PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 126, Number 9, September 1998, Pages 2747–2753
S 0002-9939(98)04600-0

M-IDEALS OF COMPACT OPERATORS ARE SEPARABLY DETERMINED

EVE OJA

(Communicated by Palle E. T. Jorgensen)

ABSTRACT. We prove that the space $K(X)$ of compact operators on a Banach space X is an M-ideal in the space $L(X)$ of bounded operators if and only if X has the metric compact approximation property (MCAP), and $K(Y)$ is an M-ideal in $L(Y)$ for all separable subspaces Y of X having the MCAP. It follows that the Kalton-Werner theorem characterizing M-ideals of compact operators on separable Banach spaces is also valid for non-separable spaces: for a Banach space X, $K(X)$ is an M-ideal in $L(X)$ if and only if X has the MCAP, contains no subspace isomorphic to ℓ_1, and has property (M). It also follows that $K(Z,X)$ is an M-ideal in $L(Z,X)$ for all Banach spaces Z if and only if X has the MCAP, and $K(\ell_1,X)$ is an M-ideal in $L(\ell_1,X)$.

INTRODUCTION

A (closed) subspace Y of a Banach space X is called an M-*ideal* if there exists a projection P on the dual space X^* such that $\operatorname{Ker} P = Y^\perp$, and $\|x^*\| = \|Px^*\| + \|x^* - Px^*\|$ for all $x^* \in X^*$.

Already more than twenty years, many authors have studied conditions for the space $K(X)$ of compact operators on X to be an M-ideal in the space $L(X)$ of bounded operators (see [8, pp. 333-336] for a brief history and references) with the main aim to characterize those Banach spaces X for which $K(X)$ is an M-ideal in $L(X)$. Some years ago, N. J. Kalton and D. Werner [10] showed that, for a separable Banach space X, $K(X)$ is an M-ideal in $L(X)$ if and only if X has the metric compact approximation property (MCAP), contains no subspace isomorphic to ℓ_1, and has property (M) (property (M), which is an internal geometric property of X, will be defined in Section 2 below). Since the method of proof of this characterization implies the separability of X, the question whether the Kalton-Werner theorem is also valid for non-separable X remained open.

In Section 1 of the present note, we prove that M-ideals of compact operators are separably determined: $K(X)$ is an M-ideal in $L(X)$ if and only if X has the MCAP, and $K(Y)$ is an M-ideal in $L(Y)$ for all separable subspaces Y of X having the MCAP. In Section 2, this enables us to extend the Kalton-Werner theorem to non-separable spaces X. This also enables us to show that $K(Z,X)$ is an M-ideal in $L(Z,X)$ for all Banach spaces Z if and only if X has the MCAP, and $K(\ell_1,X)$ is an M-ideal in $L(\ell_1,X)$ (cf. Section 3).

Received by the editors February 14, 1997.
1991 *Mathematics Subject Classification.* Primary 46B28, 47D15, 46B20.
The author was partially supported by the Estonian Science Foundation Grant 3055.

Let us fix some more notation. In a Banach space X, we denote the closed unit ball by B_X. For a set $A \subset X$, its norm closure is denoted by $\overline{A}$, its linear span by $\operatorname{span} A$, and its convex hull by $\operatorname{conv} A$. The set of all weak* strongly exposed points of B_{X^*} is denoted by $w^*\text{-sexp}\, B_{X^*}$, and the identity operator of X is denoted by I_X or simply by I. Recall that X is said to have the MCAP if there is a net in $B_{K(X)}$ converging strongly to I. (This means that $K(X)$ contains a left 1-approximate unit (cf. e.g. [8, p. 294]).)

1. M-ideals of compact operators

The following characterization of M-ideals of compact operators will be needed below to prove that M-ideals of compact operators are separably determined.

Theorem 1. *Let X be a Banach space. Then $K(X)$ is an M-ideal in $L(X)$ if and only if X an M-ideal in X^{**}, and for all $S \in B_{K(X)}$ there is a net (K_α) in $B_{K(X)}$ such that $K_\alpha \to I$ strongly and*

$$\limsup \|S + I - K_\alpha\| \le 1.$$

Proof. The *necessity* is well known (cf. [11] or [8, p. 291] together with [19] or [8, p. 299]). *Sufficiency.* Recall that $K(X)$ is an M-ideal in $L(X)$ if and only if $K(X)$ is an M-ideal in $\mathcal{L} = \operatorname{span}(K(X) \cup \{I\})$ (cf. [14] or [8, p. 299], or [9] for separable X). Recall also that the MCAP of X implies the existence of a linear norm preserving extension operator $\Phi : K(X)^* \to \mathcal{L}^*$ (cf. e.g. [12]). This makes it possible to consider the topology $\sigma = \sigma(\mathcal{L}, \Phi(K(X)^*))$. In [18], it is essentially proved (for a simpler proof cf. [6]) that $K(X)$ is an M-ideal in $\mathcal{L}$ if and only if for all $S \in B_{K(X)}$ and $T \in B_{\mathcal{L}}$ there is a net (L_α) in $B_{K(X)}$ such that (L_α) is σ-convergent to T and

$$\limsup \|S + T - L_\alpha\| \le 1.$$

Consider $T = K + \lambda I \in B_{\mathcal{L}}$ (with $K \in K(X)$). Note that $|\lambda| \le 1$ because otherwise K would be invertible. Therefore $\lambda = re^{i\varphi}$ with $r \in [0,1]$. For $e^{-i\varphi}S$, choose the net (K_α). Since X has the unique extension property (following from the fact that X is an M-ideal in X^{**}), we have $x^*(K_\alpha^* x^*) \to x^{**}(x^*)$ for all $x^* \in X^*$, $x^{**} \in X^{**}$ (cf. [5] or [8, p. 118]). Set $L_\alpha = K_\alpha T = K_\alpha K + \lambda K_\alpha$. Then $L_\alpha \in B_{K(X)}$,

$$x^{**}(L_\alpha^* x^*) = (K^{**}x^{**})(K_\alpha^* x^*) + \lambda x^{**}(K_\alpha^* x^*) \to x^{**}(T^* x^*),$$

i.e. $(x^{**} \otimes x^*)(L_\alpha - T) \to 0$ for all $x^{**} \otimes x^* \in \mathcal{L}^*$, and (since $\|K_\alpha K - K\| \to 0$)

$$\begin{aligned}
\limsup \|S + T - L_\alpha\| &= \limsup \|S + re^{i\varphi} I - re^{i\varphi} K_\alpha\| \\
&= \limsup \|e^{-i\varphi} S + rI - rK_\alpha\| \\
&\le r \limsup \|e^{-i\varphi} S + I - K_\alpha\| + 1 - r \\
&\le 1.
\end{aligned}$$

We know that $B_{X^*} = \overline{\operatorname{conv}}\,(w^*\text{-sexp}\, B_{X^*})$ and X^* has the Radon–Nikodým property (because X is an M-ideal in X^{**}; cf. e.g. [8, pp. 126, 127], the latest implying $K(X)^* = \overline{X^{**} \otimes X^*}$ by [4, Theorem 1]. Hence, $L_\alpha \to T$ in the σ-topology whenever $(\Phi g)(L_\alpha - T) \to 0$ for all $g = x^{**} \otimes x^* \in K(X)^*$ with $x^{**} \in X^{**}$ and $x^* \in w^*\text{-sexp}\, B_{X^*}$. By the proof of [12, Lemma 3.4 (a)], such a g has a unique norm-preserving extension (to the whole $L(X)$). Thus, $\Phi g = x^{**} \otimes x^* \in \mathcal{L}^*$, and the result follows. $\square$

Remark 1. It is known (cf. [14] or [8, p. 299], or [9] for separable X) that $K(X)$ is an M-ideal in $L(X)$ if and only if there is a net (K_α) in $B_{K(X)}$ such that both $K_\alpha \to I_X$ and $K_\alpha^* \to I_{X^*}$ strongly, and $\limsup \|S+I-K_\alpha\| \leq 1$ for all $S \in B_{K(X)}$.

Remark 2. The following is clear from the proof of Theorem 1: if X is an M-ideal in X^{**} (in particular, if $K(X)$ is an M-ideal in $L(X)$), and $K_\alpha \to I$ strongly for some net $K_\alpha \in B_{K(X)}$, then $K_\alpha \to I$ in the $\sigma(L(X), \Phi(K(X)^*))$-topology for any linear norm preserving extension operator $\Phi : K(X)^* \to L(X)^*$.

The next theorem is the main result of the present note.

Theorem 2. *Let X be a Banach space. Then $K(X)$ is an M-ideal in $L(X)$ if and only if X has the MCAP, and $K(Y)$ is an M-ideal in $L(Y)$ for all separable subspaces Y of X having the MCAP.*

Proof. The *necessity* is well known (cf. [14] or [8, p. 301], or [9] for separable X). *Sufficiency*. We shall apply Theorem 1. To prove that X is an M-ideal in X^{**}, we need to show that every separable subspace of X is an M-ideal in its bidual (cf. [13] or [8, p. 115]). Consider a separable subspace Y of X. Since X has the MCAP, Y is contained in a separable subspace Z of X having the MCAP (the proof of this fact is the same as of the similar fact for the metric approximation property (cf. e.g. [17, p. 606])). Thus, $K(Z)$ is an M-ideal in $L(Z)$, which implies that Z is an M-ideal in Z^{**}. But then also its subspace Y is an M-ideal in Y^{**}.

Let us now make the following observation. If $K(Y)$ is an M-ideal in $L(Y)$ for a separable Banach space Y, and $K_n \to I_Y$ strongly for some sequence $(K_n) \subset B_{K(Y)}$, then, for all $S \in B_{K(Y)}$ and $\varepsilon > 0$, there is some $K \in \operatorname{conv}\{K_1, K_2, \dots\}$ such that $\|S - K + I_Y\| \leq 1 + \varepsilon/2$. [Due to Remark 2, the proof of this fact is the same as of the similar assertion about Banach spaces being M-ideals in their biduals in [13, Proposition 2.8, (i) $\Longrightarrow$ (ii)] (cf. also [8], p. 113), only using instead of the weak* topology the $\sigma(L(Y), \Phi(K(Y)^*))$-topology (where $\Phi : K(Y)^* \to L(Y)^*$ is the (unique linear) norm preserving extension operator).]

We denote by s_{op} the strong operator topology on $L(X)$, and suppose that the condition of Theorem 1 is not satisfied: for some $S \in B_{K(X)}$, there is no such net. Then there are $\varepsilon > 0$ and a convex s_{op} neighbourhood U_0 of I such that

$$\|S - K + I\| > 1 + \varepsilon \quad \forall K \in B_{K(X)} \cap U_0. \tag{1}$$

For all $n \in \mathbb{N}$, denote by Λ_n a finite $\varepsilon/4$-net in the subset $\{(\lambda_1, \dots, \lambda_n) : \lambda_k \geq 0, \lambda_1 + \cdots + \lambda_n = 1\}$ of ℓ_1^n. Let $(K_\alpha)_{\alpha \in \mathcal{A}}$ be a net in $B_{K(X)}$ converging to I in the s_{op}. We shall follow some ideas from the proofs of Proposition 2.8, (iii) $\Longrightarrow$ (iv), in [13] (cf. [8, p. 114]) and Theorem 18.2 in [17, p. 606] to pick a sequence $\alpha_1, \alpha_2, \dots$ in $\mathcal{A}$ and to define a separable subspace $Y \subset X$ so that $S(X) \subset Y$, $K_n(X) \subset Y$ for all $K_n = K_{\alpha_n}$, $K_n y \to y$ for all $y \in Y$, and $\|(S - K + I)|_Y\| > 1 + \varepsilon/2$ for all $K \in \operatorname{conv}\{K_1, K_2, \dots\}$. This will contradict the observation above, and complete the proof.

To begin, choose $K_1 = K_{\alpha_1} \in U_0$ such that $\|K_1 x - x\| < 1$ for all $x \in S(B_X)$. Assume that a convex s_{op} neighbourhood $U_{n-1} \subset U_{n-2}$ (where $U_{-1} = U_0$) and $K_n = K_{\alpha_n} \in U_{n-1}$ have been chosen. Consider $S_\lambda \in L(X), \lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda_n$, defined by $S_\lambda = S - (\lambda_1 K_1 + \cdots + \lambda_n K_n) + I$. For all $\lambda \in \Lambda_n$, select $x_\lambda \in B_X$ such that $\|S_\lambda x_\lambda\| > \|S_\lambda\| - \varepsilon/4$, and denote $C_n = \{x_\lambda : \lambda \in \Lambda_n\}$. Put

$$F_n = (1 + \varepsilon) B_{L(X)} + \operatorname{conv}\{K_1, \dots, K_n\} - S.$$

Since F_n is closed in the s_{op} and does not contain I (by (1)), there is a convex s_{op} neighbourhood $U_n \subset U_{n-1}$ of I such that $U_n \cap F_n = \emptyset$, which means

$$\|S - K + L\| > 1 + \varepsilon \quad \forall K \in \operatorname{conv}\{K_1, \dots, K_n\}, \quad \forall L \in U_n. \tag{2}$$

Choose $K_{n+1} = K_{\alpha_{n+1}} \in U_n$ such that

$$\|K_{n+1}x - x\| < \frac{1}{n+1} \quad \forall x \in C_1 \cup \cdots \cup C_n \cup S(B_X) \cup K_1(B_X) \cup \cdots \cup K_n(B_X).$$

Put $Y = \{x \in X : \lim K_n x = x\}$. It is straightforward that Y is closed, $S(X) \subset Y$, and $K_n(X) \subset Y$, $C_n \subset Y$ for all $n \in \mathbb{N}$. Moreover, if $K \in \operatorname{conv}\{K_1, K_2, \dots\}$, then $K \in \operatorname{conv}\{K_1, \dots, K_n\}$ for some n, and since $\|(S - K + I) - S_\lambda\| < \varepsilon/4$ for some $\lambda \in \Lambda_n$,

$$\|(S - K + I)|_Y\| > \|S_\lambda|_Y\| - \varepsilon/4 > \|S_\lambda\| - \varepsilon/2 > 1 + \varepsilon/2$$

by (2) and the fact that $I \in U_n$. □

2. Kalton-Werner theorem

Recall (cf. [9]) that a Banach space X is said to have *property* (M) if

$$\limsup \|x + x_n\| = \limsup \|y + x_n\|$$

whenever $\|x\| = \|y\|$, and (x_n) is a weakly null sequence in X; if

$$\limsup \|x^* + x_n^*\| = \limsup \|y^* + x_n^*\|$$

whenever $\|x^*\| = \|y^*\|$, and (x_n^*) is a weak*-null sequence in X^*, then X is said to have *property* (M^*). We also need the strong version of property (M^*), which we call *property* (sM^*), defined by bounded weak*-null nets (x_λ^*) instead of weak*-null sequences (x_n^*). It is shown in [14], that if X is separable, then properties (M^*) and (sM^*) are equivalent. The Kalton–Werner theorem mentioned above asserts that if a Banach space X is separable, then $K(X)$ is an M-ideal in $L(X)$ if and only if X has the MCAP, contains no subspace isomorphic to ℓ_1, and has property (M).

Theorem 3. *For a Banach space X the following assertions are equivalent.*

(a) *$K(X)$ is an M-ideal in $L(X)$.*
(b) *X has the MCAP, and has property (sM^*).*
(c) *X has the MCAP, is weakly compactly generated, and has property (M^*).*
(d) *X has the MCAP, contains no subspace isomorphic to ℓ_1, and has property (M).*

Proof. (a) $\Longrightarrow$ (b) is well known (cf. [7] and [14] or e.g. [8, p. 299]; cf. [9] for separable case).

(b) $\Longrightarrow$ (c). Property (sM^*) implies that X is an M-ideal in X^{**} (cf. [14] or [8, p. 297]; cf. [9] for separable case). But then X is weakly compactly generated (cf. [3] or e.g. [8, p. 142]).

(c) $\Longrightarrow$ (d). Let Y be an arbitrary separable subspace of X. We have to show that Y is not isomorphic to ℓ_1, and has property (M). Since X is weakly compactly generated, there exists a separable subspace Z containing Y, and a norm-one projection P of X onto Z (cf. e.g. [2, p. 149]). But then Z^* isometrically embeds into X^* by means of the formula $z^* \in Z^* \mapsto z^*P \in X^*$. This implies that Z also has property (M^*). By [9], since Z is separable, Z has property (M), and is an M-ideal in Z^{**}, in particular (cf. [11] or e.g. [8, p. 126]), Z is an Asplund space (i.e. every separable subspace of Z has a separable dual). Hence, Y has property (M), and is not isomorphic to ℓ_1.

(d) $\Longrightarrow$ (a) follows from Theorem 2 and the Kalton-Werner theorem. $\square$

Remark. In (c) of Theorem 3, the condition that X is weakly compactly generated clearly may be replaced by the separable complementation property (i.e. every separable subspace of X is contained in a separable subspace which is a range of a norm-one projection on X).

A Banach space X is said to have the compact approximation property (CAP) if there is a net in $K(X)$ converging strongly to the identity. Since a reflexive space with the CAP has the MCAP (cf. [1] or [5]), we can refine Theorem 3 for reflexive X as follows.

Corollary 4. *For a reflexive Banach space X, the following assertions are equivalent.*

(a) *$K(X)$ is an M-ideal in $L(X)$.*
(b) *X has the CAP, and has property (sM^*).*
(c) *X has the CAP, and has property (M^*).*
(d) *X has the CAP, and has property (M).*

Remark. For separable reflexive spaces X, Corollary 4 was obtained in [10]. The equivalence (a) $\Longleftrightarrow$ (b) of Corollary 4 was established in [12] (using an entirely different proof).

3. (M_p)-SPACES

Let $1 \leq p \leq \infty$. Following [15] (cf. also [8, p. 306]), we say that a Banach space X is an (M_p)-*space* if $K(X \oplus_p X)$ is an M-ideal in $L(X \oplus_p X)$. Note that (M_1)-spaces are finite dimensional [16], [8, p. 306], and therefore not of interest in the present context. Since every separable subspace of $X \oplus_p X$ is contained in $Y \oplus_p Y$ for some separable subspace Y of X with the MCAP whenever X has the MCAP (cf. the proof of Theorem 2), the next result follows immediately from Theorem 2.

Corollary 5. *Let $1 < p \leq \infty$. A Banach space X is an (M_p)-space if and only if X has the MCAP, and all separable subspaces of X with the MCAP are (M_p)-spaces.*

In [10], N. J. Kalton and D. Werner characterized separable (M_p)-spaces using the following stronger version of property (M). A Banach space X is said to have *property* (m_p) if

$$\limsup \|x + x_n\| = \|(\|x\|, \limsup \|x_n\|)\|_p$$

(where $\|\cdot\|_p$ denotes the ℓ_p^2-norm) whenever (x_n) is a weakly null sequence in X. For separable Banach spaces X, the following result was obtained in [10].

Corollary 6. *Let $1 < p \leq \infty$. For a Banach space X, the following assertions are equivalent.*

(a) *X is an (M_p)-space.*
(b) *$K(X)$ is an M-ideal in $L(X)$, and X has property (m_p).*
(c) *X has the MCAP, contains no subspace isomorphic to ℓ_1, and has property (m_p).*

(d) *X has the MCAP, and every separable subspace of X is almost isometric (in the sense of Banach-Mazur distance) to a subspace of an ℓ_p-sum of finite-dimensional spaces when $p < \infty$, respectively, to a subspace of c_0 when $p = \infty$.*

Proof. The equivalence (b) $\iff$ (c) is clear from Theorem 3 since (m_p) implies (M); (c) $\iff$ (d) is clear from Theorems 3.3 and 3.5 in [10] since considered ℓ_p-sums are reflexive, and subspaces of c_0 fail to contain a copy of ℓ_1. The equivalence (a) $\iff$ (c) is proved in [10] for separable X; it extends to the general case by Corollary 5 using the fact that if X has the MCAP, then every separable subspace of X is contained in a separable subspace having the MCAP. $\square$

Finally, we come to the most important application of this paper – a characterization of (M_∞)-spaces. The class of (M_∞)-spaces was introduced and studied by R. Payá and W. Werner in [16], where it is proved that X is an (M_∞)-space if and only if $K(Z, X)$ is an M-ideal in $L(Z, X)$ for *every* Banach space X. One of the main results of [13] states that a separable Banach space Y with the MCAP is an (M_∞)-space if and only if $K(\ell_1, Y)$ is an M-ideal in $L(\ell_1, Y)$. As we now see, this is also true for non-separable spaces.

Corollary 7. *A Banach space X is an (M_∞)-space if and only if X has the MCAP, and $K(\ell_1, X)$ is an M-ideal in $L(\ell_1, X)$.*

Proof. The *necessity* is clear from the above. The *sufficiency* immediately follows from Corollary 5 and the result of [13] stated just before Corollary 7, because the M-ideal property of $K(\ell_1, X)$ in $L(\ell_1, X)$ implies that $K(\ell_1, Y)$ is an M-ideal in $L(\ell_1, Y)$ for all subspaces Y of X [13]. $\square$

Acknowledgment

The author wishes to thank M. Põldvere for interesting discussions on the topic of this paper.

References

1. C.-M. Cho and W. B. Johnson, *A characterization of subspaces X of ℓ_p for which $K(X)$ is an M-ideal in $L(X)$*, Proc. Amer. Math. Soc. **93** (1985), 466–470. MR **86h:**46026
2. J. Diestel, *Geometry of Banach Spaces – Selected Topics*, Lecture Notes in Math., vol. 485, Springer-Verlag, Berlin, Heidelberg, and New York, 1975. MR **57:**1079
3. M. Fabian and G. Godefroy, *The dual of every Asplund space admits a projectional resolution of the identity*, Studia Math. **91** (1988), 141–151. MR **90b:**46032
4. M. Feder and P. Saphar, *Spaces of compact operators and their dual spaces*, Israel J. Math. **21** (1975), 38–49. MR **51:**13762
5. G. Godefroy and P. Saphar, *Duality in spaces of operators and smooth norms on Banach spaces*, Illinois J. Math. **32** (1988), 672–695. MR **89j:**47026
6. R. Haller and E. Oja, *Geometric characterizations of positions of Banach spaces in their biduals*, Arch. Math. **69** (1997), 227–233. CMP 97:16
7. P. Harmand and Å. Lima, *Banach spaces which are M-ideals in their biduals*, Trans. Amer. Math. Soc. **283** (1984), 253–264. MR **86b:**46016
8. P. Harmand, D. Werner, and W. Werner, *M-ideals in Banach spaces and Banach algebras*, Lecture Notes in Math., vol. 1547, Springer-Verlag, Berlin and Heidelberg, 1993. MR **94k:**46022
9. N. J. Kalton, *M-ideals of compact operators*, Illinois J. Math. **37** (1993), 147–169. MR **94b:**46028
10. N. J. Kalton and D. Werner, *Property (M), M-ideals, and almost isometric structure of Banach spaces*, J. reine angew. Math. **461** (1995), 137–178. MR **96m:**46022

11. Å. Lima, *On M-ideals and best approximation*, Indiana Univ. Math. J. **31** (1982), 27–36. MR **83b:**46021
12. Å. Lima, *Property (wM^*) and the unconditional metric compact approximation property*, Studia Math. **113** (1995), 249–263. MR **96c:**46019
13. Å. Lima, E. Oja, T. S. S. R. K. Rao, and D. Werner, *Geometry of operator spaces*, Michigan Math. J. **41** (1994), 473–490. MR **95h:**46027
14. E. Oja, *A note on M-ideals of compact operators*, Acta et Comment. Univ. Tartuensis **960** (1993), 75–92. MR **95a:**46026
15. E. Oja and D. Werner, *Remarks on M-ideals of compact operators on $X \oplus_p X$*, Math. Nachr. **152** (1991), 101–111. MR **92g:**47055
16. R. Payá and W. Werner, *An approximation property related to M-ideals of compact operators*, Proc. Amer. Math. Soc. **111** (1991), 993–1001. MR **91g:**46018
17. I. Singer, *Bases in Banach spaces* II, Editura Acad. R. S. România, Springer-Verlag, Bucureşti, 1981. MR **82k:**46024
18. D. Werner, *M-ideals and the "basic inequality"*, J. Approx. Th. **76** (1994), 21–30. MR **95i:**47080
19. W. Werner, *Inner M-ideals in Banach algebras*, Math. Ann. **291** (1991), 205–223. MR **93b:**46094

Institute of Pure Mathematics, Tartu University, Vanemuise 46, EE2400 Tartu, Estonia

E-mail address: eveoja@math.ut.ee