

ENDOMORPHISMS OF FINITE FULL TRANSFORMATION SEMIGROUPS

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ABSTRACT. We describe all endomorphisms of finite full transformation semigroups and count their number.

A *full transformation semigroup* \mathcal{T}_X on a set X is the set X^X of all transformations (i.e., self-maps) $X \rightarrow X$ of X with composition of transformations as multiplication. This is an important object in semigroup theory, combinatorics, many-valued logic, etc. Various properties of \mathcal{T}_X are known. In particular, Schreier [4] proved in 1936 that automorphisms of \mathcal{T}_X are inner: for every automorphism α there exists a uniquely determined element $g \in \mathcal{G}_X \subset \mathcal{T}_X$ of the symmetric group \mathcal{G}_X on X such that $\alpha(t) = gtg^{-1}$ for all $t \in \mathcal{T}_X$. Here the juxtaposition gt stands for the composition $g \circ t$ and the composition acts from the right to the left: $g \circ t(x) = g(t(x))$ for every $x \in X$. Thus the automorphism group of \mathcal{T}_X is naturally isomorphic to \mathcal{G}_X .

Surprisingly, no one has considered endomorphisms of \mathcal{T}_X . Our paper seems to be the first attempt at filling that gap. We consider the finite case only, that is, X is a finite set of cardinality n for $n \geq 0$.

We introduce a few notations and terms. Endomorphisms that are not automorphisms are called *proper*. The kernel congruence $\ker(\varepsilon)$ of an endomorphism ε is defined by $(s, t) \in \ker(\varepsilon) \Leftrightarrow \varepsilon(s) = \varepsilon(t)$ for any $s, t \in \mathcal{T}_X$. Δ_A is the identity relation on a set A . If $\varepsilon' = \varepsilon|_{\mathcal{G}_X}$ is the restriction of ε to \mathcal{G}_X , then $\ker(\varepsilon')$ also stands for the corresponding normal subgroup of \mathcal{G}_X . The *second projection* (also called the *range*) of $t \in \mathcal{T}_X$ is the set $\text{pr}_2 t = t(X)$. In particular, $\text{pr}_2(st) \subset \text{pr}_2 s$. The *rank* of t is the cardinality $|\text{pr}_2(t)|$ of $\text{pr}_2(t)$.

We can assume that $X = \{1, 2, \dots, n\}$ and write \mathcal{T}_n instead of \mathcal{T}_X . Analogously, \mathcal{G}_n stands for \mathcal{G}_X . We consider \mathcal{G}_n as a subgroup of \mathcal{G}_{n+1} consisting of all permutations that fix the point $n+1$. Also, \mathcal{A}_X denotes the alternating group on X . For $n = 3$ or $n \geq 5$, \mathcal{A}_n is the only nontrivial normal subgroup of \mathcal{G}_n , while \mathcal{G}_4 contains another nontrivial normal subgroup \mathcal{K} , Klein's four-group. For every $x \in X$, c_x denotes the constant transformation in \mathcal{T}_X that maps all elements of X onto x . For example, $c_4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 4 & 4 \end{pmatrix}$ in \mathcal{T}_4 .

Our main results are the following Theorem and Corollary. Their proof is followed by a Proposition that is another corollary to our main theorem.

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Theorem. (A) Choose a permutation g of X and define $\alpha^g(t) = gtg^{-1}$ for all $t \in \mathcal{T}_X$. Then α^g is an automorphism of \mathcal{T}_X .

(E) Choose $\beta, \gamma \in \mathcal{T}_X$ such that $\beta^3 = \beta$ and $\beta\gamma = \gamma\beta = \gamma^2 = \gamma$. If

$$\varepsilon_{\beta,\gamma}(t) = \begin{cases} \beta, & \text{for } t \in \mathcal{G}_X \setminus \mathcal{A}_X, \\ \beta^2, & \text{for } t \in \mathcal{A}_X, \\ \gamma, & \text{for } t \in \mathcal{T}_X \setminus \mathcal{G}_X, \end{cases}$$

then $\varepsilon_{\beta,\gamma}$ is an endomorphism of \mathcal{T}_X .

If $\beta^2 = \beta = \gamma$, then $\varepsilon_{\beta,\gamma}$ is a constant endomorphism that maps \mathcal{T}_X onto a trivial semigroup $\{\gamma\}$. If $\beta^2 = \beta \neq \gamma$, then $\varepsilon_{\beta,\gamma}$ is an endomorphism of rank 2 that maps \mathcal{T}_X onto a two-element semilattice $\{\beta, \gamma\}$ with $\gamma < \beta$. If $\beta \neq \beta^2 \neq \gamma$, then $\varepsilon_{\beta,\gamma}$ is an endomorphism of rank 3 that maps \mathcal{T}_X onto a three-element semigroup $\{\beta, \beta^2, \gamma\}$, where γ is a zero element and $\{\beta, \beta^2\}$ a two-element subgroup.

Conversely, every automorphism of \mathcal{T}_X has the form (A) and every proper endomorphism the form (E), except that \mathcal{T}_4 has 24 additional endomorphisms σ^g , $g \in \mathcal{G}_4$, defined as follows:

Each of the six cosets of \mathcal{K} in \mathcal{G}_4 contains exactly one element of \mathcal{G}_3 . If $t \in \mathcal{G}_4$, let $\sigma(t)$ be that (only) element of $\mathcal{K}t \cap \mathcal{G}_3$, and if $t \in \mathcal{T}_4 \setminus \mathcal{G}_4$, let $\sigma(t) = c_4$. Then $\sigma^g = \alpha^g \sigma$, that is, $\sigma^g(t) = g\sigma(t)g^{-1}$. In particular, $\sigma = \sigma^e$, where e is the identity element of \mathcal{G}_4 .

Corollary. Every proper endomorphism of \mathcal{T}_n has rank 1, 2, or 3, except that \mathcal{T}_4 also has additional endomorphisms of rank 7. There are

$$n! \sum_{m=1}^n \frac{m^{n-m}}{(n-m)!m!}$$

endomorphisms of rank 1,

$$n! \sum_{m=2}^n \sum_{r=1}^{m-1} \frac{m^{n-m} r^{m-r}}{(n-m)!(m-r)!r!}$$

endomorphisms of rank 2, and

$$n! \sum_{m=3}^n \sum_{k=1}^{\lfloor \frac{m-1}{2} \rfloor} \sum_{r=1}^{m-2k} \frac{m^{n-m} r^{m-k-r}}{2^k (n-m)!(m-2k-r)!k!r!}$$

endomorphisms of rank 3.

For $n = 4$, there are 24 endomorphisms of rank 7.

Thus \mathcal{T}_n has

$$n! \left[1 + \sum_{m=1}^n \sum_{k=0}^{\lfloor \frac{m-1}{2} \rfloor} \sum_{r=1}^{m-2k} \frac{m^{n-m} r^{m-k-r}}{2^k (n-m)!(m-2k-r)!k!r!} \right]$$

endomorphisms for $n > 1, n \neq 4$ and

$$n! \left[2 + \sum_{m=1}^n \sum_{k=0}^{\lfloor \frac{m-1}{2} \rfloor} \sum_{r=1}^{m-2k} \frac{m^{n-m} r^{m-k-r}}{2^k (n-m)!(m-2k-r)!k!r!} \right] = 345$$

endomorphisms for $n = 4$.

In particular, the orders of \mathcal{T}_n and $\text{End}(\mathcal{T}_n)$ for $1 \leq n \leq 9$ are:

$n =$	1	2	3	4	5	6	7	8	9
$ \mathcal{T}_n = n^n =$	1	4	27	256	3, 125	46, 656	823, 543	16, 777, 216	387, 420, 489
$ \text{End}(\mathcal{T}_n) =$	1	7	40	345	3, 226	38, 503	529, 614	8, 219, 025	141, 633, 028

Problem. Find an asymptotic estimate for $|\text{End}(\mathcal{T}_n)|_{n \rightarrow \infty}$, where $\text{End}(\mathcal{T}_n)$ is the semigroup of all endomorphisms of \mathcal{T}_n . It seems that the estimate can be found by the Laplace method and arguments from [2]. Is it true that $\frac{|\text{End}(\mathcal{T}_n)|}{|\mathcal{T}_n|}$ converges monotonically to 0? It appears to be so.

Constant (that is, of rank 1) endomorphisms of \mathcal{T}_n are in one-to-one correspondence with idempotents of \mathcal{T}_n . Harris and Schoenfeld (see [3]) found an asymptotic estimate for the number of idempotents in \mathcal{T}_n .

The proofs of the Theorem and Corollary follow from Lemmas 1–5.

Lemma 1. *Let $G \subset \mathcal{T}_X$ be a subgroup of \mathcal{T}_X for a finite or infinite X . All elements $g \in G$ have the same range $Y \subset X$, and the mapping $g \mapsto \bar{g} = g|_Y$, where $g|_Y$ is the restriction of g to Y , is an isomorphism of G onto a group of permutations of Y .*

Proof. Here “permutation” means a bijection of Y onto itself. Lemma 1 is folklore, and we give its brief proof for completeness’ sake.

For every $g, h \in G$ there exists $x \in G$ such that $gx = h$, and hence $\text{pr}_2 h = \text{pr}_2 gx \subset \text{pr}_2 g$. Analogously, $\text{pr}_2 g \subset \text{pr}_2 h$, so that $\text{pr}_2 h = \text{pr}_2 g$.

Let e be the identity element of G . Then $\text{pr}_2 \bar{g} \subset Y = \text{pr}_2 ge = g(\text{pr}_2 e) = g(Y) = \text{pr}_2 \bar{g}$, whence $\text{pr}_2 \bar{g} = Y$. Thus \bar{g} is a permutation of Y , and so $g\Delta_Y = \bar{g} = \Delta_Y \bar{g}$. It follows that $\bar{h}\bar{g} = h\Delta_Y \bar{g} = hg\Delta_Y = \overline{hg}$ and $g \mapsto \bar{g}$ is a homomorphism of G onto a group of permutations of Y . This homomorphism is an isomorphism because $\bar{g} = \bar{e} \Rightarrow g^2 = \bar{g}g = \bar{e}g = eg = g \Rightarrow g = e$. □

Lemma 2. *Every endomorphism of \mathcal{T}_n injective on \mathcal{G}_n is an automorphism.*

Proof. If ε is an endomorphism injective on \mathcal{G}_n , then \mathcal{G}_n is isomorphic to its image $G = \varepsilon(\mathcal{G}_n)$ under ε , and so $|G| = n!$. By Lemma 1, G is isomorphic to a group of permutations of a subset $Y \subset X$, where $Y = \text{pr}_2 e$ for e the identity element of G . Thus $n! = |G| \leq |Y|!$. It follows that $Y = X$, and hence $G = \mathcal{G}_n$.

Since $c_x g = c_x$ for all $g \in \mathcal{G}_n$, we obtain $\varepsilon(c_x)\varepsilon(g) = \varepsilon(c_x g) = \varepsilon(c_x)$. Here $\varepsilon(g)$ may be an arbitrary element of \mathcal{G}_n , and hence $\varepsilon(c_x)h = \varepsilon(c_x)$ for all $h \in \mathcal{G}_n$. This is possible only if $\varepsilon(c_x)$ is a constant endomorphism, say, $\varepsilon(c_x) = c_{f(x)}$, where $f(x)$ is a suitable element of X .

Obviously, $tc_x = c_{t(x)}$ for all $t \in \mathcal{T}_n$. Thus $c_{f(t(x))} = \varepsilon(c_{t(x)}) = \varepsilon(tc_x) = \varepsilon(t)\varepsilon(c_x) = \varepsilon(t)c_{f(x)} = c_{\varepsilon(t)(f(x))}$, so that $f(t(x)) = \varepsilon(t)(f(x))$ for all $x \in X$. Therefore, $ft = \varepsilon(t)f$ and, for $x, y \in X$ and $g \in \mathcal{G}_n$, $f(x) = f(y) \Rightarrow f(g(x)) = \varepsilon(g)(f(x)) = \varepsilon(g)(f(y)) = f(g(y))$, and hence $f(x) = f(y) \Rightarrow f(g(x)) = f(g(y))$ for every $g \in \mathcal{G}_n$. It follows that either $x = y$, and hence f is injective, or $f(x) = f(y)$ for all $x, y \in X$, and hence f is a constant transformation. In the latter case $f = c_a$ for some $a \in X$, and our formula $f(t(x)) = \varepsilon(t)(f(x))$ becomes $a = \varepsilon(t)(a)$ for every $t \in \mathcal{T}_n$. Thus a is a fixed point of $\varepsilon(t)$ for every t , and hence for every $g \in \mathcal{G}_n$, which is possible only if $X = \{a\}$. Therefore, f is injective for every finite X , and so $f, f^{-1} \in \mathcal{G}_n$. Now the formula $ft = \varepsilon(t)f$ can be rewritten as $\varepsilon(t) = ft f^{-1}$, which shows that ε is an automorphism of \mathcal{T}_n . □

Recall that every ideal of \mathcal{T}_n is of the form $I_k = \{t \in \mathcal{T}_n : \text{rank}(t) < k\}$ with $2 \leq k \leq n+1$ (see [1]). Thus \mathcal{T}_n is a disjoint union of its group of units \mathcal{G}_n and the maximal ideal I_n .

Lemma 3. *Every proper endomorphism ε of \mathcal{T}_n , $n \neq 4$, maps \mathcal{G}_n into elements $\alpha, \beta \in \mathcal{T}_n$ with $\alpha = \beta^2$ and $\beta^3 = \beta$. Also, ε maps $I_n = \mathcal{T}_n \setminus \mathcal{G}_n$ into a single idempotent $\gamma \in \mathcal{T}_X$ such that $\beta\gamma = \gamma\beta = \gamma$. Thus $\text{pr}_2 \varepsilon$ has cardinality 1, 2, or 3. Respectively, $\text{pr}_2 \varepsilon$ is a trivial semigroup $\{\gamma\}$, a semilattice $\{\alpha, \beta\}$ of order 2, or a two-element group $\{\alpha, \beta\}$ with zero γ adjoined.*

Proof. By Lemma 2, ε is not injective on \mathcal{G}_n , and so $\ker(\varepsilon)$ does not induce an identity congruence on \mathcal{G}_n . It follows from the description of congruence relations on \mathcal{T}_n (see [1]) that one of the equivalence classes of $\ker(\varepsilon)$ is either I_n or I_{n+1} . In the latter case $I_{n+1} = \mathcal{T}_n$, and hence ε maps \mathcal{T}_n onto the trivial semigroup $\{\gamma\}$ for some idempotent $\gamma \in \mathcal{T}_n$.

If $\ker(\varepsilon) = I_n$, then ε maps $I_n = \mathcal{T}_n \setminus \mathcal{G}_n$ into an idempotent γ . Also, $\ker(\varepsilon)$ decomposes \mathcal{G}_n into cosets modulo a nontrivial normal subgroup N of \mathcal{G}_n . If $n \neq 4$, then $N = \mathcal{G}_n$ or $N = \mathcal{A}_n$, the alternating group. In the former case ε maps \mathcal{G}_n into an idempotent $\alpha \in \mathcal{T}_n$, and $\varepsilon(\mathcal{T}_n)$ is a two-element semilattice $\{\alpha, \gamma\}$ with γ as the zero element. In the latter case \mathcal{G}_n/N is a two-element group, and hence ε maps N into an idempotent α and $\mathcal{G}_n \setminus N$ into β such that $\beta^2 = \alpha$ and $\beta^3 = \alpha\beta = \beta$. \square

It remains to describe proper endomorphisms of \mathcal{T}_4 .

Lemma 4. *All proper endomorphisms of \mathcal{T}_4 are either endomorphisms of ranks 1, 2 or 3 described in Lemma 3 or endomorphisms ε of rank 7, where $\text{pr}_2 \varepsilon$ is isomorphic to the symmetric group \mathcal{G}_3 with zero adjoined.*

There are 24 endomorphisms σ^g of rank 7; they correspond to 24 permutations $g \in \mathcal{G}_4$ of the symmetric group \mathcal{G}_4 and have the structure described in the Theorem.

Proof. The only difference with our proof of Lemma 3 is that \mathcal{G}_4 contains another nontrivial normal subgroup $N = \mathcal{K}$, which is Klein's four-group. It is easy to see that the factor group $\mathcal{G}_n/\mathcal{K}$ is isomorphic to \mathcal{G}_3 , the symmetric group of degree 3. Obviously, \mathcal{K} has index 6 in \mathcal{G}_4 and the complement I_4 of \mathcal{G}_4 in \mathcal{T}_4 forms a congruence class modulo $\ker(\varepsilon)$, where ε is our endomorphism. Thus the rank of ε is 7 and $\text{pr}_2 \varepsilon$ is isomorphic to \mathcal{G}_3^0 (the group \mathcal{G}_3 with zero adjoined).

The rest of the proof is based on two facts: (1) \mathcal{T}_4 contains exactly four subsemigroups S isomorphic to \mathcal{G}_3^0 , and (2) each of these four semigroups has exactly six automorphisms.

(1) We know that S is a group G with zero adjoined. Let t be that zero. Then t is an idempotent of \mathcal{T} of rank r , where $1 \leq r \leq 4$. Also, $gt = t$ for every $g \in G$, that is, each of the r elements of $\text{pr}_2 t$ is a fixed point of each g . Applying Lemma 1 we see that G is isomorphic to a group \bar{G} of permutations of a subset $Y \subset \{1, 2, 3, 4\}$, and r elements of $\text{pr}_2 t$ are fixed points of all elements $\bar{g} \in \bar{G}$. Thus the elements of \bar{G} can actually permute only $|Y| - r$ points, and so $3! = |G| \leq (|Y| - r)! \leq (4 - 1)!$. It follows that $|Y| - r = 3$, and hence $r = 1$ and $|Y| = 4$, which implies $Y = X$. Thus $t = c_x$ for some $x \in X$. Since $|G| = 6$, G is the group of all permutations of X with a fixed point x . We have four choices for x , which give us four choices for S .

(2) Assume that $x = 4$. Then elements of G actually permute only the three elements of $\{1, 2, 3\}$, and, since we identified \mathcal{G}_3 with an appropriate subgroup of

\mathcal{G}_4 , we see that $S = \mathcal{G}_3 \cup \{c_4\}$. Every automorphism of S leaves c_4 fixed and induces an automorphism of \mathcal{G}_3 . Since \mathcal{G}_3 has precisely six (inner) automorphisms, S has six automorphisms too.

Now it is easy to see that all endomorphisms of rank 7 have the form $\varepsilon^g = \alpha^g \varepsilon$ for $g \in \mathcal{G}_4$, so that $\varepsilon^g(t) = \varepsilon(t)^g = g\varepsilon(t)g^{-1}$ for every $t \in \mathcal{T}_4$. Clearly, we obtain different endomorphisms for different g . If we choose $\varepsilon = \sigma^\varepsilon$ as described in the Theorem, all other endomorphisms of rank 7 are $(\sigma^\varepsilon)^g = \sigma^g$. □

Our Theorem follows from Lemmas 3 and 4. □

Now we are ready to count the number of endomorphisms of \mathcal{T}_n . Since each automorphism of \mathcal{T}_n is inner, their total number is $n!$ and it remains to find the number of proper endomorphisms.

By Lemma 3, each proper endomorphism ε , $n \neq 4$, is determined by its range. The range is determined by two elements β and γ such that $\beta^3 = \beta$ and $\beta\gamma = \gamma\beta = \gamma^2 = \gamma$. These relations between β and γ are characterized in the following lemma.

Lemma 5. *Three (not necessarily distinct) transformations β , $\alpha = \beta^2$ and γ are the range of an endomorphism of \mathcal{T}_n if and only if*

- (i) *the restriction $\bar{\beta} = \beta|_Y$ of β to $Y = \text{pr}_2 \beta$ is an involution, that is, $\bar{\beta}$ is a permutation of Y such that $\bar{\beta}^2 = \text{id}$;*
- (ii) *every element x of $\text{pr}_2 \gamma$ is a fixed point of both γ and β ;*
- (iii) *if $x \notin Y$, then $\gamma(x) = \gamma(\beta(x))$;*
- (iv) *if (x, y) is a transposition in $\bar{\beta}$, that is, $\beta(x) = y$ and $\beta(y) = x$ for $x \neq y$, then $\gamma(x) = \gamma(y)$.*

Proof. Necessity. (i) It follows from $\beta^3 = \beta$ that $\{\beta, \beta^2\}$ is a two-element subgroup of \mathcal{T}_n . By Lemma 1, $\bar{\beta} = \beta|_Y$ is a permutation of Y and $\bar{\beta}^2 = \Delta_Y$, that is, $\bar{\beta}$ is an involution of Y . We do not exclude the case when $\bar{\beta} = \Delta_Y$.

(ii) Let $x \in \text{pr}_2 \gamma$. If $x = \gamma(y)$ for some $y \in X$ and $\delta\gamma = \gamma$ for $\delta \in \mathcal{T}_n$, then $\delta(x) = \delta\gamma(y) = \gamma(y) = x$. Thus $\beta\gamma = \gamma\gamma = \gamma$ implies $\gamma(x) = \beta(x) = x$.

(iii) This condition follows from $\gamma = \gamma\beta$.

(iv) If (x, y) is a transposition of $\bar{\beta}$, then $\gamma(x) = \gamma(\beta(y)) = \gamma(y)$.

Sufficiency. Suppose that (i)–(iv) hold. Then (i) implies $\beta^3 = \bar{\beta}^2\beta = \Delta_Y\beta = \beta$. Since $\gamma(x) \in \text{pr}_2 \gamma$, (ii) implies $\gamma^2(x) = \gamma(\gamma(x)) = \gamma(x)$ for all $x \in X$, that is, $\gamma^2 = \gamma$. Also by (ii), $\beta(\gamma(x)) = \gamma(x)$ for all $x \in X$, and hence $\beta\gamma = \gamma$.

It remains to prove that $\gamma\beta = \gamma$ or, equivalently, $\gamma(\beta(x)) = \gamma(x)$ for all $x \in X$. By (iv) this is so for $x \notin Y$. If $x \in Y$, then, by (i), x is either a fixed point of $\bar{\beta}$ (and hence of β) or a part of a transposition (x, y) of $\bar{\beta}$. In the former case $\gamma(\beta(x)) = \gamma(x)$. In the latter case, by (iv), $\gamma(\beta(x)) = \gamma(y) = \gamma(x)$. Thus $\gamma\beta = \gamma$. □

Proof of the Corollary. Proper endomorphisms of \mathcal{T}_n are in one-to-one correspondence with the pairs $\{\beta, \gamma\}$ that satisfy the conditions of Lemma 5. We count the number of these pairs in the following way.

First we classify these pairs for a given $Y = \text{pr}_2 \beta$. If Y is an m -element subset of X , then $1 \leq m \leq n$. We will choose $\bar{\beta}$ with $\text{pr}_2 \bar{\beta} = Y$, then extend $\bar{\beta}$ to β , and then choose an appropriate γ . Then we calculate the number of choices and add these numbers for all possible choices of Y .

By Lemma 5.(i), $\bar{\beta}$ is a permutation of Y whose cycles are either fixed points or transpositions. Thus $\bar{\beta}$ is completely determined by its transpositions. If there are k transpositions, they move $2k$ elements of Y . Let Z be the set of these $2k$

elements. There are $\binom{m}{2k}$ choices for Z . Obviously, $k \geq 0$. By Lemma 5.(ii) and by $\text{pr}_2 \gamma \subset Y$, we see that $2k < m$. Thus $0 \leq k \leq \lfloor \frac{m-1}{2} \rfloor$, where $\lfloor r \rfloor$ denotes the integral part of r .

To calculate the number of choices for k transpositions in Z , split Z into k disjoint *ordered* pairs of distinct elements. Each ordered pair has two components, and there are $\binom{2k}{k}$ ways of choosing the set of k first components for these pairs. To form ordered pairs, bijectively map k first components onto k remaining elements of Z . This can be done in $k!$ ways. Therefore, there are $\binom{2k}{k}k!$ different ways of choosing these k ordered pairs.

Each two-element subset can be turned into an ordered pair in two ways, and hence k disjoint transpositions can be turned into 2^k different sets of ordered pairs. Thus there are

$$\binom{m}{2k} \frac{(2k)!}{2^k k!} = \frac{m!}{2^k (m-2k)! k!}$$

ways to choose $\bar{\beta}$ of rank m .

To extend $\bar{\beta}$ to β we have to define $\beta(x) \in Y$ for all $x \in X \setminus Y$. Since $X \setminus Y$ contains $n - m$ elements, there are m^{n-m} ways of extending each $\bar{\beta}$. Thus, given Y , there are $\frac{m^{n-m} m!}{2^k (m-2k)! k!}$ choices for β . We can choose Y in $\binom{n}{m}$ different ways. It follows that there are

$$\binom{n}{m} \frac{m^{n-m} m!}{2^k (m-2k)! k!} = \frac{m^{n-m} n!}{2^k (n-m)! (m-2k)! k!}$$

choices for β of rank m .

Given β of rank m with k transpositions, we now choose γ . Let $\text{pr}_2 \gamma = W \subset Y$, with W containing r elements. By Lemma 4.(ii), W consists of fixed points of β . There are $m - 2k$ fixed points, so that $1 \leq r \leq m - 2k$, and there are $\binom{m-2k}{r}$ choices for W .

By Lemma 5.(ii), $\gamma(x) = x$ for every $x \in W$. It remains to define $\gamma(x) \in W$ for $x \notin W$. By Lemma 5.(iii), we need to define $\gamma(x)$ for $x \in Y \setminus W$ only. The set $Y \setminus W$ contains $m - r$ elements and includes our $2k$ transposed elements. To define γ for these $2k$ elements, by Lemma 5.(iv), we have to define γ only for k of these elements. It follows that we can define $\gamma(x)$ arbitrarily only for $m - r - k$ values of x . Thus, given W , there are r^{m-k-r} choices for γ .

It follows that there are $\binom{m-2k}{r} r^{m-k-r}$ choices for γ of rank r . Varying r between 1 and $m - 2k$, we obtain $\sum_{r=1}^{m-2k} \binom{m-2k}{r} r^{m-k-r}$ choices for γ and

$$\begin{aligned} & \frac{m^{n-m} n!}{2^k (n-m)! (m-2k)! k!} \sum_{r=1}^{m-2k} \binom{m-2k}{r} r^{m-k-r} \\ &= \sum_{r=1}^{m-2k} \frac{m^{n-m} r^{m-k-r} n!}{2^k (n-m)! (m-2k-r)! k! r!} \end{aligned}$$

choices for $\{\beta, \gamma\}$ with β of rank m .

It follows that the total number of proper endomorphisms of \mathcal{T}_n is

$$\sum_{m=1}^n \sum_{k=0}^{\lfloor \frac{m-1}{2} \rfloor} \sum_{r=1}^{m-2k} \frac{m^{n-m} r^{m-k-r} n!}{2^k (n-m)! (m-2k-r)! k! r!}.$$

This and part (A) prove the last two claims of the Corollary.

Endomorphisms of rank 1 are characterized by the condition $\beta = \gamma$, that is, $k = 0$ and $r = m$, which yields the first claim of the Corollary. The number of endomorphisms of rank 1 coincides with the number of idempotents of \mathcal{T}_n , and thus we recover a result of Tainiter (see [5]).

Endomorphisms of rank 2 are characterized by the condition $\beta^2 = \beta \neq \gamma$, that is, $k = 0$ and $r < m$. Thus $m \geq 2$, which produces the second claim of the Corollary.

Endomorphisms of rank 3 are characterized by the condition $k \neq 0$, which gives us the third claim of the Corollary and completes its proof. \square

Remark. The Theorem and Corollary make it possible to classify and count various special endomorphisms. For example, an endomorphism ε is called a *retraction* if it is idempotent (that is, $\varepsilon^2 = \varepsilon$). The image of a retraction is called a *retract*. We can easily describe retractions and retracts of \mathcal{T}_n .

Proposition. *An endomorphism ε of \mathcal{T}_n is a retraction if and only if it is one of the following:*

- (i) ε is an endomorphism of rank 1;
- (ii) ε maps elements of \mathcal{G}_n into the identity element e of \mathcal{G}_n , and elements of $\mathcal{T}_n \setminus \mathcal{G}_n$ into any idempotent γ different from e ;
- (iii) ε maps elements of the alternating group \mathcal{A}_n into e , the remaining permutations of $\mathcal{G}_n \setminus \mathcal{A}_n$ into an odd permutation $\beta \in \mathcal{G}_n$, which is an involution (that is, $\beta^2 = e$) with fixed points, and ε maps elements of $\mathcal{T}_n \setminus \mathcal{G}_n$ into $\gamma \in \mathcal{T}_n$ such that $\gamma^2 = \beta\gamma = \gamma\beta = \gamma$.

Also, \mathcal{T}_4 has four additional retractions. They are endomorphisms σ^g of rank 7 with $g \in \mathcal{K}$.

The number of retractions of rank 1 is $n! \sum_{m=1}^n \frac{m^{n-m}}{(n-m)!m!}$. The number of retractions of rank 2 is $n! \sum_{m=1}^{n-1} \frac{m^{n-m}}{(n-m)!m!}$. The number of retractions of rank 3 is

$$n! \sum_{k=1}^{\lfloor \frac{n+1}{4} \rfloor} \sum_{r=1}^{n-4k-2} \frac{r^{n-k-r}}{2^{2k+1}(n-4k-r-2)!(2k+1)!r!}.$$

The number of retractions of rank 7 for $n = 4$ is 4.

Proof. Necessity. It is clear that endomorphisms described in (i) and (ii) are retractions. If ε is an endomorphism of type (iii), then $\beta \neq \beta^2 = e$ because β is odd and e even. Also, $\gamma \neq e$ because $\gamma\beta = \gamma$ and $e\beta = \beta \neq \gamma$. Thus ε has rank 3. Since β is an odd permutation, it does not belong to \mathcal{A}_n , and hence $\varepsilon(\beta) = \beta$. Also, $\varepsilon(e) = e$, and $\varepsilon(\gamma) = \gamma$ because $\gamma \notin \mathcal{G}_n$. Thus $\varepsilon^2 = \varepsilon$ and ε is a retraction.

It remains to check that, if $n = 4$, σ^g are retractions for $g \in \mathcal{K}$. For every $t \in I_4$, $\sigma^g(t) = g\sigma(t)g^{-1} = gc_4g^{-1} = c_{g(4)} \in I_4$, and hence $\sigma^g(\sigma^g(t)) = \sigma^g(c_{g(4)}) = \sigma^g(t)$. If $t \in \mathcal{G}_4$ and $g \in \mathcal{K}$, then $t(tg^{-1})^{-1} = tgt^{-1} \in \mathcal{K}$, whence $\mathcal{K}t = \mathcal{K}tg^{-1}$. By our definition of σ , $\sigma(t) \in \mathcal{K}t$, and hence $\sigma^g(t) = g\sigma(t)g^{-1} \in g\mathcal{K}tg^{-1} = g\mathcal{K}t = \mathcal{K}t$, because $g\mathcal{K} = \mathcal{K}$. It follows that $\mathcal{K}\sigma^g(t) = \mathcal{K}t$ for every $t \in \mathcal{G}_4$, and so $\mathcal{K}\sigma^g(\sigma^g(t)) = \mathcal{K}\sigma^g(t)$, which means that $\sigma^g(\sigma^g(t)) = \sigma^g(t)$, because \mathcal{K} is the kernel of the group homomorphism $\sigma_{|\mathcal{G}_4}^g : \mathcal{G}_4 \rightarrow \mathcal{G}_4$. Thus $(\sigma^g)^2 = \sigma^g$ and σ^g is a retraction of \mathcal{T}_4 .

Sufficiency. Let ε be a retraction of \mathcal{T}_n . If $\text{rank}(\varepsilon) = 1$, then ε belongs to class (i).

Let $\text{rank}(\varepsilon) = 2$. By the Theorem the image of ε is $\{\beta, \gamma\}$, where β and γ are distinct idempotents. If $\beta \in I_n$, then $\gamma = \beta\gamma \in I_n$, and hence $\beta = \varepsilon(e) = \varepsilon^2(e) = \varepsilon(\varepsilon(e)) = \varepsilon(\beta) = \gamma$, which is a contradiction. Thus $\beta \in \mathcal{G}_n$. The only idempotent of \mathcal{G}_n is e , and so $\beta = e$ and ε belongs to class (ii).

Let $\text{rank}(\varepsilon) = 3$. By the Theorem the image of ε is $\{\beta, \beta^2, \gamma\}$. If $\beta \in I_n$, then $\beta^2 \in I_n$ and we obtain a contradiction as in the case of ε of rank 2. Thus $\beta \in \mathcal{G}_n$, and hence $\alpha = \beta^2 = e$ because α is an idempotent element of \mathcal{G}_n . It follows that β is an involution in \mathcal{G}_n . By Lemma 5.(ii), β has fixed points. If it is an even permutation, then $\beta \in \mathcal{A}_n$, and hence $\varepsilon(\beta) = \beta$. Thus, for every $t \in \mathcal{G}_n \setminus \mathcal{A}_n$, $\beta = \varepsilon(t) = \varepsilon^2(t) = \varepsilon(\beta) = \alpha$, which is a contradiction. Therefore, β is odd, and so ε belongs to class (iii).

Let $\text{rank}(\varepsilon) = 7$ (and hence $n = 4$ and $\varepsilon = \sigma^g$ for some $g \in \mathcal{G}_4$). For every $t \in \mathcal{G}_4$, $\sigma^g(\sigma^g(t)) = \sigma^g(t)$, which means $g\sigma(\sigma^g(t))g^{-1} = g\sigma(t)g^{-1}$, and this implies $\sigma(\sigma^g(t)) = \sigma(t)$. In the proof of necessity we saw that $\sigma^2 = \sigma$, and so $\sigma(t) = \sigma(\sigma^g(t)) = \sigma(g\sigma(t)g^{-1}) = \sigma(g)\sigma^2(t)\sigma(g^{-1}) = \sigma(g)\sigma(t)\sigma(g^{-1}) = \sigma(gtg^{-1})$. Thus $\sigma(gtg^{-1}t^{-1}) = \sigma(gtg^{-1})\sigma(t)^{-1} = e$. Therefore, $gtg^{-1}t^{-1} \in \ker(\sigma|_{\mathcal{G}_4}) = \mathcal{K}$ for all $t \in \mathcal{G}_4$.

Suppose that g is a transposition $(i j)$. If t is a 3-cycle $(i j k)$, then

$$(i j k) = (i j)(i j k)(j i)(i k j) = gtg^{-1}t^{-1} \in \mathcal{K}$$

(recall that we apply the factors in the product $gtg^{-1}t^{-1}$ from the right to the left). If g is a 3-cycle $(i j k)$, choose $t = (i j)$. We obtain

$$(i k j) = (i j k)(i j)(i k j)(j i) = gtg^{-1}t^{-1} \in \mathcal{K}.$$

If g is a 4-cycle $(i j k l)$ (where $\{i, j, k, l\} = \{1, 2, 3, 4\}$), choose $t = (i j)$. We obtain

$$(i k j) = (i j k l)(i j)(i l k j)(j i) = gtg^{-1}t^{-1} \in \mathcal{K}.$$

It follows that if g is a 2-, 3-, or 4-cycle, then \mathcal{K} contains a 3-cycle, which is not true. Thus either $g = e$ or g is a product of two disjoint transpositions, i.e., $g \in \mathcal{K}$.

It is clear that the number of retractions of rank 1 is the number of idempotents of \mathcal{T}_n , and the number of retractions of rank 2 is the number of idempotents less 1 (because retractions of rank 2 are in one-to-one correspondence with idempotents $\gamma \in I_n$). We skip a proof that the number of retractions of rank 3 is indeed given by the formula in the Proposition. The number of retractions of rank 7 is the order of Klein's four-group \mathcal{K} , that is, 4. \square

A description of all retracts of \mathcal{T}_n easily follows from the Proposition.

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