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A NOTE ON SEQUENCES LYING IN THE RANGE OF A VECTOR MEASURE VALUED IN THE BIDUAL

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ABSTRACT. Let X be a Banach space. It is unknown if every subset A of X lying in the range of an X^{**}-valued measure is actually contained in the range of an X-valued measure. In this paper we solve this problem in the case when we consider only vector measures of bounded variation.

INTRODUCTION

Let X be a Banach space. It is still unknown if every subset A of X lying in the range of an X^{**}-valued measure is actually contained in the range of an X-valued measure. In this paper we only consider vector measures of bounded variation, and solve this problem by exhibiting a sequence (x_n) in the $\mathcal{L}_\infty$ Banach space Y of Bourgain and Delbaen [BD] that lies in the range of a Y^{**}-valued measure of bounded variation but is not contained in the range of a Y-valued measure with bounded variation. The proof is based in the following result that we prove in section 2:

A bounded sequence (x_n) in X lies inside the range of an X^{**}-valued measure with bounded variation iff $(\alpha_n x_n)$ is contained in the range of an X-valued measure of bounded variation for every $(\alpha_n) \in c_0$.

We start by explaining some basic notation used in this paper. In general, our operator and vector measure terminology and notation follow [Ps] and [DU]. We only consider real Banach spaces. If X is such a space, B_X will denote its closed unit ball. The phrase "range of an X-valued measure" always means a set of the form $rg(F) = \{F(A) \ : \ A \in \Sigma\}$, where Σ is a σ-algebra of subsets of a set Ω and $F : \Sigma \to X$ is countably additive.

THE MAIN STEP

For simplicity, we denote by $R_{bv}(X)$ the vector space of all sequences (x_n) in X lying in the range of an X-valued measure of bounded variation. We have obtained the following result.

Theorem 1. *Let (x_n) be a bounded sequence in X. The following statements are equivalent:*

i) $(x_n) \in R_{bv}(X^{**})$,

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ii) $(\alpha_n x_n) \in R_{bv}(X)$ *for every* $(\alpha_n) \in c_0$,
iii) $(\alpha_n x_n) \in R_{bv}(X^{**})$ *for every* $(\alpha_n) \in c_0$.

Proof. i)⇒ii) Suppose (x_n) is a sequence lying in the range of a vector measure valued in the bidual space X^{**} and having bounded variation. By [Pi1, Lemma 2] the operator $\Sigma e_n^* \otimes x_n : \ell_1 \to X$ is integral (here (e_n^*) is the unit basis of ℓ_∞). If (α_n) is a null sequence of real numbers, the operator $\Sigma e_n^* \otimes \alpha_n x_n : \ell_1 \to X$ is the composition of $\Sigma e_n^* \otimes x_n$ with the diagonal compact operator $(\beta_n) \in \ell_1 \to (\alpha_n \beta_n) \in \ell_1$. So Grothendieck's Theorem [DU, p.252] tells us that $\Sigma e_n^* \otimes (\alpha_n x_n) : \ell_1 \to X$ is nuclear. Again it follows from [Pi1, Lemma 2] that $(\alpha_n x_n)$ belongs to $R_{bv}(X)$.

ii)⇒iii) This is obvious.

iii)⇒i) By hypotheses we can consider the linear map

$$U : (\alpha_n) \in c_0 \longrightarrow \sum_{n=1}^{\infty} e_n^* \otimes \alpha_n x_n \in I(\ell_1, X).$$

It is a standard argument to prove that U has closed graph. Since $\ell_1^* \simeq \ell_\infty$ has the metric approximation property, $\mathcal{N}(\ell_1, X)$ is isomorphically isometric to a subspace of $I(\ell_1, X)$ [J, p.410]. As U maps each finite sequence $(\alpha_1, \alpha_2, \dots, \alpha_n, 0, \dots)$ into a nuclear operator, it is easy to prove that the rank of U is contained in $\mathcal{N}(\ell_1, X)$. We also denote by U the operator

$$(\alpha_n) \in c_0 \longrightarrow \Sigma e_n^* \otimes \alpha_n x_n \in \mathcal{N}(\ell_1, X).$$

Its dual operator U^* takes $\mathcal{L}(X, \ell_1^{**})$ into ℓ_1; in particular, $\mathcal{K}(X, \ell_1)$ into ℓ_1. Let us prove that $U^*(\Sigma x_n^* \otimes e_n) = (\langle x_n, x_n^* \rangle)_n$ for every $T = \Sigma x_n^* \otimes e_n \in \mathcal{K}(X, \ell_1)$ (here (e_n) is the unit basis of ℓ_1). Using the trace duality, we obtain the following equalities:

$$U^*(T)_m = \langle u_m, U^*(T) \rangle = \langle U(u_m), T \rangle = \langle e_m^* \otimes x_m, T \rangle$$

$$= tr(T \circ (e_m^* \otimes x_m)) = \langle (\langle x_n, x_n^* \rangle), e_m^* \rangle = \langle x_m, x_m^* \rangle$$

for all $m \in \mathbb{N}$ and for all $T \in \mathcal{K}(X, \ell_1)$, (u_m) being the unit basis of c_0. To conclude the proof it suffices to notice that the adjoint of the operator

$$U^* : \mathcal{K}(X, \ell_1) \longrightarrow \ell_1$$

is defined by

$$(\alpha_n) \in \ell_\infty \longrightarrow \Sigma e_n^* \otimes \alpha_n x_n \in I(\ell_1, X^{**}),$$

which can be proved in the same way using the trace duality. Then we have obtained that $\Sigma e_n^* \otimes x_n : \ell_1 \to X^{**}$ is integral and, therefore, so is $\Sigma e_n^* \otimes x_n : \ell_1 \to X$. □

If we consider countable additive vector measures, there is not an analogous theorem. To see this, let X be a non-reflexive $\mathcal{L}_\infty$ space. By the non-reflexivity, there is a bounded sequence (x_n) in X that is not contained in the range of any X^{**}-valued measure. Nevertheless, by [PR, Theorem 3.6] $(\alpha_n x_n)$ lies inside the range of an X-valued measure for every $(\alpha_n) \in c_0$.

Using the same method as in the proof of Theorem 1, we are going to prove a partial result in this general context. As in [PR], we denote by $R(X)$ the vector space of all sequences $(x_n) \in X$ lying in the range of a vector measure. If (x_n) belongs to $R(X)$, we put $\|(x_n)\|_r = inf\|m\|$, where the infimum is taken over all vector measures m satisfying

$$\{x_n : n \in \mathbb{N}\} \subset rg(m).$$

$(R(X), \|\cdot\|_r)$ is itself a Banach space.

Proposition 2. *Suppose X is a Banach space satisfying the condition $\mathcal{K}(X, \ell_1) \subset \Pi_1(X, \ell_1)$, and (x_n) is a bounded sequence in X. The following statements are equivalent:*

i) $(x_n) \in R(X)$,
ii) $(\alpha_n x_n) \in R(X)$ *for every* $(\alpha_n) \in c_0$,
iii) $(x_n) \in R_{bv}(X^{**})$.

Proof. First of all, notice that the inclusion $\Pi_1(X, \ell_1) \subset \mathcal{K}(X, \ell_1)$ holds for every Banach space X.

i)⇒ii) This is obvious since the closed convex hull of a range is the range of another vector measure [DU, p.274].

ii)⇒iii) In this part of the proof we follow the ideas of Theorem 1. We consider the linear map

$$U : (\alpha_n) \in c_0 \longrightarrow (\alpha_n x_n) \in R(X).$$

It is continuos because its graph is closed.

If $S = \Sigma x_n^* \otimes e_n \in \Pi_1(X, \ell_1)$, by [Pi2, Proposition 2] the linear form

$$\psi_s : (z_n) \in R(X) \longrightarrow \sum_{n=1}^{\infty} \langle z_n, x_n^* \rangle \in \mathbb{R}$$

is well-defined, continuous, and $\|\psi_s\| \leq \pi_1(S)$. Then the linear map

$$S = \sum_{n=1}^{\infty} x_n^* \otimes e_n \in \Pi_1(X, \ell_1) \rightarrow U^*(\psi_s) \in \ell_1$$

is continuous. In the same way as in Theorem 1 we can prove that this map is defined by

$$\Sigma x_n^* \otimes e_n \in \Pi_1(X, \ell_1) \rightarrow (\langle x_n, x_n^* \rangle) \in \ell_1.$$

Since $K(X, \ell_1) = \Pi_1(X, \ell_1)$, dualizing again we have

$$(\alpha_n) \in \ell_\infty \rightarrow \Sigma \alpha_n x_n \otimes e_n^* \in I(\ell_1, X^{**}).$$

This shows that $\Sigma x_n \otimes e_n^* : \ell_1 \rightarrow X$ is integral. So $(x_n) \in R_{bv}(X^{**})$.

iii)⇒i) In [Pi1] it is proved that every sequence in X lying in the range of a vector measure of bounded variation and valued in some superspace of X actually belongs to $R(X)$. □

In [MR] it is proved that the condition $K(X, Y) \subset \Pi_1(X, Y)$ implies that $\mathcal{L}(X, Y) = \Pi_1(X, Y)$, whenever X or Y has the bounded approximation property. So, we have the following equivalence:

$$K(X, \ell_1) \subset \Pi_1(X, \ell_1) \Longleftrightarrow \mathcal{L}(X, \ell_1) = \Pi_1(X, \ell_1).$$

Recall that a Banach space X is called a G.T.-space if $\mathcal{L}(X, \ell_2) = \Pi_1(X, \ell_2)$ [Ps]. In [G, Theorem 2] it is proved that the following statements are equivalent:

i) $\mathcal{L}(X, \ell_1) = \Pi_1(X, \ell_1)$,
ii) $\mathcal{L}(X^*, \ell_1) = \Pi_1(X^*, \ell_1)$.

From this equivalence the next result follows easily.

Proposition 3. *Let X be a Banach space. The following statements are equivalent:*
i) $\mathcal{L}(X, \ell_1) = \Pi_1(X, \ell_1)$,
ii) *X and X^* are G.T.-spaces.*

In fact, if a Banach space X satisfies one of the two above conditions, then we have

$$\mathcal{L}(X, \ell_1) = \Pi_1(X, \ell_1) = \Pi_2(X, \ell_1).$$

The counterexample

Let Y be the $\mathcal{L}_\infty$ space of Bourgain and Delbaen [BD]. They proved that Y satisfies the Radon-Nikodým property and has a normalized unconditional basis (e_n). We are going to show that (e_n) lies inside the range of a Y^{**}-valued measure of bounded variaton, but it does not belong to $R_{bv}(Y)$. In view of Theorem 1, if we prove that $(\alpha_n e_n) \in R_{bv}(Y^{**})$ for every $(\alpha_n) \in c_0$, then the sequence (e_n) will belong to $R_{bv}(Y^{**})$. Notice that it suffices to consider null sequences (α_n) such that $\alpha_n \neq 0$ for all $n \in \mathbb{N}$. Take such a sequence (α_n). Since Y is an $\mathcal{L}_\infty$-space, it follows from [PR] that $(\alpha_n e_n)$ lies in the range of a Y-valued measure. As $(\alpha_n e_n)$ is an unconditional basis of Y, by [AD] we have that $(\alpha_n e_n)$ is a weakly ℓ_2-summable sequence. Then the map

$$A : (\beta_n) \in \ell_1 \longrightarrow \sum_{n=1}^{\infty} \beta_n \alpha_n e_n \in Y$$

is absolutely summing because it admits the factorization $A = B \circ I$, I being the inclusion map from ℓ_1 to ℓ_2 and B the operator defined by $B(\beta_n) = \sum_{n=1}^{\infty} \beta_n \alpha_n e_n$. Now recall that absolutely summing and integral operators into an $\mathcal{L}_\infty$-space are the same [SR]. So, by [Pi1] the sequence $(\alpha_n e_n)$ lies in the range of a Y^{**}-valued measure having bounded variation. Then we have proved that (e_n) belongs to $R_{bv}(Y^{**})$. A final note in [Pi1] tells us that (e_n) is actually in the range of a Y-valued measure. Therefore, (e_n) is an unconditional basis of Y lying in the range of a vector measure. Again a call to [AD] tells us that (e_n) is a weakly ℓ_2-summable sequence.

Finally, suppose (e_n) is contained in the range of a Y-valued measure with bounded variation. Then the operator $T : \ell_1 \longrightarrow Y$, defined by $T(\beta_n) = \Sigma \beta_n e_n \in Y$ for all $(\beta_n) \in \ell_1$, should be Pietsch-integral [Pi1]. Since Y is a Radon-Nikodým space, it follows from [DU, Theorem VII.4.8] that T is even nuclear. Nevertheless, T is not a compact operator.

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