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CHARACTER DEGREES AND LOCAL SUBGROUPS OF π-SEPARABLE GROUPS

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Abstract. Let G be a finite $\{p,q\}$-solvable group for different primes p and q. Let $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$ be such that $PQ = QP$. We prove that every $\chi \in \mathrm{Irr}(G)$ of p'-degree has q'-degree if and only if $\mathbf{N}_G(P) \subseteq \mathbf{N}_G(Q)$ and $\mathbf{C}_{Q'}(P) = 1$.

1. Introduction

The main result we present is the equivalence of a local group theoretic condition about Sylow normalizers of a finite group G and a global condition on the character degrees of G, namely:

Theorem A. *Let G be a finite $\{p,q\}$-solvable group for different primes p and q. Let $P \in Syl_p(G)$ and $Q \in Syl_q(G)$ be such that $PQ = QP$. Then every $\chi \in \mathrm{Irr}(G)$ of p'-degree has q'-degree if and only if $\mathbf{N}_G(P) \subseteq \mathbf{N}_G(Q)$ and $\mathbf{C}_{Q'}(P) = 1$.*

Theorem A is no longer true if we remove the q-solvability assumption, even though Hall $\{p,q\}$-subgroups exist in this case, and we will provide an example below. The question does remain open if Theorem A is valid for q-solvable groups. It is our impression that this is heavily related to the validity of McKay's conjecture and similar results.

We note that Theorem A can be applied to prove Ito's Theorem for q-solvable groups G that each $\chi \in \mathrm{Irr}(G)$ has q'-degree if and only if G has a normal abelian Sylow q-subgroup. Just choose a prime p not dividing $|G|$.

We later will discuss where Therorem A can and cannot be extended to sets of primes, and apply such information to coprime automorphism groups.

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2. Preliminaries

Our techniques are reasonably elementary. We repeatedly use Glauberman's Lemma and some easy consequences of the Glauberman-Isaacs correspondence. Chapter 13 of [Is] is a good reference for this material. We summarize most of what is needed from this in the next lemma, which we will use often without reference. If

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A acts on G, we let $\mathrm{Irr}_A(G)$ denote the set of those $\chi \in \mathrm{Irr}(G)$ that are A-invariant. We let $\mathrm{Irr}_{p'}(G)$ denote the set of those $\chi \in \mathrm{Irr}(G)$ of p'-degree, and similarly define $\mathrm{Irr}_{\pi'}(G)$ for a set π of primes.

Lemma 1. *Let $G = KH$ and $H \cap K = L$, where $L, K \lhd G$ and $(|G/K|, |K/L|) = 1$. Set $C/L = \mathbf{C}_{K/L}(H)$.*

(a) If $\theta \in \mathrm{Irr}(K)$ is H-invariant, then θ_L has an H-invariant irreducible constituent, and C transitively permutes the H-invariant irreducible constituents of θ_L.

(b) If $\phi \in \mathrm{Irr}(L)$ is H-invariant, then ϕ^K has an H-invariant irreducible contituent.

(c) If $L = 1$, there is a bijection ${}^* : \mathrm{Irr}_H(K) \to \mathrm{Irr}(C)$ such that χ^* is a constituent of χ_C.

(d) If $L = 1$ and $B \subseteq H$, then $\mathrm{Irr}_B(K) = \mathrm{Irr}_H(K)$ if and only if $\mathbf{C}_K(B) = C$.

Proof. Part (a) follows from Glauberman's Lemma (Lemma (13.8) and Corollary (13.9) of [Is]). Part (c) is a weak form of the Glauberman-Isaacs correspondence (see discussion following Theorem (13.25) of [Is]), while part (b) is Theorem (13.31) of [Is]. Finally, part (d) follows from part (c) and Glauberman's lemma (specifically Corollary (13.10) of [Is]) and a detailed proof of (d) is given in Lemma (2.2) of [Na]. □

We now give Theorem A in the special case $G = PQ$. We will give it later in Lemma 4, for when G has a normal $\{p, q\}$-complement.

Lemma 2. *Suppose that $G = PQ$ with $P \in Syl_p(G)$ and $Q \in Syl_q(G)$ for distinct primes p and q. Then every irreducible character of G with p'-degree has q'-degree if and only if $Q \lhd G$ and $\mathbf{C}_{Q'}(P) = 1$.*

Proof. The hypothesis about character degrees implies that p divides the degree of every non-linear irreducible character of G, in which case Thompson's Theorem (Theorem (12.2) of [Is]) shows that G has a normal p-complement. Thus, for either direction, we have that Q is normal in G, so that Q' is also normal in G and G/Q is a p-group. Note that if $\chi \in \mathrm{Irr}_{p'}(G)$, then χ restricts irreducibly to Q. Also, each P-invariant irreducible character of Q lies under some irreducible character of G of p'-degree (see Corollary 6.28 of [Is]). With use of Lemma 1(c), we have that $\mathbf{C}_{Q'}(P) = 1$ iff $1_{Q'}$ is the only P-invariant irreducible character of Q'.

Now, suppose that $\mathbf{C}_{Q'}(P) = 1$ and let $\chi \in \mathrm{Irr}(G)$ of p'-degree. Then $\theta = \chi_Q \in \mathrm{Irr}(Q)$ is P-invariant. By Lemma 1(a), there exists a P-invariant $\phi \in \mathrm{Irr}(Q')$ under θ. Then $\phi = 1_{Q'}$, and we deduce that θ is linear. Hence, q does not divide $\chi(1)$.

Assume now that every irreducible character of G of p'-degree has q'-degree. If $\xi \in \mathrm{Irr}_P(Q)$, we have that ξ extends to a p'-degree character of $\hat{\xi}$ of G. By hypothesis, q does not divide $\hat{\xi}(1) = \xi(1)$ and thus we deduce that ξ is linear, as desired. □

If π is a set of prime numbers, recall that property D_π for a group G is that a Hall π-subgroup H exists and that every π-subgroup of G is conjugate to a subgroup of H.

Proposition 3. *Assume that G satisfies D_π and let H be a Hall π-subgroup. If P is a Sylow p-subgroup of H, then $\mathbf{N}_H(P)$ is a Hall π-subgroup of $\mathbf{N}_G(P)$.*

Proof. Let $q \in \pi$ and Q be a Sylow q-subgroup of $\mathbf{N}_H(P)$. Choose $Q \subseteq Q_1 \in \mathrm{Syl}_q(\mathbf{N}_G(P))$ so that for some $g \in G$, we have $(QP)^g \subseteq (Q_1P)^g \subseteq H$. Since P^g is H-conjugate to P, it is no loss to assume that $g \in \mathbf{N}_G(P)$. Then, $Q^g \subseteq Q_1^g \subseteq \mathbf{N}_H(P)$ and so $Q = Q_1$, proving the proposition. □

Lemma 4. *Suppose that G has a normal $\{p, q\}$-complement K, where p and q are distinct primes. Let $P \in Syl_p(G)$ and $Q \in Syl_q(G)$ be such that $PQ = QP$. Then every irreducible character of G with p'-degree has q'-degree if and only if $\mathbf{N}_G(P) \subseteq \mathbf{N}_G(Q)$ and $\mathbf{C}_{Q'}(P) = 1$.*

Proof. Write $H = PQ$. Observe that if G satisfies the hypotheses on character degrees, so does $H = PQ \cong G/K$. Hence, to prove either direction, we may assume by Lemma 2 (and our hypothesis) that P normalizes Q and $\mathbf{C}_{Q'}(P) = 1$ or, equivalently, that every $\beta \in \mathrm{Irr}(H)$ of p'-degree has q'-degree.

Observe, by Proposition 3, that $\mathbf{N}_G(P) = \mathbf{N}_H(P)\mathbf{C}_K(P) \subseteq H\mathbf{C}_K(P)$ and $\mathbf{N}_G(Q) = H\mathbf{C}_K(Q)$. Thus $\mathbf{N}_G(P) \subseteq \mathbf{N}_G(Q)$ if and only if $\mathbf{C}_K(P) \subseteq \mathbf{C}_K(Q)$ if and only if $\mathbf{C}_K(P) = \mathbf{C}_K(H)$ if and only if every P-invariant irreducible character of K is H-invariant, by Lemma 1(d). Hence, it suffices to show that each $\chi \in \mathrm{Irr}_{p'}(G)$ has q'-degree if and only if every P-invariant irreducible character of K is H-invariant.

Suppose first that every P-invariant irreducible character of K is H-invariant. Let $\chi \in \mathrm{Irr}(G)$ with p'-degree, and let $\theta \in \mathrm{Irr}(K)$ be under χ. By the Clifford correspondence, we have that the inertia group in G of θ has p'-index. Hence, replacing θ by some G-conjugate, we may assume that θ is P-invariant. By hypothesis, we have that θ is H-invariant, and thus there exists $\hat{\theta} \in \mathrm{Irr}(G)$ such that $\hat{\theta}_K = \theta$ by Corollary (6.28) of [Is]. By Gallagher's Theorem (Corollary (6.17) of [Is]), we have that $\chi = \beta\hat{\theta}$, for some $\beta \in \mathrm{Irr}(G/K)$. Now, β has p'-degree (because $\beta(1)$ divides $\chi(1)$) and thus, by the first paragraph of the proof, we have that β has q'-degree. Hence, χ has q'-degree, as desired.

Suppose now that every $\chi \in \mathrm{Irr}(G)$ of p'-degree has q'-degree. We wish to show that every $\theta \in \mathrm{Irr}_P(K)$ is H-invariant. If $T = I_G(\theta)$, then we have that θ extends to T. Let $\eta \in \mathrm{Irr}(T)$ be an extension. By the Clifford correspondence, we have that $\eta^G \in \mathrm{Irr}(G)$. Now, since $P \subseteq T$ and η has p'-degree, it follows that η^G has p'-degree. Hence, η^G has q'-degree, and it follows that $T = G$, as desired. □

McKay conjectured that $|\mathrm{Irr}_{2'}(G)| = |\mathrm{Irr}_{2'}(\mathbf{N}_G(P))|$ whenever G is simple and P is a Sylow 2-subgroup of G. Isaacs sugested this for all primes and at least solvable groups, and proved this for groups of odd order in [Is 1]. This was extended to solvable G in [Wo 1], then to p-solvable in [OW], and to sets of primes and even Brauer characters in [Wo 2] (under appropriate separability conditions). We use the McKay "conjecture" frequently in the proof of Theorem A, which is also Theorem 6. What we need from these results appears just below.

If $L \triangleleft G$ and $\phi \in \mathrm{Irr}(L)$, we set

$$\mathrm{Irr}_{\pi'}(G \mid \phi) = \mathrm{Irr}_{\pi'}(G) \cap \mathrm{Irr}(G \mid \phi)\,.$$

Theorem 5. *Suppose that $L \triangleleft G$ and that G/L is a π-separable group. Assume that H/L is a Hall π-subgroup of G/L and $\phi \in \mathrm{Irr}(L)$ is H-invariant. Then*

$$|\,\mathrm{Irr}_{\pi'}(G \mid \phi)| = |\,\mathrm{Irr}_{\pi'}(N \mid \phi)|,$$

where $N/L = \mathbf{N}_{G/L}(H/L)$.

Proof. When π is a singleton $\{p\}$, this is proven as Theorem (15.10) of [MW] using the methods of [OW]. The same argument works just as well for an arbitrary set of primes. It is proven in its full generality in [Wo 2]. $\square$

3. Main results

This is Theorem A of the Introduction.

Theorem 6. *Suppose that G is π-separable, where $\pi = \{p, q\}$ for distinct primes p and q, and let $PQ = QP$ with $P \in Syl_p(G)$ and $Q \in Syl_q(G)$. Then* $\mathrm{Irr}_{p'}(G) \subseteq \mathrm{Irr}_{q'}(G)$ *if and only if* $\mathbf{N}_G(P) \subseteq \mathbf{N}_G(Q)$ *and* $\mathbf{C}_{Q'}(P) = 1$.

Proof. We argue by induction on $|G|$. Let $M \triangleleft G$ such that G/M is a π'-group.

First, we claim that $\mathbf{N}_G(P) \subseteq \mathbf{N}_G(Q)$ if and only if $\mathbf{N}_M(P)$ is contained in $\mathbf{N}_M(Q)$. To see this, assume that $\mathbf{N}_M(P)$ is contained in $\mathbf{N}_M(Q)$. Let $U = \mathbf{N}_G(Q)$. Since P and Q are contained in M, the Frattini argument yields that $G = MU$ and $G = M\mathbf{N}_G(P)$. Also, because $P \subseteq M \cap U$ by hypothesis, again by the Frattini argument we have that $U = (U \cap M)\mathbf{N}_U(P)$. Thus, $G = M\mathbf{N}_U(P)$. Then

$$|G : \mathbf{N}_U(P)| = |M : M \cap \mathbf{N}_U(P)| = |M : \mathbf{N}_M(P)| = |G : \mathbf{N}_G(P)| \,.$$

So, $\mathbf{N}_G(P) = \mathbf{N}_U(P) \subseteq \mathbf{N}_G(Q)$, as claimed.

Because G/M is a π'-group, we have that $\mathrm{Irr}_{p'}(G) \subseteq \mathrm{Irr}_{q'}(G)$ if and only if $\mathrm{Irr}_{p'}(M) \subseteq \mathrm{Irr}_{q'}(M)$. Of course, $PQ \subseteq M$. If $M < G$, we employ induction and the last paragraph to conclude that $\mathrm{Irr}_{p'}(G) \subseteq \mathrm{Irr}_{q'}(G)$ iff $\mathbf{N}_M(P) \subseteq \mathbf{N}_M(Q)$ and $\mathbf{C}_{Q'}(P) = 1$ iff $\mathbf{N}_G(P) \subseteq \mathbf{N}_G(Q)$ and $\mathbf{C}_{Q'}(P) = 1$. Thus we may assume that $\mathbf{O}^{\pi'}(G) = G$.

We now let $K = \mathbf{O}^{\pi}(G) < G$ and $L = \mathbf{O}^{\pi'}(K)$. By Lemma 4, we may assume that $1 < L < K$. We let $H/L = LPQ/L$, a Hall π-subgroup of G/L, and note that $KH = G$ and $K \cap H = L$. We set $C/L = \mathbf{C}_{K/L}(H) = \mathbf{N}_{K/L}(H/L)$. Letting $N/L = \mathbf{N}_{G/L}(H/L)$, we have that $HC = N$ and $N \cap K = C$. Note that $N < G$, because $\mathbf{O}^{\pi'}(G) = G$.

We claim that we may assume that $\mathbf{N}_G(P) \subseteq N$. If we assume the hypothesis that $\mathbf{N}_G(P) \subseteq \mathbf{N}_G(Q)$, then $\mathbf{N}_G(P) \subseteq \mathbf{N}_G(LPQ) = \mathbf{N}_G(H) = N$. To complete the claim, we may assume that $\mathrm{Irr}_{p'}(G) \subseteq \mathrm{Irr}_{q'}(G)$. Now we can apply Lemma 4 to G/L to conclude that $\mathbf{N}_{G/L}(LP/L) \subseteq \mathbf{N}_{G/L}(LQ/L)$. Because $\mathbf{N}_{G/L}(LP/L) = L\mathbf{N}_G(P)/L$, it follows that $L\mathbf{N}_G(P)$ normalizes $(LP/L)(LQ/L) = H/L$. Then $L\mathbf{N}_G(P) \subseteq N$, establishing the claim.

We also observe that $C/L = \mathbf{C}_{K/L}(P)$. One containment is trivial, as C/L centralizes H/L. For the other, note that $\mathbf{C}_{K/L}(P) \subseteq \mathbf{N}_{G/L}(LP/L) \cap K/L = (L\mathbf{N}_G(P)/L) \cap K/L \subseteq N/L \cap K/L = C/L$.

Let $\phi \in \mathrm{Irr}(L)$ be H-invariant. In particular, ϕ is LP-invariant. Set

$$J/L = \mathbf{N}_{G/L}(LP/L) = L\mathbf{N}_G(P)/L \,,$$

so that $J \subseteq N$ by the previous paragraphs. Since G/L is p-solvable and π-solvable, we apply Theorem 5 for the prime p and the set π to conclude that

$$|\mathrm{Irr}_{p'}(G \,|\, \phi)| = |\mathrm{Irr}_{p'}(J \,|\, \phi)| = |\mathrm{Irr}_{p'}(N \,|\, \phi)|$$

and also

$$|\mathrm{Irr}_{\pi'}(G \,|\, \phi)| = |\mathrm{Irr}_{\pi'}(N \,|\, \phi)| \,.$$

Trivially, $\mathrm{Irr}_{p'}(G\,|\,\phi) \subseteq \mathrm{Irr}_{q'}(G\,|\,\phi)$ iff $\mathrm{Irr}_{p'}(G\,|\,\phi) \subseteq \mathrm{Irr}_{\pi'}(G\,|\,\phi)$. Thus

$$\mathrm{Irr}_{p'}(G\,|\,\phi) \subseteq \mathrm{Irr}_{q'}(G\,|\,\phi) \quad \text{iff} \quad \mathrm{Irr}_{p'}(N\,|\,\phi) \subseteq \mathrm{Irr}_{q'}(N\,|\,\phi)\,.$$

First suppose that $\mathbf{N}_G(P) \subseteq \mathbf{N}_G(Q)$ and $\mathbf{C}_{Q'}(P) = 1$. Note that $PQ \subseteq N$ and that $\mathbf{N}_N(P) \subseteq \mathbf{N}_N(Q)$. Because $N < G$, the inductive hypothesis implies that $\mathrm{Irr}_{p'}(N) \subseteq \mathrm{Irr}_{q'}(N)$. Let $\chi \in \mathrm{Irr}_{p'}(G)$ and choose $\beta \in \mathrm{Irr}_{p'}(N)$ a constituent of χ_N (otherwise, p would divide $\chi_N(1) = \chi(1)$). Then $(qp, \beta(1)) = 1$. As H/L is the normal Hall π-subgroup of N/L, every irreducible constituent of β_L is H-invariant and the last paragraph shows that $(q, \chi(1)) = 1$. Thus $\mathrm{Irr}_{p'}(G) \subseteq \mathrm{Irr}_{q'}(G)$.

Finally, assume that $\mathrm{Irr}_{p'}(G) \subseteq \mathrm{Irr}_{q'}(G)$. Let $\alpha \in \mathrm{Irr}_{p'}(N)$. Choose an LP-invariant irreducible constituent δ of α_L. Because δ is P-invariant and $J \subseteq N$, Theorem 5 yields that

$$|\mathrm{Irr}_{p'}(G\,|\,\delta)| = |\mathrm{Irr}_{p'}(J\,|\,\delta)| = |\mathrm{Irr}_{p'}(N\,|\,\delta)| > 0,$$

and we choose $\mu \in \mathrm{Irr}_{p'}(G\,|\,\delta)$. By assumption, $\mu(1)$ is q' and thus $\mu_K \in \mathrm{Irr}(K\,|\,\delta)$. Then K/L transitively permutes the irreducible constituents of μ_L. Because $\mu(1)$ is π' (or by Lemma 1(a)), observe that μ_L has an H-invariant irreducible constituent γ, which of course is P-invariant. Because $C/L = \mathbf{C}_{K/L}(H) = \mathbf{C}_{K/L}(P)$, applying Lemma 1(a) twice shows that δ is C-conjugate to γ and hence H-invariant. By the next to last paragraph, $\mathrm{Irr}_{p'}(N\,|\,\delta) \subseteq \mathrm{Irr}_{q'}(N\,|\,\delta)$ and so $\alpha(1)$ is q'. Hence, $\mathrm{Irr}_{p'}(N) \subseteq \mathrm{Irr}_{q'}(N)$. We apply induction to conclude that $\mathbf{N}_N(P) \subseteq \mathbf{N}_N(Q)$ and $\mathbf{C}_{Q'}(P) = 1$. Since $\mathbf{N}_G(P) \subseteq N$, we have that $\mathbf{N}_G(P) \subseteq \mathbf{N}_G(Q)$, as desired. $\square$

We next discuss some related questions, including corollaries and extensions. An obvious corollary is that if G is $\{p, q\}$-separable and satisfies the character theoretic conditions in the above theorem, so does every subgroup containing a Hall $\{p, q\}$-subgroup. The following is an immediate generalization of Ito's Theorem for q-solvable groups (just take $(p, |G|) = 1$).

Corollary 7. *Suppose that G is π-separable, where $\pi = \{p, q\}$ for distinct primes p and q, and that $H = PQ = QP$ with $P \in Syl_p(G)$ and $Q \in Syl_q(G)$. Then H is abelian and $\mathbf{N}_G(P) = \mathbf{N}_G(Q)$ if and only if whenever p or q divides $\chi(1)$ with $\chi \in \mathrm{Irr}(G)$, then pq divides $\chi(1)$.*

It is perhaps interesting to notice that, as a consequence of Corollary 7, the condition $PQ = QP$ is abelian with $\mathbf{N}_G(P) = \mathbf{N}_G(Q)$ can be determined from the character degrees of G.

If p, q and r are distinct primes, there is a cyclic group PQ of order pq that acts faithfully and irreducibly on an elementary abelian r-group R. Every non-linear irreducible character of the (solvable) semi-direct product G has degree pq. So every $\chi \in \mathrm{Irr}(G)$ whose degree is coprime to pr has q'-degree. But $\mathbf{N}_G(PR) = PR$ is not contained in $QP = \mathbf{N}_G(Q)$. This tells us that, at least in one direction, Theorem 6 and even Lemma 1 cannot be generalized by replacing p by a set of primes. What fails, of course, is the appropriate generalization of Thompson's Theorem (Corollary (12.2) of [Is]), as demonstrated by the same example. But quite a bit does remain valid if p and/or q are replaced by sets of primes. We state this in the next corollary, whose proof, which we omit, can be obtained by mimicking the above proofs.

Corollary 8. *Suppose that G is π-separable and ρ-separable for disjoint sets of primes π and ρ, and let $H = PQ = QP$ with P a Hall π-subgroup and Q a Hall ρ-subgroup of G. Then:*

(a) If $\mathbf{N}_G(P) \subseteq \mathbf{N}_G(Q)$ *and* $\mathbf{C}_{Q'}(P) = 1$*, then every* $\chi \in \mathrm{Irr}(G)$ *of* π'*-degree has* ρ'*-degree.*

(b) The converse of (a) holds if G *has a normal* π*-complement or if* $|\pi| \leq 1$*.*

We now apply the above Corollary to give another proof of a result of Beltrán and Navarro, i.e., Theorem D of [BN] about coprime actions, which in turn removed the solvability hypothesis of Theorem A in [Na].

Corollary 9. *Suppose that* A *acts coprimely on a* ρ*-separable group* G*, and choose an* A*-invariant Hall* ρ*-subgroup* K *of* G*. Then every* A*-invariant irreducible character has* ρ'*-degree if and only if* $\mathbf{C}_G(A) \leq \mathbf{N}_G(K)$ *and* $\mathbf{C}_{K'}(A) = 1$*.*

Proof. Such a K exists by Glauberman's Lemma (Lemma (13.8) of [Is]). Let π be the set of prime divisors of A, so that π and ρ are disjoint. We now apply Corollary 8 to GA, which has a normal π-complement G. Now $\mathbf{C}_G(A) \leq \mathbf{N}_G(K)$ and $\mathbf{C}_{K'}(A) = 1$ iff $\mathbf{N}_{GA}(A) \leq \mathbf{N}_{GA}(K)$ and $\mathbf{C}_{K'}(A) = 1$ iff every $\chi \in \mathrm{Irr}(GA)$ of π'-degree has ρ'-degree iff every $\theta \in \mathrm{Irr}_A(G)$ has ρ'-degree, because every $\chi \in \mathrm{Irr}(GA)$ of π'-degree must restrict irreducibly to an A-invariant irreducible character of G and every A-invariant irreducible character of G extends to GA. $\square$

We do note that p-solvable groups and q-solvable groups do have a unique conjugacy class of Hall π-subgroups for $\pi = \{p, q\}$. We do not know whether Theorem A extends to q-solvable groups; however, the next example exhibits a p-nilpotent group G in which each $\chi \in \mathrm{Irr}_{p'}(G)$ has q'-degree. But while $P \leq \mathbf{N}_G(Q)$ for Sylow p and Sylow q-subgroups P and Q, we do not have $\mathbf{N}_G(P) \subseteq \mathbf{N}_G(Q)$. So at least one direction of Theorem 5 is not true for p-solvable groups. We thank A. Turull for some discussion about $\mathrm{PSL}(2, 3^5)$.

Example 10. Let $K = \mathrm{PSL}(2, 3^5)$. Then K is simple of order $2^2 \cdot 3^5 \cdot 11^2 \cdot 61$ and the degrees of its irreducible characters are 1, $242 = 2 \cdot 11^2$, $243 = 3^5$, $244 = 4 \cdot 61$, and 122. Now K has an automorphism $\langle a \rangle$ of order 5 that fixes none of the characters of degree 242. In addition, $\mathbf{C}_K(a)$ is isomorphic to A_4. If we let G be the semi-direct product $K\langle a \rangle$, then every $\chi \in \mathrm{Irr}_{5'}(G)$ has degree not divisible by 11, but the Sylow theorems show that 4 does not divide the order of the normalizer of a Sylow 11-subgroup. So if P is the Sylow 5-subgroup $\langle a \rangle$ and $Q \in \mathrm{Syl}_{11}(G)$ is P-invariant, we do not have that $\mathbf{N}_G(P) \subseteq \mathbf{N}_G(Q)$.

Regarding the converse direction, the Atlas shows there are simple groups G where every irreducible character of p'-degree also has q'-degree, but have no Hall $\{p, q\}$-subgroups (e.g. M_{11} with $p = 5$ and $q = 3$) or which have two conjugacy classes of Hall $\{p, q\}$-subgroups (e.g. $\mathrm{PSL}(2, 11)$ with $p = 2$ and $q = 3$). In this latter case, $\mathbf{N}_G(P)$ is never contained in $\mathbf{N}_G(Q)$ when $PQ = QP$. A generalization of Theorem A in this direction would apparently require some condition on existence and conjugacy of Hall $\{p, q\}$-subgroups.

References

[Atlas] J.H. Conway et al, Atlas of Finite Groups, Clarendon Press, Oxford 1985. MR **88g:**20025

[BN] A. Beltrán and G. Navarro, Actions and invariant character degrees III, J. Algebra 191, (1997), 141-148. CMP 97:11

[Is] I. M. Isaacs, "Character Theory of finite groups", corrected reprint, Dover, New York, 1994. MR **57:**417 (original ed.)

[Is 1] I. M. Isaacs, Characters of solvable and symplectic groups, Amer. J. Math. 95, (1973), 594-635. MR **48:**11270

[Na] G. Navarro, Actions and invariant character degrees, J. Algebra 160 (1993), 172-178. MR **94h:**20012

[MW] O. Manz and T. Wolf, "Representations of solvable groups", Cambridge University Press, 1993. MR **95c:**20013

[OW] T. Okuyama and M. Wajima, Irreducible characters of p-solvable groups, Proceedings of the Japan Academy 55 (1979), 309-312. MR **81g:**20013

[Wo 1] T. Wolf, Characters of p'-degree in solvable groups, Pacific J. Math., 74 (1978), 267-271. MR **57:**9823

[Wo 2] T. Wolf, Variations on McKay's conjecture, J. Algebra 135 (1990), 123-138. MR **91h:**20023

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