

THE FUGLEDE-PUTNAM THEOREM AND A GENERALIZATION OF BARRÍA'S LEMMA

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ABSTRACT. Let A and B be bounded linear operators, and let C be a partial isometry on a Hilbert space. Suppose that (1) $CA = BC$, (2) $\|A\| \geq \|B\|$, (3) $(C^*C)A = A(C^*C)$ and (4) $C(\|A\|^2 - AA^*)^{1/2} = 0$. Then we have $CA^* = B^*C$.

Let \mathcal{H} be a complex Hilbert space. An operator means a bounded linear operator on \mathcal{H} . The familiar Fuglede-Putnam theorem is stated as follows:

Theorem A (Fuglede-Putnam [3, Theorem IX.6.7]). *If A and B are normal operators on \mathcal{H} and C is an operator such that $CA = BC$, then $CA^* = B^*C$.*

Several authors have relaxed the normality hypothesis on A and B in Theorem A in various ways (for example, to hyponormality), still without restrictions on C , and have reached the same conclusion. However, it appears that few have attempted to place conditions on the operator C in order to remove the normality hypotheses on A and B . In this note we wish to generalize the following lemma of Barría from this point of view.

Lemma B (Barría [1, Lemma 2]). *Assume that $V_1^*V_2 = V_2V_1^*$, where V_1 and V_2 are isometries. Then $V_1V_2 = V_2V_1$.*

Now we state our result. The proof is elementary, depending on partially isometric extensions of contractions.

Theorem. *Let A and B be bounded linear operators, and let C be a partial isometry. Suppose that*

- (1) $CA = BC$,
- (2) $\|A\| \geq \|B\|$,
- (3) $(C^*C)A = A(C^*C)$ and
- (4) $C(\|A\|^2 - AA^*)^{1/2} = 0$.

Then we have $CA^ = B^*C$.*

Proof. We assume first that A, B and C are partial isometries. Then condition (4) becomes $C^*C \leq AA^*$. In particular, C^*C and AA^* commute. It follows from [6, Lemma 2] that $CA = BC$ is a partial isometry. Therefore, B^*B and CC^* commute

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by [6, Lemma 2] again. Then we have

$$\begin{aligned} B^*C &= B^*CAA^* = B^*BCA^* = B^*BCC^*CA^* \\ &= CC^*B^*BCA^* = C(BC)^*(BC)A^* \\ &= C(CA)^*(CA)A^* \\ &= CA^*C^*CAA^* = CC^*CA^*AA^* = CA^*. \end{aligned}$$

Thus the theorem is true for partial isometries A, B and C .

Now, let A, B and C satisfy the hypotheses of the theorem. Dividing by $\|A\|$, we may assume that $\|A\| = 1$ and $\|B\| \leq 1$.

We define operator matrices \tilde{A}, \tilde{B} and \tilde{C} by

$$\tilde{A} = \begin{bmatrix} A & (1 - AA^*)^{1/2} \\ 0 & 0 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B & (1 - BB^*)^{1/2} \\ 0 & 0 \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix}.$$

Then \tilde{A}, \tilde{B} and \tilde{C} are partial isometries on $\mathcal{H} \oplus \mathcal{H}$ and satisfy

$$\begin{aligned} \tilde{C}^*\tilde{C} &= \begin{bmatrix} C^*C & 0 \\ 0 & 0 \end{bmatrix} \leq \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} = \tilde{A}\tilde{A}^*, \\ (\tilde{C}^*\tilde{C})\tilde{A} &= \begin{bmatrix} (C^*C)A & (C^*C)(1 - AA^*)^{1/2} \\ 0 & 0 \end{bmatrix} = \tilde{A}(\tilde{C}^*\tilde{C}), \end{aligned}$$

and

$$\tilde{C}\tilde{A} = \begin{bmatrix} CA & C(1 - AA^*)^{1/2} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} BC & 0 \\ 0 & 0 \end{bmatrix} = \tilde{B}\tilde{C}.$$

Therefore, by the first paragraph of the proof, we have $\tilde{C}\tilde{A}^* = \tilde{B}^*\tilde{C}$. This implies that $CA^* = B^*C$.

Now we present an example which is not covered by the Fuglede-Putnam theorem or an existing generalization of it.

Example. Let

$$A = \begin{bmatrix} 0 & 0 & a_1 & & & & \\ & 0 & 0 & a & & & 0 \\ & & 0 & 0 & a_2 & & \\ & & & 0 & 0 & a & \\ 0 & & & \ddots & \ddots & \ddots & \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & a & & & & 0 \\ & 0 & 0 & b_1 & & & \\ & & 0 & 0 & a & & \\ & & & 0 & 0 & b_2 & \\ 0 & & & \ddots & \ddots & \ddots & \end{bmatrix}$$

and

$$C = \begin{bmatrix} 0 & 1 & 0 & & & & 0 \\ & 0 & 0 & 0 & & & \\ & & 0 & 1 & 0 & & \\ & & & 0 & 0 & 0 & \\ 0 & & & \ddots & \ddots & \ddots & \end{bmatrix},$$

where $\{a_n\}$ and $\{b_n\}$ are bounded sequences of complex numbers such that $a = \sup_n |a_n| \geq \sup_n |b_n|$.

Then B is not M -hyponormal [7], indeed, in general it is not even a dominant [9] or \mathcal{Y} -class operator. (We point out that an operator T is said to be M -hyponormal if there exists a constant $M \geq 1$ such that $(T - \lambda)(T - \lambda)^* \leq M^2(T - \lambda)^*(T - \lambda)$ for any

complex number λ . An operator T is said to be dominant if for any complex number λ there exists a number $M_\lambda \geq 1$ such that $(T - \lambda)(T - \lambda)^* \leq M_\lambda^2(T - \lambda)^*(T - \lambda)$.

For $\alpha > 0$ an operator T is said to be in \mathcal{Y}_α if there exists a number $M_\alpha \geq 1$ such that $|T^*T - TT^*|^\alpha \leq M_\alpha^2(T - \lambda)^*(T - \lambda)$ for any complex number λ . An operator T is said to be in \mathcal{Y} (or to be of \mathcal{Y} -class) if T is in \mathcal{Y}_α for some $\alpha \geq 1$.)

Therefore, neither the Fuglede-Putnam theorem nor any existing generalizations apply. However, since A, B and C satisfy the hypotheses of our theorem, we can conclude that $CA^* = B^*C$.

Remark 1. We cannot merely drop condition (3) in the Theorem. For example, let U be the unilateral shift. Take $A = B = U^*$ and $C = (U^*)^2$. Then A, B and C satisfy (1), (2) and (4), but $CA^* \neq B^*C$ immediately.

Remark 2. We cannot merely drop condition (2) in the Theorem. For example, put

$$A = \begin{bmatrix} 0 & 0 & 1 & & & & \\ & 0 & 0 & 1 & & & \\ & & 0 & 0 & 1 & & \\ & & & 0 & 0 & 1 & \\ 0 & & & & \ddots & \ddots & \ddots \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 1 & & & & \\ & 0 & 0 & 1 & & & \\ & & 0 & 0 & 1 & & \\ & & & 0 & 0 & 1 & \\ 0 & & & & \ddots & \ddots & \ddots \end{bmatrix}$$

and

$$C = \begin{bmatrix} 0 & 1 & 0 & & & & \\ & 0 & 0 & 0 & & & \\ & & 0 & 1 & 0 & & \\ & & & 0 & 0 & 0 & \\ 0 & & & & \ddots & \ddots & \ddots \end{bmatrix}.$$

Then A, B and C satisfy (1), (3) and (4), but $CA^* \neq B^*C$.

Remark 3. We cannot merely drop condition (4) in the Theorem. For example, put

$$A = \begin{bmatrix} 0 & 0 & 2 & & & & \\ & 0 & 0 & 1 & & & \\ & & 0 & 0 & 1 & & \\ & & & 0 & 0 & 1 & \\ 0 & & & & \ddots & \ddots & \ddots \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 1 & & & & \\ 0 & 0 & 0 & 1 & & & \\ & 1 & 0 & 0 & 1 & & \\ & & 0 & 0 & 0 & 1 & \\ & & & 0 & 0 & 0 & 1 \\ 0 & & & & \ddots & \ddots & \ddots \end{bmatrix}$$

and

$$C = \begin{bmatrix} 0 & 1 & 0 & & & & \\ & 0 & 0 & 0 & & & \\ & & 0 & 1 & 0 & & \\ & & & 0 & 0 & 0 & \\ 0 & & & & \ddots & \ddots & \ddots \end{bmatrix}.$$

Then A, B and C satisfy (1), (2) and (3), but $CA^* \neq B^*C$.

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