# THE FUGLEDE-PUTNAM THEOREM AND A GENERALIZATION OF BARRÍA'S LEMMA 

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#### Abstract

Let $A$ and $B$ be bounded linear operators, and let $C$ be a partial isometry on a Hilbert space. Suppose that (1) $C A=B C$, (2) $\|A\| \geq\|B\|$, (3) $\left(C^{*} C\right) A=A\left(C^{*} C\right)$ and (4) $C\left(\|A\|^{2}-A A^{*}\right)^{1 / 2}=0$. Then we have $C A^{*}=$ $B^{*} C$.


Let $\mathcal{H}$ be a complex Hilbert space. An operator means a bounded linear operator on $\mathcal{H}$. The familiar Fuglede-Putnam theorem is stated as follows:

Theorem A (Fuglede-Putnam [3, Theorem IX.6.7]). If $A$ and $B$ are normal operators on $\mathcal{H}$ and $C$ is an operator such that $C A=B C$, then $C A^{*}=B^{*} C$.

Several authors have relaxed the normality hypothesis on $A$ and $B$ in Theorem A in various ways (for example, to hyponormality), still without restrictions on $C$, and have reached the same conclusion. However, it appears that few have attempted to place conditions on the operator $C$ in order to remove the normality hypotheses on $A$ and $B$. In this note we wish to generalize the following lemma of Barría from this point of view.

Lemma B (Barría [1, Lemma 2]). Assume that $V_{1}^{*} V_{2}=V_{2} V_{1}^{*}$, where $V_{1}$ and $V_{2}$ are isometries. Then $V_{1} V_{2}=V_{2} V_{1}$.

Now we state our result. The proof is elementary, depending on partially isometric extensions of contractions.

Theorem. Let $A$ and $B$ be bounded linear operators, and let $C$ be a partial isometry. Suppose that
(1) $C A=B C$,
(2) $\|A\| \geq\|B\|$,
(3) $\left(C^{*} C\right) A=A\left(C^{*} C\right)$ and
(4) $C\left(\|A\|^{2}-A A^{*}\right)^{1 / 2}=0$.

Then we have $C A^{*}=B^{*} C$.
Proof. We assume first that $A, B$ and $C$ are partial isometries. Then condition (4) becomes $C^{*} C \leq A A^{*}$. In particular, $C^{*} C$ and $A A^{*}$ commute. It follows from [6, Lemma 2] that $C A=B C$ is a partial isometry. Therefore, $B^{*} B$ and $C C^{*}$ commute

[^0]by [6, Lemma 2] again. Then we have
\[

$$
\begin{aligned}
B^{*} C & =B^{*} C A A^{*}=B^{*} B C A^{*}=B^{*} B C C^{*} C A^{*} \\
& =C C^{*} B^{*} B C A^{*}=C(B C)^{*}(B C) A^{*} \\
& =C(C A)^{*}(C A) A^{*} \\
& =C A^{*} C^{*} C A A^{*}=C C^{*} C A^{*} A A^{*}=C A^{*}
\end{aligned}
$$
\]

Thus the theorem is true for partial isometries $A, B$ and $C$.
Now, let $A, B$ and $C$ satisfy the hypotheses of the theorem. Dividing by $\|A\|$, we may assume that $\|A\|=1$ and $\|B\| \leq 1$.

We define operator matrices $\widetilde{A}, \widetilde{B}$ and $\widetilde{C}$ by

$$
\widetilde{A}=\left[\begin{array}{cc}
A & \left(1-A A^{*}\right)^{1 / 2} \\
0 & 0
\end{array}\right], \quad \widetilde{B}=\left[\begin{array}{cc}
B & \left(1-B B^{*}\right)^{1 / 2} \\
0 & 0
\end{array}\right], \quad \widetilde{C}=\left[\begin{array}{cc}
C & 0 \\
0 & 0
\end{array}\right]
$$

Then $\widetilde{A}, \widetilde{B}$ and $\widetilde{C}$ are partial isometries on $\mathcal{H} \oplus \mathcal{H}$ and satisfy

$$
\begin{gathered}
\widetilde{C}^{*} \widetilde{C}=\left[\begin{array}{cc}
C^{*} C & 0 \\
0 & 0
\end{array}\right] \leq\left[\begin{array}{ll}
I & 0 \\
0 & 0
\end{array}\right]=\widetilde{A} \widetilde{A}^{*} \\
\left(\widetilde{C}^{*} \widetilde{C}\right) \widetilde{A}=\left[\begin{array}{cc}
\left(C^{*} C\right) A & \left(C^{*} C\right)\left(1-A A^{*}\right)^{1 / 2} \\
0 & 0
\end{array}\right]=\widetilde{A}\left(\widetilde{C}^{*} \widetilde{C}\right)
\end{gathered}
$$

and

$$
\widetilde{C} \widetilde{A}=\left[\begin{array}{cc}
C A & C\left(1-A A^{*}\right)^{1 / 2} \\
0 & 0
\end{array}\right]=\left[\begin{array}{cc}
B C & 0 \\
0 & 0
\end{array}\right]=\widetilde{B} \widetilde{C}
$$

Therefore, by the first paragraph of the proof, we have $\widetilde{C} \widetilde{A}^{*}=\widetilde{B}^{*} \widetilde{C}$. This implies that $C A^{*}=B^{*} C$.

Now we present an example which is not covered by the Fuglede-Putnam theorem or an existing generalization of it.

Example. Let

$$
A=\left[\begin{array}{ccccccc}
0 & 0 & a_{1} & & & & \\
& 0 & 0 & a & & & 0 \\
& & 0 & 0 & a_{2} & & \\
& & & 0 & 0 & a & \\
0 & & & & \ddots & \ddots & \ddots
\end{array}\right], \quad B=\left[\begin{array}{ccccccc}
0 & 0 & a & & & & 0 \\
& 0 & 0 & b_{1} & & & \\
& & 0 & 0 & a & & \\
& & 0 & 0 & b_{2} & \\
0 & & & \ddots & \ddots & \ddots
\end{array}\right]
$$

and

$$
C=\left[\begin{array}{ccccccc}
0 & 1 & 0 & & & & 0 \\
& 0 & 0 & 0 & & & \\
& & 0 & 1 & 0 & & \\
& & & 0 & 0 & 0 & \\
0 & & & & \ddots & \ddots & \ddots
\end{array}\right]
$$

where $\left\{a_{n}\right\}$ and $\left\{b_{n}\right\}$ are bounded sequences of complex numbers such that $a=$ $\sup _{n}\left|a_{n}\right| \geq \sup _{n}\left|b_{n}\right|$.

Then $B$ is not $M$-hyponormal [7], indeed, in general it is not even a dominant [9] or $\mathcal{Y}$-class operator. (We point out that an operator $T$ is said to be $M$-hyponormal if there exists a constant $M \geq 1$ such that $(T-\lambda)(T-\lambda)^{*} \leq M^{2}(T-\lambda)^{*}(T-\lambda)$ for any
complex number $\lambda$. An operator $T$ is said to be dominant if for any complex number $\lambda$ there exists a number $M_{\lambda} \geq 1$ such that $(T-\lambda)(T-\lambda)^{*} \leq M_{\lambda}^{2}(T-\lambda)^{*}(T-\lambda)$.

For $\alpha>0$ an operator $T$ is said to be in $\mathcal{Y}_{\alpha}$ if there exists a number $M_{\alpha} \geq 1$ such that $\left|T^{*} T-T T^{*}\right|^{\alpha} \leq M_{\alpha}^{2}(T-\lambda)^{*}(T-\lambda)$ for any complex number $\lambda$. An operator $T$ is said to be in $\mathcal{Y}$ (or to be of $\mathcal{Y}$-class) if $T$ is in $\mathcal{Y}_{\alpha}$ for some $\alpha \geq 1$.)

Therefore, neither the Fuglede-Putnam theorem nor any existing generalizations apply. However, since $A, B$ and $C$ satisfy the hypotheses of our theorem, we can conclude that $C A^{*}=B^{*} C$.

Remark 1. We cannot merely drop condition (3) in the Theorem. For example, let $U$ be the unilateral shift. Take $A=B=U^{*}$ and $C=\left(U^{*}\right)^{2}$. Then $A, B$ and $C$ satisfy (1), (2) and (4), but $C A^{*} \neq B^{*} C$ immediately.

Remark 2. We cannot merely drop condition (2) in the Theorem. For example, put

$$
A=\left[\begin{array}{ccccccc}
0 & 0 & 1 & & & & \\
& 0 & 0 & 1 & & & 0 \\
& & 0 & 0 & 1 & & \\
& & & 0 & 0 & 1 & \\
0 & & & & \ddots & \ddots & \ddots
\end{array}\right], \quad B=\left[\begin{array}{llllllll}
0 & 1 & 1 & & & & & \\
& 0 & 0 & 1 & & & & 0 \\
& & 0 & 0 & 1 & & & \\
& & & 0 & 0 & 1 & & \\
0 & & & \ddots & \ddots & \ddots &
\end{array}\right]
$$

and

$$
C=\left[\begin{array}{llllllll}
0 & 1 & 0 & & & & & \\
& 0 & 0 & 0 & & & & 0 \\
& & 0 & 1 & 0 & & & \\
& & & 0 & 0 & 0 & & \\
0 & & & & \ddots & \ddots & \ddots &
\end{array}\right]
$$

Then $A, B$ and $C$ satisfy (1), (3) and (4), but $C A^{*} \neq B^{*} C$.
Remark 3. We cannot merely drop condition (4) in the Theorem. For example, put

$$
A=\left[\begin{array}{ccccccc}
0 & 0 & 2 & & & & \\
& 0 & 0 & 1 & & & 0 \\
& & 0 & 0 & 1 & & \\
& & & 0 & 0 & 1 & \\
0 & & & & \ddots & \ddots & \ddots
\end{array}\right], \quad B=\left[\begin{array}{llllllll}
0 & 0 & 1 & & & & & \\
0 & 0 & 0 & 1 & & & & 0 \\
& 1 & 0 & 0 & 1 & & & \\
& & 0 & 0 & 0 & 1 & & \\
& & & 0 & 0 & 0 & 1 & \\
0 & & & & \ddots & \ddots & \ddots &
\end{array}\right]
$$

and

$$
C=\left[\begin{array}{ccccccc}
0 & 1 & 0 & & & & 0 \\
& 0 & 0 & 0 & & & \\
& & 0 & 1 & 0 & & \\
& & & 0 & 0 & 0 & \\
0 & & & & \ddots & \ddots & \ddots
\end{array}\right]
$$

Then $A, B$ and $C$ satisfy (1), (2) and (3), but $C A^{*} \neq B^{*} C$.

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