

## ***M*-IDEALS OF COMPACT OPERATORS ARE SEPARABLY DETERMINED**

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ABSTRACT. We prove that the space  $K(X)$  of compact operators on a Banach space  $X$  is an  $M$ -ideal in the space  $L(X)$  of bounded operators if and only if  $X$  has the metric compact approximation property (MCAP), and  $K(Y)$  is an  $M$ -ideal in  $L(Y)$  for all separable subspaces  $Y$  of  $X$  having the MCAP. It follows that the Kalton-Werner theorem characterizing  $M$ -ideals of compact operators on separable Banach spaces is also valid for non-separable spaces: for a Banach space  $X$ ,  $K(X)$  is an  $M$ -ideal in  $L(X)$  if and only if  $X$  has the MCAP, contains no subspace isomorphic to  $\ell_1$ , and has property  $(M)$ . It also follows that  $K(Z, X)$  is an  $M$ -ideal in  $L(Z, X)$  for all Banach spaces  $Z$  if and only if  $X$  has the MCAP, and  $K(\ell_1, X)$  is an  $M$ -ideal in  $L(\ell_1, X)$ .

### INTRODUCTION

A (closed) subspace  $Y$  of a Banach space  $X$  is called an  $M$ -ideal if there exists a projection  $P$  on the dual space  $X^*$  such that  $\text{Ker } P = Y^\perp$ , and  $\|x^*\| = \|Px^*\| + \|x^* - Px^*\|$  for all  $x^* \in X^*$ .

Already more than twenty years, many authors have studied conditions for the space  $K(X)$  of compact operators on  $X$  to be an  $M$ -ideal in the space  $L(X)$  of bounded operators (see [8, pp. 333-336] for a brief history and references) with the main aim to characterize those Banach spaces  $X$  for which  $K(X)$  is an  $M$ -ideal in  $L(X)$ . Some years ago, N. J. Kalton and D. Werner [10] showed that, for a separable Banach space  $X$ ,  $K(X)$  is an  $M$ -ideal in  $L(X)$  if and only if  $X$  has the metric compact approximation property (MCAP), contains no subspace isomorphic to  $\ell_1$ , and has property  $(M)$  (property  $(M)$ , which is an internal geometric property of  $X$ , will be defined in Section 2 below). Since the method of proof of this characterization implies the separability of  $X$ , the question whether the Kalton-Werner theorem is also valid for non-separable  $X$  remained open.

In Section 1 of the present note, we prove that  $M$ -ideals of compact operators are separably determined:  $K(X)$  is an  $M$ -ideal in  $L(X)$  if and only if  $X$  has the MCAP, and  $K(Y)$  is an  $M$ -ideal in  $L(Y)$  for all separable subspaces  $Y$  of  $X$  having the MCAP. In Section 2, this enables us to extend the Kalton-Werner theorem to non-separable spaces  $X$ . This also enables us to show that  $K(Z, X)$  is an  $M$ -ideal in  $L(Z, X)$  for all Banach spaces  $Z$  if and only if  $X$  has the MCAP, and  $K(\ell_1, X)$  is an  $M$ -ideal in  $L(\ell_1, X)$  (cf. Section 3).

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Let us fix some more notation. In a Banach space  $X$ , we denote the closed unit ball by  $B_X$ . For a set  $A \subset X$ , its norm closure is denoted by  $\overline{A}$ , its linear span by  $\text{span } A$ , and its convex hull by  $\text{conv } A$ . The set of all weak\* strongly exposed points of  $B_{X^*}$  is denoted by  $w^*\text{-sexp } B_{X^*}$ , and the identity operator of  $X$  is denoted by  $I_X$  or simply by  $I$ . Recall that  $X$  is said to have the MCAP if there is a net in  $B_{K(X)}$  converging strongly to  $I$ . (This means that  $K(X)$  contains a left 1-approximate unit (cf. e.g. [8, p. 294]).)

### 1. $M$ -IDEALS OF COMPACT OPERATORS

The following characterization of  $M$ -ideals of compact operators will be needed below to prove that  $M$ -ideals of compact operators are separably determined.

**Theorem 1.** *Let  $X$  be a Banach space. Then  $K(X)$  is an  $M$ -ideal in  $L(X)$  if and only if  $X$  is an  $M$ -ideal in  $X^{**}$ , and for all  $S \in B_{K(X)}$  there is a net  $(K_\alpha)$  in  $B_{K(X)}$  such that  $K_\alpha \rightarrow I$  strongly and*

$$\limsup \|S + I - K_\alpha\| \leq 1.$$

*Proof.* The necessity is well known (cf. [11] or [8, p. 291] together with [19] or [8, p. 299]). *Sufficiency.* Recall that  $K(X)$  is an  $M$ -ideal in  $L(X)$  if and only if  $K(X)$  is an  $M$ -ideal in  $\mathcal{L} = \text{span}(K(X) \cup \{I\})$  (cf. [14] or [8, p. 299], or [9] for separable  $X$ ). Recall also that the MCAP of  $X$  implies the existence of a linear norm preserving extension operator  $\Phi : K(X)^* \rightarrow \mathcal{L}^*$  (cf. e.g. [12]). This makes it possible to consider the topology  $\sigma = \sigma(\mathcal{L}, \Phi(K(X)^*))$ . In [18], it is essentially proved (for a simpler proof cf. [6]) that  $K(X)$  is an  $M$ -ideal in  $\mathcal{L}$  if and only if for all  $S \in B_{K(X)}$  and  $T \in B_{\mathcal{L}}$  there is a net  $(L_\alpha)$  in  $B_{K(X)}$  such that  $(L_\alpha)$  is  $\sigma$ -convergent to  $T$  and

$$\limsup \|S + T - L_\alpha\| \leq 1.$$

Consider  $T = K + \lambda I \in B_{\mathcal{L}}$  (with  $K \in K(X)$ ). Note that  $|\lambda| \leq 1$  because otherwise  $K$  would be invertible. Therefore  $\lambda = re^{i\varphi}$  with  $r \in [0, 1]$ . For  $e^{-i\varphi}S$ , choose the net  $(K_\alpha)$ . Since  $X$  has the unique extension property (following from the fact that  $X$  is an  $M$ -ideal in  $X^{**}$ ), we have  $x^*(K_\alpha^*x^*) \rightarrow x^{**}(x^*)$  for all  $x^* \in X^*$ ,  $x^{**} \in X^{**}$  (cf. [5] or [8, p. 118]). Set  $L_\alpha = K_\alpha T = K_\alpha K + \lambda K_\alpha$ . Then  $L_\alpha \in B_{K(X)}$ ,

$$x^{**}(L_\alpha^*x^*) = (K^{**}x^{**})(K_\alpha^*x^*) + \lambda x^{**}(K_\alpha^*x^*) \rightarrow x^{**}(T^*x^*),$$

i.e.  $(x^{**} \otimes x^*)(L_\alpha - T) \rightarrow 0$  for all  $x^{**} \otimes x^* \in \mathcal{L}^*$ , and (since  $\|K_\alpha K - K\| \rightarrow 0$ )

$$\begin{aligned} \limsup \|S + T - L_\alpha\| &= \limsup \|S + re^{i\varphi}I - re^{i\varphi}K_\alpha\| \\ &= \limsup \|e^{-i\varphi}S + rI - rK_\alpha\| \\ &\leq r \limsup \|e^{-i\varphi}S + I - K_\alpha\| + 1 - r \\ &\leq 1. \end{aligned}$$

We know that  $B_{X^*} = \overline{\text{conv}}(w^*\text{-sexp } B_{X^*})$  and  $X^*$  has the Radon–Nikodým property (because  $X$  is an  $M$ -ideal in  $X^{**}$ ; cf. e.g. [8, pp. 126, 127], the latest implying  $K(X)^* = \overline{X^{**} \otimes X^*}$  by [4, Theorem 1]). Hence,  $L_\alpha \rightarrow T$  in the  $\sigma$ -topology whenever  $(\Phi g)(L_\alpha - T) \rightarrow 0$  for all  $g = x^{**} \otimes x^* \in K(X)^*$  with  $x^{**} \in X^{**}$  and  $x^* \in w^*\text{-sexp } B_{X^*}$ . By the proof of [12, Lemma 3.4 (a)], such a  $g$  has a unique norm-preserving extension (to the whole  $L(X)$ ). Thus,  $\Phi g = x^{**} \otimes x^* \in \mathcal{L}^*$ , and the result follows.  $\square$

*Remark 1.* It is known (cf. [14] or [8, p. 299], or [9] for separable  $X$ ) that  $K(X)$  is an  $M$ -ideal in  $L(X)$  if and only if there is a net  $(K_\alpha)$  in  $B_{K(X)}$  such that both  $K_\alpha \rightarrow I_X$  and  $K_\alpha^* \rightarrow I_{X^*}$  strongly, and  $\limsup \|S + I - K_\alpha\| \leq 1$  for all  $S \in B_{K(X)}$ .

*Remark 2.* The following is clear from the proof of Theorem 1: if  $X$  is an  $M$ -ideal in  $X^{**}$  (in particular, if  $K(X)$  is an  $M$ -ideal in  $L(X)$ ), and  $K_\alpha \rightarrow I$  strongly for some net  $K_\alpha \in B_{K(X)}$ , then  $K_\alpha \rightarrow I$  in the  $\sigma(L(X), \Phi(K(X)^*))$ -topology for any linear norm preserving extension operator  $\Phi : K(X)^* \rightarrow L(X)^*$ .

The next theorem is the main result of the present note.

**Theorem 2.** *Let  $X$  be a Banach space. Then  $K(X)$  is an  $M$ -ideal in  $L(X)$  if and only if  $X$  has the MCAP, and  $K(Y)$  is an  $M$ -ideal in  $L(Y)$  for all separable subspaces  $Y$  of  $X$  having the MCAP.*

*Proof.* The *necessity* is well known (cf. [14] or [8, p. 301], or [9] for separable  $X$ ). *Sufficiency.* We shall apply Theorem 1. To prove that  $X$  is an  $M$ -ideal in  $X^{**}$ , we need to show that every separable subspace of  $X$  is an  $M$ -ideal in its bidual (cf. [13] or [8, p. 115]). Consider a separable subspace  $Y$  of  $X$ . Since  $X$  has the MCAP,  $Y$  is contained in a separable subspace  $Z$  of  $X$  having the MCAP (the proof of this fact is the same as of the similar fact for the metric approximation property (cf. e.g. [17, p. 606])). Thus,  $K(Z)$  is an  $M$ -ideal in  $L(Z)$ , which implies that  $Z$  is an  $M$ -ideal in  $Z^{**}$ . But then also its subspace  $Y$  is an  $M$ -ideal in  $Y^{**}$ .

Let us now make the following observation. If  $K(Y)$  is an  $M$ -ideal in  $L(Y)$  for a separable Banach space  $Y$ , and  $K_n \rightarrow I_Y$  strongly for some sequence  $(K_n) \subset B_{K(Y)}$ , then, for all  $S \in B_{K(Y)}$  and  $\varepsilon > 0$ , there is some  $K \in \text{conv}\{K_1, K_2, \dots\}$  such that  $\|S - K + I_Y\| \leq 1 + \varepsilon/2$ . [Due to Remark 2, the proof of this fact is the same as of the similar assertion about Banach spaces being  $M$ -ideals in their biduals in [13, Proposition 2.8, (i)  $\implies$  (ii)] (cf. also [8], p. 113), only using instead of the weak\* topology the  $\sigma(L(Y), \Phi(K(Y)^*))$ -topology (where  $\Phi : K(Y)^* \rightarrow L(Y)^*$  is the (unique linear) norm preserving extension operator).]

We denote by  $s_{op}$  the strong operator topology on  $L(X)$ , and suppose that the condition of Theorem 1 is not satisfied: for some  $S \in B_{K(X)}$ , there is no such net. Then there are  $\varepsilon > 0$  and a convex  $s_{op}$  neighbourhood  $U_0$  of  $I$  such that

$$(1) \quad \|S - K + I\| > 1 + \varepsilon \quad \forall K \in B_{K(X)} \cap U_0.$$

For all  $n \in \mathbb{N}$ , denote by  $\Lambda_n$  a finite  $\varepsilon/4$ -net in the subset  $\{(\lambda_1, \dots, \lambda_n) : \lambda_k \geq 0, \lambda_1 + \dots + \lambda_n = 1\}$  of  $\ell_1^n$ . Let  $(K_\alpha)_{\alpha \in \mathcal{A}}$  be a net in  $B_{K(X)}$  converging to  $I$  in the  $s_{op}$ . We shall follow some ideas from the proofs of Proposition 2.8, (iii)  $\implies$  (iv), in [13] (cf. [8, p. 114]) and Theorem 18.2 in [17, p. 606] to pick a sequence  $\alpha_1, \alpha_2, \dots$  in  $\mathcal{A}$  and to define a separable subspace  $Y \subset X$  so that  $S(X) \subset Y$ ,  $K_n(X) \subset Y$  for all  $K_n = K_{\alpha_n}$ ,  $K_n y \rightarrow y$  for all  $y \in Y$ , and  $\|(S - K + I)|_Y\| > 1 + \varepsilon/2$  for all  $K \in \text{conv}\{K_1, K_2, \dots\}$ . This will contradict the observation above, and complete the proof.

To begin, choose  $K_1 = K_{\alpha_1} \in U_0$  such that  $\|K_1 x - x\| < 1$  for all  $x \in S(B_X)$ . Assume that a convex  $s_{op}$  neighbourhood  $U_{n-1} \subset U_{n-2}$  (where  $U_{-1} = U_0$ ) and  $K_n = K_{\alpha_n} \in U_{n-1}$  have been chosen. Consider  $S_\lambda \in L(X)$ ,  $\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda_n$ , defined by  $S_\lambda = S - (\lambda_1 K_1 + \dots + \lambda_n K_n) + I$ . For all  $\lambda \in \Lambda_n$ , select  $x_\lambda \in B_X$  such that  $\|S_\lambda x_\lambda\| > \|S_\lambda\| - \varepsilon/4$ , and denote  $C_n = \{x_\lambda : \lambda \in \Lambda_n\}$ . Put

$$F_n = (1 + \varepsilon)B_{L(X)} + \text{conv}\{K_1, \dots, K_n\} - S.$$

Since  $F_n$  is closed in the  $s_{op}$  and does not contain  $I$  (by (1)), there is a convex  $s_{op}$  neighbourhood  $U_n \subset U_{n-1}$  of  $I$  such that  $U_n \cap F_n = \emptyset$ , which means

$$(2) \quad \|S - K + L\| > 1 + \varepsilon \quad \forall K \in \text{conv}\{K_1, \dots, K_n\}, \quad \forall L \in U_n.$$

Choose  $K_{n+1} = K_{\alpha_{n+1}} \in U_n$  such that

$$\|K_{n+1}x - x\| < \frac{1}{n+1} \quad \forall x \in C_1 \cup \dots \cup C_n \cup S(B_X) \cup K_1(B_X) \cup \dots \cup K_n(B_X).$$

Put  $Y = \{x \in X : \lim K_n x = x\}$ . It is straightforward that  $Y$  is closed,  $S(X) \subset Y$ , and  $K_n(X) \subset Y$ ,  $C_n \subset Y$  for all  $n \in \mathbb{N}$ . Moreover, if  $K \in \text{conv}\{K_1, K_2, \dots\}$ , then  $K \in \text{conv}\{K_1, \dots, K_n\}$  for some  $n$ , and since  $\|(S - K + I) - S_\lambda\| < \varepsilon/4$  for some  $\lambda \in \Lambda_n$ ,

$$\|(S - K + I)|_Y\| > \|S_\lambda|_Y\| - \varepsilon/4 > \|S_\lambda\| - \varepsilon/2 > 1 + \varepsilon/2$$

by (2) and the fact that  $I \in U_n$ .  $\square$

## 2. KALTON-WERNER THEOREM

Recall (cf. [9]) that a Banach space  $X$  is said to have *property*  $(M)$  if

$$\limsup \|x + x_n\| = \limsup \|y + x_n\|$$

whenever  $\|x\| = \|y\|$ , and  $(x_n)$  is a weakly null sequence in  $X$ ; if

$$\limsup \|x^* + x_n^*\| = \limsup \|y^* + x_n^*\|$$

whenever  $\|x^*\| = \|y^*\|$ , and  $(x_n^*)$  is a weak\*-null sequence in  $X^*$ , then  $X$  is said to have *property*  $(M^*)$ . We also need the strong version of property  $(M^*)$ , which we call *property*  $(sM^*)$ , defined by bounded weak\*-null nets  $(x_\lambda^*)$  instead of weak\*-null sequences  $(x_n^*)$ . It is shown in [14], that if  $X$  is separable, then properties  $(M^*)$  and  $(sM^*)$  are equivalent. The Kalton–Werner theorem mentioned above asserts that if a Banach space  $X$  is separable, then  $K(X)$  is an  $M$ -ideal in  $L(X)$  if and only if  $X$  has the MCAP, contains no subspace isomorphic to  $\ell_1$ , and has property  $(M)$ .

**Theorem 3.** *For a Banach space  $X$  the following assertions are equivalent.*

- (a)  $K(X)$  is an  $M$ -ideal in  $L(X)$ .
- (b)  $X$  has the MCAP, and has property  $(sM^*)$ .
- (c)  $X$  has the MCAP, is weakly compactly generated, and has property  $(M^*)$ .
- (d)  $X$  has the MCAP, contains no subspace isomorphic to  $\ell_1$ , and has property  $(M)$ .

*Proof.* (a)  $\implies$  (b) is well known (cf. [7] and [14] or e.g. [8, p. 299]; cf. [9] for separable case).

(b)  $\implies$  (c). Property  $(sM^*)$  implies that  $X$  is an  $M$ -ideal in  $X^{**}$  (cf. [14] or [8, p. 297]; cf. [9] for separable case). But then  $X$  is weakly compactly generated (cf. [3] or e.g. [8, p. 142]).

(c)  $\implies$  (d). Let  $Y$  be an arbitrary separable subspace of  $X$ . We have to show that  $Y$  is not isomorphic to  $\ell_1$ , and has property  $(M)$ . Since  $X$  is weakly compactly generated, there exists a separable subspace  $Z$  containing  $Y$ , and a norm-one projection  $P$  of  $X$  onto  $Z$  (cf. e.g. [2, p. 149]). But then  $Z^*$  isometrically embeds into  $X^*$  by means of the formula  $z^* \in Z^* \mapsto z^*P \in X^*$ . This implies that  $Z$  also has property  $(M^*)$ . By [9], since  $Z$  is separable,  $Z$  has property  $(M)$ , and is an  $M$ -ideal in  $Z^{**}$ , in particular (cf. [11] or e.g. [8, p. 126]),  $Z$  is an Asplund space

(i.e. every separable subspace of  $Z$  has a separable dual). Hence,  $Y$  has property  $(M)$ , and is not isomorphic to  $\ell_1$ .

(d)  $\implies$  (a) follows from Theorem 2 and the Kalton-Werner theorem.  $\square$

*Remark.* In (c) of Theorem 3, the condition that  $X$  is weakly compactly generated clearly may be replaced by the separable complementation property (i.e. every separable subspace of  $X$  is contained in a separable subspace which is a range of a norm-one projection on  $X$ ).

A Banach space  $X$  is said to have the compact approximation property (CAP) if there is a net in  $K(X)$  converging strongly to the identity. Since a reflexive space with the CAP has the MCAP (cf. [1] or [5]), we can refine Theorem 3 for reflexive  $X$  as follows.

**Corollary 4.** *For a reflexive Banach space  $X$ , the following assertions are equivalent.*

- (a)  $K(X)$  is an  $M$ -ideal in  $L(X)$ .
- (b)  $X$  has the CAP, and has property  $(sM^*)$ .
- (c)  $X$  has the CAP, and has property  $(M^*)$ .
- (d)  $X$  has the CAP, and has property  $(M)$ .

*Remark.* For separable reflexive spaces  $X$ , Corollary 4 was obtained in [10]. The equivalence (a)  $\iff$  (b) of Corollary 4 was established in [12] (using an entirely different proof).

### 3. $(M_p)$ -SPACES

Let  $1 \leq p \leq \infty$ . Following [15] (cf. also [8, p. 306]), we say that a Banach space  $X$  is an  $(M_p)$ -space if  $K(X \oplus_p X)$  is an  $M$ -ideal in  $L(X \oplus_p X)$ . Note that  $(M_1)$ -spaces are finite dimensional [16], [8, p. 306], and therefore not of interest in the present context. Since every separable subspace of  $X \oplus_p X$  is contained in  $Y \oplus_p Y$  for some separable subspace  $Y$  of  $X$  with the MCAP whenever  $X$  has the MCAP (cf. the proof of Theorem 2), the next result follows immediately from Theorem 2.

**Corollary 5.** *Let  $1 < p \leq \infty$ . A Banach space  $X$  is an  $(M_p)$ -space if and only if  $X$  has the MCAP, and all separable subspaces of  $X$  with the MCAP are  $(M_p)$ -spaces.*

In [10], N. J. Kalton and D. Werner characterized separable  $(M_p)$ -spaces using the following stronger version of property  $(M)$ . A Banach space  $X$  is said to have property  $(m_p)$  if

$$\limsup \|x + x_n\| = \|(\|x\|, \limsup \|x_n\|)\|_p$$

(where  $\|\cdot\|_p$  denotes the  $\ell_p^2$ -norm) whenever  $(x_n)$  is a weakly null sequence in  $X$ . For separable Banach spaces  $X$ , the following result was obtained in [10].

**Corollary 6.** *Let  $1 < p \leq \infty$ . For a Banach space  $X$ , the following assertions are equivalent.*

- (a)  $X$  is an  $(M_p)$ -space.
- (b)  $K(X)$  is an  $M$ -ideal in  $L(X)$ , and  $X$  has property  $(m_p)$ .
- (c)  $X$  has the MCAP, contains no subspace isomorphic to  $\ell_1$ , and has property  $(m_p)$ .

- (d)  $X$  has the MCAP, and every separable subspace of  $X$  is almost isometric (in the sense of Banach-Mazur distance) to a subspace of an  $\ell_p$ -sum of finite-dimensional spaces when  $p < \infty$ , respectively, to a subspace of  $c_0$  when  $p = \infty$ .

*Proof.* The equivalence (b)  $\iff$  (c) is clear from Theorem 3 since  $(m_p)$  implies  $(M)$ ; (c)  $\iff$  (d) is clear from Theorems 3.3 and 3.5 in [10] since considered  $\ell_p$ -sums are reflexive, and subspaces of  $c_0$  fail to contain a copy of  $\ell_1$ . The equivalence (a)  $\iff$  (c) is proved in [10] for separable  $X$ ; it extends to the general case by Corollary 5 using the fact that if  $X$  has the MCAP, then every separable subspace of  $X$  is contained in a separable subspace having the MCAP.  $\square$

Finally, we come to the most important application of this paper – a characterization of  $(M_\infty)$ -spaces. The class of  $(M_\infty)$ -spaces was introduced and studied by R. Payá and W. Werner in [16], where it is proved that  $X$  is an  $(M_\infty)$ -space if and only if  $K(Z, X)$  is an  $M$ -ideal in  $L(Z, X)$  for every Banach space  $X$ . One of the main results of [13] states that a separable Banach space  $Y$  with the MCAP is an  $(M_\infty)$ -space if and only if  $K(\ell_1, Y)$  is an  $M$ -ideal in  $L(\ell_1, Y)$ . As we now see, this is also true for non-separable spaces.

**Corollary 7.** *A Banach space  $X$  is an  $(M_\infty)$ -space if and only if  $X$  has the MCAP, and  $K(\ell_1, X)$  is an  $M$ -ideal in  $L(\ell_1, X)$ .*

*Proof.* The *necessity* is clear from the above. The *sufficiency* immediately follows from Corollary 5 and the result of [13] stated just before Corollary 7, because the  $M$ -ideal property of  $K(\ell_1, X)$  in  $L(\ell_1, X)$  implies that  $K(\ell_1, Y)$  is an  $M$ -ideal in  $L(\ell_1, Y)$  for all subspaces  $Y$  of  $X$  [13].  $\square$

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