{Ad}\, w \cdot \Phi$ is an isomorphism between $N \subset M$ and $N_1 \subset M_1$. Q.E.D.

Recall that if R_0 is the hyperfinite type II$_1$ factor, then by [11] in the finite case, and by [20] in the amenable case, G-kernels of R_0 are classified up to conjugacy by their obstructions as defined in [11, 25], which are elements of $H^3(G, \mathbf{T})$: the third cohomology group of G with coefficients in the unit circle $\mathbf{T}$. It would thus be an interesting problem to classify G-kernels on subfactors by their cohomological invariants, their standard invariants (cf. [18]) and, in the case of type III$_\lambda$ factors, their Connes-Takesaki modules.

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