

## DISTRIBUTIONS SUPPORTED IN A HYPERSURFACE AND LOCAL $h^p$

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ABSTRACT. We give a necessary condition for a distribution with compact support in a hypersurface to be in the local Hardy space  $h^p(\mathbf{R}^n)$ . We apply this condition to prove a result distinguishing two types of Hardy spaces of distributions on a smooth domain  $\Omega \subset \mathbf{R}^n$ .

### 1. INTRODUCTION

The aim of this paper is to solve a problem which arose in the study of various local Hardy spaces of distributions on a smooth bounded domain  $\Omega \subset \mathbf{R}^n$ . In [CKS], Chang, Krantz and Stein defined the spaces  $h_r^p(\Omega)$  and  $h_z^p(\Omega)$ ,  $0 < p \leq 1$ , (related to the Hardy spaces defined by Miyachi [M] and Jonsson, Sjögren, and Wallin [JSW]) and proceeded to prove regularity results for the Dirichlet and Neumann problems in terms of these spaces. Subsequent work by Chang, Stein and the author ([CDS]) revealed that for certain values of  $p$ , namely  $p \neq \frac{n}{n+k}$ , where  $k = 0, 1, 2, \dots$ , these two spaces can be identified. However, it was proved that in the one-dimensional case  $n = 1$ , the spaces are not equivalent for the values  $p = 1, \frac{1}{2}, \frac{1}{3}, \dots$ . Thus it remained to determine whether the two spaces were different for the “critical” values of  $p$  for  $n > 1$ .

The space  $h_r^p(\Omega)$  consists of those distributions on  $\Omega$  which are the restriction to  $\Omega$  of distributions in  $h^p(\mathbf{R}^n)$ , the local Hardy spaces defined by Goldberg (see [G].) The space  $h_z^p(\Omega)$  is a subspace of  $h_r^p(\Omega)$ , consisting of those distributions on  $\Omega$  which are the restriction to  $\Omega$  of distributions in  $h^p(\mathbf{R}^n)$  which also vanish outside  $\bar{\Omega}$ . Note that two such extensions of the same element of  $h_z^p(\Omega)$  differ by a distribution in  $h^p(\mathbf{R}^n)$  which is supported on  $\partial\Omega$ . Thus in order to understand  $h_z^p(\Omega)$ , it is important to understand the nature of distributions in  $h^p(\mathbf{R}^n)$  which are supported in a smooth hypersurface, or (by a choice of local coordinates) in a hyperplane.

Some simple sufficient conditions can be stated for a distribution with compact support in a hyperplane to be in  $h^p(\mathbf{R}^n)$  (see for example [S], 5.18, which implies that  $f \in h^p(\mathbf{R}^n)$  if its order is strictly less than  $\frac{1}{p} - n$ ). Here we prove a necessary condition, namely that the order of the distribution “in the normal direction” must be strictly less than  $n(\frac{1}{p} - 1)$  (Theorem 1). We do this by means of a lower bound on the integral of the maximal function of such a distribution on planes parallel to

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the hyperplane (Lemma 1). It is this lemma that allows us to construct a counter-example showing that for  $p = \frac{n}{n+k}$ ,  $k = 0, 1, 2, \dots$ ,  $h_r^p(\Omega) \neq h_z^p(\Omega)$  (Theorem 2).

## 2. $h^p$ DISTRIBUTIONS SUPPORTED IN A HYPERPLANE

For  $n \geq 2$ , write  $\mathbf{R}^n = \mathbf{R}^{n-1} \times \mathbf{R}$ , and let  $\Pi = \mathbf{R}^{n-1} = \{x_n = 0\}$  be our chosen hyperplane. If  $f$  is a distribution in  $\mathbf{R}^n$  of order  $K$  with compact support in  $\Pi$ , then

$$(1) \quad f(\phi) = \sum_{k=0}^K f_k(\phi_k),$$

where each  $f_k$  is a distribution of compact support and order  $K - k$  in  $\mathbf{R}^{n-1}$ , and

$$\phi_k(x') = \partial_{x_n}^k \phi(x', x_n)|_{x_n=0}$$

(see [H], Theorem 2.3.5). Let  $N$  be the largest integer such that  $f_N$  is non-zero. We will call  $N$  the *order of  $f$  in the normal direction*. Then we have the following:

**Theorem 1.** *A distribution  $f$  in  $\mathbf{R}^n$  with compact support in  $\Pi$  is in the local Hardy space  $h^p(\mathbf{R}^n)$ ,  $0 < p \leq 1$ , only if its order  $N$  in the normal direction satisfies*

$$N < n(1/p - 1).$$

Note that for  $p = 1$ , this agrees with the fact that the only distribution supported in  $\Pi$  which is in  $h^1$  is the zero distribution, since elements of  $h^1$  are functions.

To characterize the spaces  $h^p(\mathbf{R}^n)$ , we will use a “local grand maximal function”  $m(f)$  defined using “normalized bump functions”. Such a function  $\varphi$  is smooth ( $\mathcal{C}^\infty$ ), supported in a ball of radius  $r$ , and satisfies

$$|\partial^\alpha \varphi| \leq r^{-n-|\alpha|}$$

for all multi-exponents  $\alpha = (\alpha_1, \dots, \alpha_n)$  with

$$|\alpha| = \alpha_1 + \dots + \alpha_n \leq N_p + 1,$$

where  $N_p$  is the greatest integer in  $n(1/p - 1)$ .

For a tempered distribution  $f$ , we define

$$m(f)(x) = \sup_{\varphi_r^x} f(\varphi_r^x),$$

where the supremum is taken over all normalized bump functions  $\varphi_r^x$  supported in balls of radii  $r \leq 1$  containing  $x$ . Then  $f \in h^p(\mathbf{R}^n)$  if and only if

$$m(f) \in L^p(\mathbf{R}^n).$$

That this is equivalent to the definition given by Goldberg ([G]) can be seen from the remarks in [S], 5.17.

In view of this characterization, the theorem will follow from the following

**Lemma 1.** *Let  $f$  be a distribution in  $\mathbf{R}^n$  with compact support in  $\Pi$ , and let  $N$  be the order of  $f$  in the normal direction. Then for  $0 < p \leq 1$ , the local grand maximal function  $m(f)$  satisfies*

$$\int_{\mathbf{R}^{n-1}} m(f)^p(x', x_n) dx' \geq C x_n^{n-1-(n+N)p}$$

for all sufficiently small  $x_n > 0$ .

Note that when  $N \geq n(1/p - 1)$ ,  $n - 1 - (n + N)p \leq -1$  so the lemma shows that  $m(f)^p$  is not locally integrable near  $x_n = 0$ . This proves the theorem.

*Proof of Lemma 1.* Write

$$f(\phi) = \sum_{k=0}^N f_k(\phi_k),$$

as in equation 1, with  $f_N$  nonzero. We may assume the support of  $f$  lies in the interior of the unit cube  $Q = [0, 1]^{n-1}$ . Since  $f_N$  is nonzero, there is a smooth function  $\phi$  such that  $f_N(\phi) = \epsilon > 0$ . We may also take the support of  $\phi$  to lie inside  $Q$ . By changing  $\epsilon$  if necessary, we normalize  $\phi$  so that

$$\sup_{|\gamma| \leq N_p + 1} |\partial_{x'}^\gamma \phi| \leq 1.$$

Consider the dyadic cubes  $Q_\alpha^j$ , with  $j = 0, 1, 2, \dots$  and  $\alpha = (\alpha_1, \dots, \alpha_{n-1})$ ,  $0 \leq \alpha_i \leq 2^j - 1$ , where the side length of  $Q_\alpha^j$  is  $2^{-j}$  and its closest corner to the origin lies at the point  $(\alpha_1/2^j, \dots, \alpha_{n-1}/2^j)$ . Thus  $Q = Q_{(0, \dots, 0)}^0$ .

For each  $j$ , we associate with the cubes  $Q_\alpha^j$  a partition of unity  $\{\eta_\alpha^j\}$ , where each  $\eta_\alpha^j$  is a smooth function supported in the double of the cube  $Q_\alpha^j$ , i.e. the cube  $\widetilde{Q}_\alpha^j$  with the same center but double the side length, and such that

$$\sum_\alpha \eta_\alpha^j = 1$$

on  $Q$ . Furthermore, we may assume that, for every  $j$  and  $\alpha$ ,

$$|\partial_{x'}^\gamma (\eta_\alpha^j)| \leq 2^{j|\gamma|}$$

for all derivatives of order  $|\gamma| \leq N_p + 1$ .

Set

$$\phi_\alpha^j = A_{n,p} 2^{j(n-1)} \phi \eta_\alpha^j,$$

where the constant  $A_{n,p}$  is chosen so that

$$|\partial_{x'}^\gamma (\phi_\alpha^j)| \leq 2^{j(n-1+|\gamma|)}$$

for all  $|\gamma| \leq N_p + 1$ . Consider a smooth one-variable function  $\psi(t)$  with

$$\psi(t) = \frac{1}{N!} t^N$$

for  $t \in [-1/2, 1/2]$ ,  $\psi(t) = 0$  for  $|t| > 1$ , and  $|d_t^m \psi(t)| \leq 1$  for  $m \leq N_p + 1$ . Let

$$\psi_j(t) = 2^j \psi(2^j t),$$

and

$$\Phi_\alpha^j(x', x_n) = C_{n,p} \phi_\alpha^j(x') \psi_j(x_n),$$

where again we chose the constant  $C_{n,p}$  so that

$$|\partial^\beta (\Phi_\alpha^j)| \leq (2^{-j} \sqrt{n})^{-(n+|\beta|)}$$

for all  $|\beta| \leq N_p + 1$ . Since  $\Phi_\alpha^j$  is supported in the cube  $\widetilde{Q}_\alpha^j \times [-2^{-j}, 2^{-j}]$  of side length  $2^{-j+1}$ , hence in a ball of radius  $2^{-j} \sqrt{n}$ , this makes  $\Phi_\alpha^j$  a normalized bump function.

Now note that

$$f(\Phi_\alpha^j) = f_N((\Phi_\alpha^j)_N) = C_{n,p} 2^{j(N+1)} f_N(\phi_\alpha^j).$$

Thus for  $x \in \widetilde{Q}_\alpha^j \times [-2^{-j}, 2^{-j}]$ ,

$$m(f) \geq C_{n,p} 2^{j(N+1)} |f_N(\phi_\alpha^j)|.$$

Recall that, for every  $j$ ,

$$\sum_\alpha f_N(\phi_\alpha^j) = A_{n,p} 2^{j(n-1)} f_N(\phi) = A_{n,p} 2^{j(n-1)} \epsilon.$$

Therefore for  $2^{-(j+1)} \leq x_n \leq 2^{-j}$ ,

$$\begin{aligned} \int_{\mathbf{R}^{n-1}} m(f)^p(x', x_n) dx' &\geq C_{n,p} \sum_\alpha \left( 2^{j(N+1)} |f_N(\phi_\alpha^j)| \right)^p |Q_\alpha^j| \\ &\geq C_{n,p} 2^{j(N+1)p} 2^{-j(n-1)} \left| \sum_\alpha f_N(\phi_\alpha^j) \right|^p \\ &= C'_{n,p} 2^{j(N+1)p} 2^{-j(n-1)} 2^{j(n-1)p} \epsilon^p \\ &= C'_{n,p} \epsilon^p 2^{-j[n-1-(n+N)p]} \\ &\geq C_{n,p,N,\epsilon} x_n^{n-1-(n+N)p}. \end{aligned}$$

This proves the lemma. □

We will now give an example to show that Theorem 1 does not hold with  $N$  replaced by  $K$ , the total order of the distribution.

**Example 1.** For  $0 < p < 1$ , there exists a distribution  $f \in h^p(\mathbf{R}^n)$  with compact support contained in a hyperplane, such that the order  $K$  of  $f$  satisfies

$$K \geq N_p,$$

where  $N_p$  is the greatest integer in  $n(1/p - 1)$ .

Take the hyperplane  $\Pi = \mathbf{R}^{n-1}$  as above. In  $\mathbf{R}^{n-1}$ , consider cubes  $Q_j$  with centers  $c_j$  and side lengths  $\delta_j$ , such that the double cubes  $2Q_j$  are pairwise disjoint and  $\sum \delta_j < \infty$ . Take functions  $a_j$  of the  $n - 1$  variables  $x' = x_1, \dots, x_{n-1}$  (respectively) supported in  $Q_j$  and satisfying

$$\begin{aligned} \|a_j\|_\infty &\leq \delta_j^{1-n/p}, \\ \int a_j(x') (x')^\alpha dx' &= 0 \end{aligned}$$

for all  $\alpha = (\alpha_1, \dots, \alpha_{n-1})$  with  $|\alpha| \leq N_p$ , and

$$\int a_j(x') (x_1 - (c_j)_1)^{N_p+1} dx' = C_n \delta_j^{n-n/p+N_p+1}$$

for some  $C_n > 0$ .

Define linear functionals  $\tau_j$  on  $\mathcal{S}(\mathbf{R}^n)$  by

$$\tau_j(\varphi) = \int_{\mathbf{R}^{n-1}} a_j(x') \varphi(x', 0) dx'$$

for every  $\varphi \in \mathcal{S}(\mathbf{R}^n)$ . By expanding  $\varphi(x', 0)$  in a Taylor expansion around  $c_j$ , one can see that

$$|\tau_j(\varphi)| \leq C \|\varphi\|_{C^{N_p+1}} \delta_j^{n-n/p+N_p+1} \leq C_{n,p} \|\varphi\|_{C^{N_p+1}}$$

since  $\delta_j \rightarrow 0$  and  $n - n/p + N_p + 1 > 0$ . This shows the  $\tau_j$  are continuous on  $\mathcal{S}(\mathbf{R}^n)$ , and we can define  $f \in \mathcal{S}'(\mathbf{R}^n)$  by

$$f = \sum \lambda_j \tau_j,$$

where the sequence  $\lambda_j$  is chosen to satisfy  $\sum \lambda_j^p < \infty$  and  $\lambda_j/\delta_j \rightarrow \infty$  as  $j \rightarrow \infty$  (for example take  $\delta_j = 2^{-j}$ ,  $\lambda_j = j^{-1-1/p}$ ).

To see that the order of  $f$  is at least  $N_p$ , take a sequence of smooth functions  $\varphi_j \in \mathcal{S}(\mathbf{R}^n)$  with  $\varphi_j(x', 0)$  supported in  $2Q_j$ ,

$$|\partial^\alpha \varphi_j| \leq 1$$

for  $|\alpha| \leq N_p - 1$ , and

$$\varphi_j(x', 0) = \frac{1}{(N_p + 1)!} (x_1 - (c_j)_1)^{N_p + 1} \delta_j^{-2}$$

on  $Q_j$ . Thus

$$f(\varphi_j) = \lambda_j \int_{Q_j} a_j(x') \varphi_j(x', 0) dx' = C_{n,p} \lambda_j \delta_j^{n-n/p+N_p+1-2} \rightarrow \infty$$

as  $j \rightarrow \infty$ , since  $n - n/p + N_p + 1 - 2 \leq -1$ .

Finally, to see that  $f \in h^p(\mathbf{R}^n)$ , consider the local grand maximal function  $m(f)$ . It suffices to show  $\|m(\tau_j)\|_{L^p(\mathbf{R}^n)} \leq C$  uniformly in  $j$ . Take  $x \in \mathbf{R}^n$ . Note that if  $\varphi_t^x$  is a normalized bump function supported in a ball of radius  $t$  containing  $x$ , then  $\tau_j(\varphi_t^x) \neq 0$  only if  $t \geq x_n/2$ , and  $t\varphi_t^x(y', 0)$  is a normalized bump function in  $n - 1$  variables. Thus

$$m(\tau_j)(x) \leq \frac{2}{x_n} m_{\mathbf{R}^{n-1}}(a_j)(x'),$$

where  $m_{\mathbf{R}^{n-1}}$  denotes the local grand maximal function on  $\mathbf{R}^{n-1}$ . Recalling that  $m_{\mathbf{R}^{n-1}}$  is bounded on  $L^2(\mathbf{R}^{n-1})$ , we get that

$$\begin{aligned} \int_{2Q_j \times [-\delta_j, \delta_j]} m(\tau_j)^p(x) dx &\leq C \int_{-\delta_j}^{\delta_j} x_n^{-p} \int_{2Q_j} m_{\mathbf{R}^{n-1}}(a_j)^p(x') dx' \\ &\leq C \delta_j^{1-p} \|m(a_j)\|_{L^2(\mathbf{R}^{n-1})}^p |2Q_j|^{1-p/2} \\ &\leq C \delta_j^{1-p} \|a_j\|_{L^2(\mathbf{R}^{n-1})}^p |2Q_j|^{1-p/2} \\ &\leq C \delta_j^{1-p} \delta_j^{p-n} |Q_j| \\ &= C. \end{aligned}$$

Note that here we used the fact that  $p < 1$ .

For  $x \notin 2Q_j \times [-\delta_j, \delta_j]$ , if  $\varphi_t^x$  is a normalized bump function supported in a ball of radius  $t$  containing  $x$ , then as above

$$\begin{aligned} |\tau_j(\varphi_t^x)| &\leq C \|\varphi_t^x\|_{C^{N_p+1}} \delta_j^{n-n/p+N_p+1} \\ &\leq C t^{-n-(N_p+1)} \delta_j^{n-n/p+N_p+1} \\ &\leq C |x - (c_j, 0)|^{-n-(N_p+1)} \delta_j^{n-n/p+N_p+1}, \end{aligned}$$

so

$$\begin{aligned} & \int_{x \notin 2Q_j \times [-\delta_j, \delta_j]} m(\tau_j)^p(x) dx \\ & \leq C \delta_j^{np-n+(N_p+1)p} \int_{|x-(c_j,0)| \geq \delta_j} |x - (c_j, 0)|^{-np-(N_p+1)p} dx \\ & = C. \end{aligned}$$

This completes the example.

### 3. THE SPACES $h_r^p(\Omega)$ AND $h_z^p(\Omega)$

Let  $\Omega$  be a bounded domain in  $\mathbf{R}^n$ , with smooth boundary. Let us recall the definitions of the spaces  $h_r^p(\Omega)$  and  $h_z^p(\Omega)$ , given in [CKS]:

$$h_r^p(\Omega) = \{f \in \mathcal{D}'(\Omega) : \exists F \in h^p(\mathbf{R}^n), F|_\Omega = f\}$$

and

$$h_z^p(\Omega) = \{f \in \mathcal{D}'(\Omega) : \exists F \in h^p(\mathbf{R}^n), F|_\Omega = f, F|_{\mathbf{R}^n \setminus \overline{\Omega}} = 0\}.$$

Clearly  $h_z^p(\Omega)$  is a subspace of  $h_r^p(\Omega)$ . It was shown in [CDS] that in fact  $h_r^p(\Omega) = h_z^p(\Omega)$  for all values of  $p$  with  $\frac{n}{n+j+1} < p < \frac{n}{n+j}$ ,  $j = 0, 1, \dots$ . We will now show that this is not true when  $n(1/p - 1)$  is an integer.

**Theorem 2.** For  $p = 1, \frac{n}{n+1}, \frac{n}{n+2}, \dots$ ,

$$h_r^p(\Omega) \neq h_z^p(\Omega).$$

*Proof.* Assume  $0 \in \partial\Omega$ , and let  $U \subset V$  be open neighborhoods of 0. We may take these so that there is a smooth choice of coordinates in  $V$  in which  $V = B(0, 2)$ ,  $U = B(0, 1)$  and  $\Omega \cap V = \mathbf{R}_+^n \cap B(0, 2)$ . Here  $\mathbf{R}_+^n$  denotes the open upper half-space  $\{x : x_n > 0\}$ .

We will construct a distribution  $f \in h_r^p(\Omega)$ , supported in  $U$ , such that there is no distribution  $G \in h^p(\mathbf{R}^n)$  satisfying

$$G|_\Omega = f$$

and

$$G|_{\mathbf{R}^n \setminus \overline{\Omega}} = 0.$$

Note that the space of distributions in  $h_r^p(\Omega)$  which are supported in  $U$  does not depend on the choice of coordinates.

Let us construct  $f$  as follows. For  $j = 1, 2, \dots$ , consider the cylinder

$$S_j = B^{n-1}(0, 2^{-(j+1)}) \times [2^{-(j+1)}, 2^{-j}],$$

where  $B^{n-1}(0, r)$  denotes the  $(n - 1)$ -dimensional ball of radius  $r$  in the variables  $x' = x_1, \dots, x_{n-1}$ . Note that  $S_j$  is contained in the (much larger) cube

$$Q_j = [-2^{-(j+1)}, 2^{-(j+1)}]^{n-1} \times [2^{-(j+1)}, 3 \cdot 2^{-(j+1)}]$$

of volume  $|Q_j| = 2^{-jn}$ . Set

$$a_j = |Q_j|^{-1/p} \chi_{S_j}.$$

If  $\lambda_j = j^{-(1+1/p)}$ , then  $\sum \lambda_j^p < \infty$  so by the atomic decomposition for  $h_r^p$  (see [CKS], Theorem 2.7,) the distribution  $f \in \mathcal{D}'(\Omega)$  defined by

$$f = \sum_{j=1}^{\infty} \lambda_j a_j$$

is an element of  $h^p_r(\Omega)$ . Furthermore,  $f$  is supported in  $\Gamma \cap B(0, 1)$ , where  $\Gamma$  is the cone

$$\{x \in \mathbf{R}^n : |x'| \leq x_n < 1\}.$$

In order to extend  $f$  to a distribution  $F \in \mathcal{S}'(\mathbf{R}^n)$ , we define distributions  $\tau_j$  by

$$\tau_j(\varphi) = \int a_j(x) \left[ \varphi(x) - \sum_{|\alpha| \leq N_p - 1} \frac{1}{\alpha!} \partial^\alpha \varphi(0) x^\alpha \right] dx$$

for every  $\varphi \in \mathcal{S}(\mathbf{R}^n)$ . Since

$$\begin{aligned} |\tau_j(\varphi)| &= \left| \int a_j(x) \left[ \varphi(x) - \sum_{|\alpha| \leq N_p - 1} \frac{1}{\alpha!} \partial^\alpha \varphi(0) x^\alpha \right] dx \right| \\ &\leq C 2^{jn/p} \|\varphi\|_{C^{N_p}} \int_{Q_j} |x|^{N_p} dx \\ &\leq C 2^{j(n/p - N_p - n)} \|\varphi\|_{C^{N_p}} \\ &= C \|\varphi\|_{C^{N_p}}, \end{aligned}$$

and  $\sum \lambda_j < \infty$ , we can define a distribution  $F \in \mathcal{S}'(\mathbf{R}^n)$  by

$$F = \sum_{j=1}^{\infty} \lambda_j \tau_j.$$

This distribution is supported in  $B(0, 1) \cap \overline{\Omega}$  and satisfies

$$F|_{\Omega} = \sum_{j=1}^{\infty} \lambda_j \tau_j|_{\Omega} = \sum_{j=1}^{\infty} \lambda_j a_j = f.$$

Now suppose there was a distribution  $G \in \mathcal{S}'(\mathbf{R}^n)$  with  $G$  supported on  $\overline{\Omega}$  and  $G|_{\Omega} = f$ . We want to show that  $G$  cannot be an element of  $h^p(\mathbf{R}^n)$ . Since  $h^p(\mathbf{R}^n)$  is closed under multiplication by smooth functions of compact support, we may assume  $G$  is supported in  $V = B(0, 2)$ .

Consider the distribution  $G - F \in \mathcal{S}'(\mathbf{R}^n)$ . Since both  $G$  and  $F$  are supported on  $\overline{\Omega}$ , and  $G|_{\Omega} = f = F|_{\Omega}$ , we must have that  $G - F$  is supported on  $\partial\Omega$ , and in particular on  $B^{n-1}(0, 2) = B(0, 2) \cap \partial\mathbf{R}^n_+$ . Thus  $G - F$  is of the form of equation (1).

Let  $N$  denote the order of  $G - F$  in the normal direction.

*Claim 1.* If  $N < N_p = n(1/p - 1)$ , then the distribution  $G$  does not belong to  $h^p(\mathbf{R}^n)$ , since its local grand maximal function satisfies

$$m(G)(x) \geq C|x|^{-n/p} \left( \log \frac{1}{|x|} \right)^{-1/p}$$

for  $x \in \Gamma$ ,  $x \neq 0$ .

To get the lower bound for  $m(G)$ , we will construct a special test function  $\phi$ . First take a smooth function  $\psi$  of one variable, supported in  $(-1, 2)$ , with  $\psi(y) \geq 0$  for  $y \geq 0$ ,

$$\psi(y) = \frac{y^{N_p}}{N_p!} \text{ for } y \in [0, 1]$$

and

$$\left| \frac{d^k}{dy^k}(\psi) \right| \leq 1$$

for  $k \leq N_p + 1$ . Now take  $\varphi$  to be a smooth function of  $n - 1$  variables, supported in  $B^{n-1}(0, 2)$ , with  $\varphi \geq 0$ ,  $\varphi = 1$  on  $B^{n-1}(0, 1)$  and

$$|\partial^\alpha(\varphi)| \leq 1$$

for  $|\alpha| \leq N_p + 1$ . Defining  $\phi$  by

$$\phi(x) = \varphi(x')\psi(x_n),$$

we see that  $\phi$  is supported in  $B(0, 3)$ ,

$$\phi(x) = \frac{x_n^{N_p}}{N_p!}$$

for  $x \in B^{n-1}(0, 1) \times [0, 1]$ , and

$$|\partial^\alpha(\phi)| \leq 1$$

for  $|\alpha| \leq N_p + 1$ .

Suppose  $x \in \Gamma$ ,  $x \neq 0$ . Let  $x^* = (x', 0)$ , and define

$$\phi_x(y) = 3^{-n-N_p}|x|^{-n}\phi\left(\frac{y-x^*}{|x|}\right).$$

Then  $\phi_x$  is supported in the ball  $B(x^*, 3|x|)$ , with  $\phi_x \geq 0$  on  $\mathbf{R}_+^n$  and

$$\phi_x(y) = \frac{y_n^{N_p}}{N_p!(3|x|)^{n+N_p}}$$

for  $y \in S_x = B^{n-1}(x', |x|) \times [0, |x|]$ . Furthermore

$$|\partial^\alpha(\phi_x)| \leq (3|x|)^{-n-|\alpha|}$$

for  $|\alpha| \leq N_p + 1$ . Thus  $\phi_x$  is a normalized bump function.

Note that, for  $k < N_p$ ,

$$\frac{\partial^k}{\partial y_n^k} \phi_x(y', y_n)|_{y_n=0} = 3^{-n-N_p}|x|^{-n-k}\varphi\left(\frac{y'-x'}{|x|}\right)\frac{\partial^k \psi}{\partial y_n^k}(0) = 0,$$

so by equation (1), if  $N < N_p$ ,

$$(G - F)(\phi_x) = 0.$$

Moreover,  $\tau_j(\phi_x) = \int a_j \phi_x$  for all  $j$ . Thus

$$\begin{aligned} G(\phi_x) &= F(\phi_x) \\ &\geq \sum_{S_j \subset S_x} \lambda_j \int_{S_j} a_j(y)\phi_x(y)dy \\ &= \frac{1}{N_p!(3|x|)^{n+N_p}} \sum_{S_j \subset S_x} \lambda_j \int_{S_j} a_j(y)y_n^{N_p}dy \\ &= C_{N_p}(3|x|)^{-n-N_p} \sum_{S_j \subset S_x} \lambda_j 2^{j(n/p-N_p-n)} \\ &\geq C'_{N_p}|x|^{-n-N_p} \sum_{j \geq -\log_2 x_n+2} j^{-1/p-1} \\ &\approx |x|^{-n/p} \left(\log \frac{1}{|x|}\right)^{-1/p}. \end{aligned}$$

Here we have used the fact that for  $x \in \Gamma$ , if  $2^{-j} \leq (\sqrt{2} - 1)x_n$ , then

$$S_j = B^{n-1}(0, 2^{-(j+1)}) \times [2^{-(j+1)}, 2^{-j}] \subset S_x = B^{n-1}(x', |x|) \times [0, |x|].$$

*Claim 2.* If  $N \geq N_p$ , then the distribution  $G$  does not belong to  $h^p(\mathbf{R}^n)$ , since its local grand maximal function satisfies

$$\int_{\mathbf{R}^{n-1}} m(G)^p(x', x_n) dx' \geq Cx_n^{-1}$$

for all sufficiently small  $x_n > 0$ .

By Lemma 1, with  $N \geq N_p$ , the distribution  $G - F$  satisfies

$$\int_{\mathbf{R}^{n-1}} m(G - F)^p(x', x_n) dx' \geq Cx_n^{n-1-(n+N_p)p} = Cx_n^{-1}$$

for all sufficiently small  $x_n > 0$ . Since

$$\int_{\mathbf{R}^{n-1}} m(G)^p(x', x_n) dx' \geq \int_{\mathbf{R}^{n-1}} m(G - F)^p(x', x_n) dx' - \int_{\mathbf{R}^{n-1}} m(F)^p(x', x_n) dx',$$

it suffices to prove that

$$\int_{\mathbf{R}^{n-1}} m(F)^p(x', x_n) dx' = o(x_n^{-1})$$

as  $x_n \rightarrow 0$ . In fact, we will show that

$$\int_{\mathbf{R}^{n-1}} m(F)^p(x', x_n) dx' \leq C(x_n \log(1/x_n))^{-1}$$

for sufficiently small  $x_n > 0$ .

To prove the upper bound on the maximal function of  $F$ , take  $x \in \mathbf{R}_+^n$ , and let  $\varphi_t^x$  be a bump function supported in a ball of radius  $t \leq 1$  containing  $x$ . Since  $F$  is supported in  $B(0, 1)$ , we can assume  $|x| < 3$ .

If  $t \geq |x|/4$ , write

$$F(\varphi_t^x) = \sum_{j=1}^{\infty} \lambda_j \tau_j(\varphi_t^x).$$

Note that if  $j < -\log_2(t) - 4$ , then  $2^{-(j+1)} > 8t > 2t + x_n$ , so  $\varphi_t^x = 0$  on  $S_j$  and

$$\begin{aligned} \sum_{j < -\log_2(t) - 4} |\lambda_j \tau_j(\varphi_t^x)| &= \sum_{j < -\log_2(t) - 4} \left| \lambda_j 2^{jn/p} \sum_{|\alpha| \leq N_p - 1} \frac{1}{\alpha!} \partial^\alpha \varphi_t^x(0) \int_{S_j} x^\alpha dx \right| \\ &\leq C \sum_{j < -\log_2(t) - 4} \lambda_j \sum_{|\alpha| \leq N_p - 1} \frac{1}{\alpha!} t^{-n-|\alpha|} 2^{j(n/p-n-|\alpha|)} \\ &\leq C \sum_{|\alpha| \leq N_p - 1} \frac{1}{\alpha!} t^{-n-|\alpha|} \sum_{j < -\log_2(t) - 4} \frac{2^{j(n/p-n-|\alpha|)}}{j^{1+1/p}} \\ &\leq C \sum_{|\alpha| \leq N_p - 1} \frac{1}{\alpha!} t^{-n-|\alpha|} \frac{t^{-(n/p-n-|\alpha|)}}{\log(1/t)^{1/p}} \\ &\leq Ct^{-n/p} (\log(1/t))^{-1/p}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \sum_{j \geq -\log_2(t)-4} |\lambda_j \tau_j(\varphi_t^x)| &\leq C \sum_{j \geq -\log_2(t)-4} \frac{1}{j^{1+1/p}} \|\varphi_t^x\|_{C^{N_p}} \\ &\leq Ct^{-n-N_p} \sum_{j \geq -\log_2(t)-4} \frac{1}{j^{1+1/p}} \\ &\leq Ct^{-n/p} (\log(1/t))^{-1/p}. \end{aligned}$$

Thus

$$\begin{aligned} |F(\varphi_t^x)| &\leq Ct^{-n/p} (\log(1/t))^{-1/p} \\ &\leq C|x|^{-n/p} (\log(1/|x|))^{-1/p}, \end{aligned}$$

since  $(|x|/t)^{n/p} (\log|x|/\log t)^{1/p}$  remains bounded when  $|x| \ll t$ .

If  $t < |x|/4$ , then  $\varphi_t^x$  vanishes in a neighborhood of 0, and  $S_j \cap \text{supp}(\varphi_t^x) \neq \emptyset$  only if  $j \leq 3/2 - \log_2|x|$ , so

$$\begin{aligned} |F(\varphi_t^x)| &\leq C \sum_{j \leq -C \log|x|} \lambda_j \left| \int a_j \varphi_t^x \right| \\ &\leq C \sum_{j \leq -C \log|x|} \frac{1}{j^{1+1/p}} 2^{jn/p} \|\varphi_t^x\|_{L^1} \\ &\leq C|x|^{-n/p} (\log(1/|x|))^{-1/p}. \end{aligned}$$

In both cases we are assuming of course that  $|x|$  is sufficiently small, i.e.  $|x| \leq a \ll 1$ , so that  $\log(1/|x|)$  is bounded below. Otherwise we just have

$$|F(\varphi_t^x)| \leq C|x|^{-n/p} \sum \lambda_j \leq C_a.$$

Now integrating in the first  $n - 1$  variables, we have

$$\begin{aligned} \int_{\mathbf{R}^{n-1}} m(F)^p(x', x_n) dx' &= \int_{|x'| \leq x_n} m(F)^p(x', x_n) dx' + \int_{x_n \leq |x'| \leq a} m(F)^p(x', x_n) dx' \\ &\quad + \int_{a \leq |x'| \leq 3} m(F)^p(x', x_n) dx' \\ &\leq C \int_{|x'| \leq x_n} x_n^{-n} (\log(1/x_n))^{-1} dx' \\ &\quad + C \int_{x_n \leq |x'| \leq a} |x'|^{-n} (\log(1/|x'|))^{-1} dx' + C \\ &\leq Cx_n^{-1} (\log(1/x_n))^{-1} + C \int_{x_n}^a \frac{dr}{r^2(-\log r)} + C. \end{aligned}$$

But

$$\begin{aligned} \int_{x_n}^a \frac{dr}{r^2(-\log r)} &= \int_{x_n}^a \frac{dr}{r^2 \log^2 r} + x_n^{-1} \left( \log \frac{1}{x_n} \right)^{-1} + C \\ &\leq (-\log a)^{-1} \int_{x_n}^a \frac{dr}{r^2(-\log r)} + x_n^{-1} (\log(1/x_n))^{-1} + C \end{aligned}$$

so

$$\int_{x_n}^a \frac{dr}{r^2(-\log r)} \leq Cx_n^{-1}(\log(1/x_n))^{-1} + C$$

for  $a$  sufficiently small.

This concludes the proof of Theorem 2.  $\square$

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