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CONTINUITY OF LIE MAPPINGS OF THE SKEW ELEMENTS OF BANACH ALGEBRAS WITH INVOLUTION

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Abstract. Let A and B be centrally closed prime complex Banach algebras with linear involution. If A is semisimple, then any Lie derivation of the skew elements of A is continuous and any Lie isomorphism from the skew elements of B onto the skew elements of A is continuous.

The Lie product $[a, b] = ab - ba$ induces on any Banach algebra A a Lie structure of great interest for their intimate connections with the geometry of manifolds modeled on Banach spaces. In case A has a linear involution $*$, then the *skew* elements are the linear subspace $K_A = \{a \in A\colon a^* = -a\}$ which is a Lie subalgebra of A. A *Lie derivation* of K_A is a linear mapping d from K_A to itself which satisfies $d([a, b]) = [d(a), b] + [a, d(b)]$ for all $a, b \in K_A$. If B is another Banach algebra with linear involution, then a Lie isomorphism from K_B onto K_A is a linear bijection ϕ from K_B onto K_A satisfying $\phi([a, b]) = [\phi(a), \phi(b)]$ for all $a, b \in K_B$.

Examples 1. Let H be a complex Hilbert space. Let $L(H)$ denote the primitive C^*-algebra of all continuous linear operators on H, and for each $a \in L(H)$, let $a^\bullet$ denote the usual adjoint operator of a.

1. If J is a conjugation of H, then it is easy to check that the mapping $*$ from $L(H)$ to itself defined by $a^* = Ja^\bullet J$ is a linear involution on $L(H)$. If J is an anticonjugation of H, then the mapping $a^* = -Ja^\bullet J$ is a linear involution on $L(H)$. The skew elements relative to the preceding involutions are classical complex Banach-Lie algebras of bounded operators (see [3]).

2. Let us denote by C_∞ the set of all compact linear operators on H and let $\|\cdot\|_\infty$ be the usual operator norm. For $1 \le p < \infty$, let C_p denote the usual class of those compact linear operators a on H for which $\|a\|_p = (\sum_{n=1}^{\infty} \mu_n^p)^{1/p} < \infty$, where $\{\mu_n\}$ is the sequence of eigenvalues of the operator $(a^\bullet a)^{1/2}$ arranged in decreasing order and repeated according to multiplicity. According to [2, Lemmas XI.9, XI.10, and XI.14], C_p is a two-sided ideal of $L(H)$ which becomes a complex Banach algebra for the norm $\|\cdot\|_p$. Since C_p contains all the continuous linear operators with finite-dimensional range, we deduce that C_p is primitive. The involutions introduced in the preceding example leave invariant C_p, and their skew elements are classical complex Banach-Lie algebras of compact operators (see [3]).

It was proved in [3] that Lie derivations and Lie $\bullet$-automorphisms of all the preceding Banach-Lie algebras are continuous.

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The purpose of this paper is to prove the following results.

Theorem 1. *Let A be a centrally closed prime semisimple complex Banach algebra with linear involution, and let d be a Lie derivation of K_A. Then d is continuous.*

Theorem 2. *Let A and B be centrally closed prime complex Banach algebras with linear involution, and assume that A is in addition semisimple. If ϕ is a Lie isomorphism from K_B onto K_A, then ϕ is continuous.*

We recall that a prime algebra A is *centrally closed* if any linear mapping f from a two-sided ideal I of A to A satisfying $f(ab) = af(b)$ and $f(ba) = f(b)a$ for all $a \in A$ and $b \in I$ is a multiple of the identity operator.

Examples 2. 1. Any primitive complex Banach algebra is centrally closed (see [6, Theorem 12]).

2. [7, Proposition 2.5] shows that prime C^*-algebras are centrally closed.

From now on we assume A and B to be Banach algebras satisfying the requirements in Theorems 1 and 2, d stands for a Lie derivation of K_A, and ϕ denotes a Lie isomorphism from K_B onto K_A.

It should be noted that the involutions are the identity on the center of A and B and consequently they are involutions of the first kind. On account of either [1, Remark 3] or [9, Remark 1.3] (both of them based on the proof of [4, Theorem 2.2]), we deduce that the subalgebras $\langle K_A \rangle$ of A and $\langle K_B \rangle$ of B generated by K_A and K_B, respectively, contain nonzero two-sided ideals I_A of A and I_B of B, respectively.

Proof of Theorem 1. If A has finite dimension, then d is continuous.

Assume that A has infinite dimension. From [9, Theorem 1.1] we deduce that d can be extended to an ordinary linear derivation D of $\langle K_A \rangle$. Since A is prime, it follows that I_A is an essential ideal of A, i.e. $I_A \cap J \neq 0$ whenever J is a nonzero two-sided ideal of A. Consequently, the restriction of D to I_A is an essentially defined derivation of A in the sense of [10] and therefore it is closable. Let $\{a_n\}$ be a sequence in K_A satisfying $\lim a_n = 0$ and $\lim d(a_n) = a$ for some $a \in K$. For any $b \in I_A$, $\{a_n b\}$ is a sequence in I_A converging to zero and

$$\lim D(a_n b) = \lim(d(a_n)b + a_n D(b)) = ab.$$

Hence $ab = 0$ for all $b \in I_A$. Since A is prime, we conclude that $a = 0$. From Johnson's uniqueness-of-norm theorem [5] we deduce that the involution of A is continuous. Accordingly, K_A is closed in A. The closed graph theorem now shows that d is continuous. □

Proof of Theorem 2. If K_B has finite dimensions, then ϕ is continuous.

If K_B is infinite-dimensional, then so is K_A and therefore A and B have infinite dimension. From [1, Theorem 3] we deduce that ϕ can be extended to an ordinary linear isomorphism Φ from $\langle K_B \rangle$ onto $\langle K_A \rangle$.

We claim that B is semisimple. Let $\mathrm{Rad}(C)$ stand for the Jacobson radical of any subalgebra C of either A or B. Since I_A is a two-sided ideal of both A and $\langle K_A \rangle$, it follows that

$$0 = \mathrm{Rad}(A) \cap I_A = \mathrm{Rad}(I_A) = \mathrm{Rad}(\langle K_A \rangle) \cap I_A.$$

Since A is prime we conclude that $\mathrm{Rad}(\langle K_A \rangle) = 0$. As $\langle K_B \rangle$ is isomorphic to $\langle K_A \rangle$ we deduce that $\mathrm{Rad}(\langle K_B \rangle) = 0$. On the other hand,

$$0 = \mathrm{Rad}(\langle K_B \rangle) \cap I_B = \mathrm{Rad}(I_B) = \mathrm{Rad}(B) \cap K_B,$$

which shows that $\mathrm{Rad}(B) = 0$, as claimed.

As in the preceding proof we deduce that K_B is closed in B.

Let $\{a_n\}$ be a sequence in I_B such that $\lim a_n = 0$ and $\lim \Phi a_n = \Phi a$ for some $a \in I_B$. We claim that $r(\Phi a) = 0$. To this end we follow the pattern established in [8]. For any element c in either A or B let $r(c)$ denote its spectral radius. Moreover, if C is a subalgebra of either A or B containing c, then we will denote by $\mathrm{Sp}(c, C)$ the spectrum of c in C. For all $n \in \mathbf{N}$ and $z \in \mathbf{C}$ set $p_n(z) = z\Phi(a_n) + (\Phi(a) - \Phi(a_n))$ and note that

$$r(p_n(z)) \leq \|p_n(z)\| \leq |z|\,\|\Phi(a_n)\| + \|\Phi(a) - \Phi(a_n)\|.$$

On the other hand, it is immediate that

$$\mathrm{Sp}(\Phi(za_n + (a - a_n)), A) \subset \mathrm{Sp}(za_n + (a - a_n), I_B).$$

Since I_B is a two-sided ideal of B we see that

$$\mathrm{Sp}(za_n + (a - a_n), I_B) = \mathrm{Sp}(za_n + (a - a_n), B)$$

and consequently

$$\begin{aligned} r(p_n(z)) = r(\Phi(za_n + (a - a_n))) &\leq r(za_n + (a - a_n)) \\ &\leq |z|\,\|a_n\| + \|a - a_n\|. \end{aligned}$$

The rest of the proof of our claim runs as in [8].

Let $\{a_n\}$ be a sequence in K_B such that $\lim a_n = 0$ and $\lim \phi(a_n) = b$ for some $b \in K_A$. Choose $a \in K_B$ such that $\phi(a) = b$. For all $a' \in I_B$ and $c \in I_A$, $\{a_n a' \Phi^{-1} c\}$ is a sequence in I_B converging to zero and $\lim \Phi(a_n a' \Phi^{-1} c) = \Phi(aa' \Phi^{-1} c)$. According to the above claim, we have $r(\Phi(aa')c) = 0$. Consequently, $\Phi(aa')I_A$ lies in the radical of A and hence $\Phi(aa') = 0$. From this we conclude that $aI_B = 0$, which shows that $a = 0$ and therefore $b = \phi(a) = 0$. The closed graph theorem shows that ϕ is continuous. □

On account of Examples 2.1 and 2.2, our theorems now yield the following results.

Corollary 1. *Lie derivations and Lie automorphisms of the skew elements of a primitive complex Banach algebra with linear involution are continuous.*

Corollary 2. *Lie derivations and Lie automorphisms of the skew elements of a prime C^*-algebra with linear involution are continuous.*

References

1. K. I. Beidar, W. S. Martindale 3rd, and A. V. Mikhalev, *Lie isomorphisms in prime rings with involution*, J. Algebra **169** (1994), 304–327. MR **95m:**16021
2. N. Dunford and J. T. Schwartz, *Linear operators* II, Interscience, New York-London, 1963. MR **90g:**47001b
3. P. de la Harpe, *Classical Banach-Lie algebras and Banach Lie groups of operators in Hilbert space*, Lect. Notes in Math., **285**, Springer-Verlag, Berlin 1972. MR **57:**16372
4. I. N. Herstein, *Topics in ring theory*, Univ. of Chicago Press, Chicago, 1969. MR **42:**6018
5. B. E. Johnson, *The uniqueness of the* (*complete*) *norm topology*, Bull. Amer. Math. Soc. **73** (1967), 537–539. MR **35:**2142
6. W. S. Martindale 3rd, *Lie isomorphisms of prime rings*, Trans. Amer. Math. Soc. **142** (1969), 437–455. MR **40:**4308
7. M. Mathieu, *Elementary operators on prime C^*-algebras* I, Math. Ann. **284** (1989), 223–244. MR **90h:**46092
8. T. J. Ransford, *A short proof of Johnson's uniqueness-of-norm theorem*, Bull. London Math. Soc. **21** (1989), 487–488. MR **90g:**46069

9. G. A. Swain, *Lie derivations of the skew elements of prime rings with involution*, J. Algebra **184** (1996), 679–704. MR **97f:**16059
10. A. R. Villena, *Essentially defined derivations on semisimple Banach algebras*, Proc. Edinburgh Math. Soc. (2) **40** (1997), no. 1, 175–179. MR **98a:**46057

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