

SHARP LOG-SOBOLEV INEQUALITIES

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ABSTRACT. We show existence of a wide variety of Log-Sobolev inequalities in which the constant is exactly that required by the Poincaré inequality which may be inferred from the Log-Sobolev.

We are given a smooth compact Riemannian manifold M , intrinsic gradient ∇ , and volume element $d\mu$ — we assume W.L.O.G. that $\mu(M) = 1$ — and a smooth positive function m , giving rise to a probability measure $md\mu = dm$, for which we have a log-Sobolev inequality (LSI):

$$(1) \quad \rho \int_M |\nabla f|^2 dm \geq \int_M |f|^2 \ln |f|^2 dm - \int_M |f|^2 dm \ln \int_M |f|^2 dm.$$

It is well known that $\rho \geq 2/\lambda$, where λ is the first non-zero eigenvalue of the Schrödinger operator

$$f \rightarrow \Delta f + \nabla f \cdot \frac{\nabla m}{m},$$

(Δ the usual Laplacian) attached to the Dirichlet form in (1).

We call the LSI sharp if $\rho = 2/\lambda$. Many examples of sharp inequalities are known, the most familiar arising from M the sphere with the usual metric scaled to give M unit volume, and $m = 1$.

We will show here that for every compact homogeneous Riemannian manifold, there are a continuum of choices of m for which sharp LSI's exist.

We follow the notation and conclusions of [1], which we now briefly review. For every M as described initially there is a least constant $\rho_0(M)$, the hypercontractive constant for M , such that

$$(2) \quad \rho_0(M) \int |\nabla f|^2 d\mu \geq \int |f|^2 \ln |f|^2 d\mu - \int |f|^2 d\mu \ln \int |f|^2 d\mu.$$

For any positive $\rho < \rho_0(M)$ there is a minimum $\alpha(\rho)$, called the defect, such that

$$(3) \quad \rho \int |\nabla f|^2 d\mu \geq \int |f|^2 \ln |f|^2 d\mu - \int |f|^2 d\mu \ln \int |f|^2 d\mu - \alpha(\rho) \int |f|^2 d\mu,$$

with the inequality an equality for a real non-trivial minimizing function $f = J_\rho$ satisfying $\int J_\rho^2 d\mu = 1$ and the non-linear P.D.E.

$$(4) \quad \rho \Delta J_\rho + J_\rho \ln J_\rho^2 - \alpha(\rho) J_\rho = 0.$$

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(If $\rho_0(M) > 2/\lambda$, where λ is the first non-trivial eigenvalue of the Laplacian, then there is known [1] to be a non-trivial ($\neq 1$) minimizer for $\rho = \rho_0(M)$ as well.)

If one replaces f in the defective LSI (3) by fJ_ρ , it is thrown into the non-defective form:

$$(5) \quad \rho \int |\nabla f|^2 J_\rho^2 d\mu \geq \int |f|^2 \ln |f|^2 J_\rho^2 d\mu - \int |f|^2 J_\rho^2 d\mu \ln \int |f|^2 J_\rho^2 d\mu,$$

i.e., a version of our equation (1).

We know, as noted earlier, that in equation (5), $\rho \geq 2/\tau$ where τ is the first non-trivial eigenvalue of the Schrödinger operator attached to the Dirichlet form in (5); i.e., the operator

$$(6) \quad f \rightarrow \Delta f + 2 \frac{\nabla J_\rho}{J_\rho} \nabla f.$$

Our principal result is

Theorem. *If M is a compact homogeneous Riemannian manifold, then (5) above is a sharp LSI.*

(Thus we have distinct choices of probability measure with sharp LSI's for M for each $\rho < \rho_0(M)$, and for $\rho = \rho_0(M)$ in some cases.)

Proof. In the P.D.E. for an eigenvector of the operator (6)

$$\Delta f + 2 \frac{\nabla J_\rho}{J_\rho} \nabla f + \theta f = 0,$$

make the Liouville substitution $f = g/J_\rho$, which throws it into the form

$$\Delta g - \frac{\Delta J_\rho}{J_\rho} g + \theta g = 0,$$

which becomes using the differential equation (4) for J_ρ ,

$$(7) \quad \Delta g + \frac{1}{\rho} [\ln J_\rho^2 - \alpha(\rho)] g + \theta g = 0.$$

The eigenvector J_ρ belongs to the smallest eigenvalue $\theta = 0$.

Next take the differential equation (4) satisfied by J_ρ and apply a Killing vector X . X commutes with the Laplacian, and so we obtain

$$\Delta(XJ_\rho) + (2/\rho)(1 + \ln J_\rho)(XJ_\rho) - \frac{\alpha(\rho)}{\rho}(XJ_\rho) = 0$$

which is a version of (7) with $\theta = 2/\rho$ and $g = XJ_\rho$. Now since J_ρ is not constant, and M is homogeneous, XJ_ρ cannot be zero for all Killing vectors. So for the first non-trivial eigenvalue of (6), we have $\tau \leq 2/\rho$. Since we know $\rho \geq 2/\tau$, we must have $\rho = 2/\tau$, and the LSI is sharp.

This completes the proof of our theorem. \square

It would be quite interesting to get sharp inequalities on the line, other than the usual one.

REFERENCES

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