

SOME THETA FUNCTION IDENTITIES RELATED TO THE ROGERS-RAMANUJAN CONTINUED FRACTION

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ABSTRACT. In his first and second letters to Hardy, Ramanujan made several assertions about the Rogers-Ramanujan continued fraction $F(q)$. In order to prove some of these claims, G. N. Watson established two important theorems about $F(q)$ that he found in Ramanujan's notebooks. In his lost notebook, after stating a version of the quintuple product identity, Ramanujan offers three theta function identities, two of which contain as special cases the celebrated two theorems of Ramanujan proved by Watson. Using addition formulas, the quintuple product identity, and a new general product formula for theta functions, we prove these three identities of Ramanujan from his lost notebooks.

1. INTRODUCTION

In his notebooks [7], [1, p. 34], Ramanujan defined his general theta function $f(a, b)$ by

$$f(a, b) := \sum_{k=-\infty}^{\infty} a^{k(k+1)/2} b^{k(k-1)/2}, \quad |ab| < 1.$$

This function has the same generality as the classical theta function $\vartheta(z, q)$ [11, pp. 463, 464]. After Ramanujan, define $f(-q) := f(-q, -q^2)$.

In his famous letters to Hardy [6, pp. 29, 57], Ramanujan defined the Rogers-Ramanujan continued fraction

$$F(q) := \frac{1}{1 + \frac{q}{1 + \frac{q^2}{1 + \frac{q^3}{1 + \dots}}}}, \quad |q| < 1,$$

and evaluated it for certain values of q . The following two identities are two of the most important theorems about $F(q)$.

Theorem 1.1. For $|q| < 1$,

$$(1.1) \quad \frac{1}{F(q)} - q^{1/5} - q^{2/5}F(q) = \frac{f(-q^{1/5})}{f(-q^5)}.$$

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Theorem 1.2. For $|q| < 1$,

$$(1.2) \quad \frac{1}{F^5(q)} - 11q - q^2 F^5(q) = \frac{f^6(-q)}{f^6(-q^5)}.$$

They were found by Watson [10] in Ramanujan's notebooks. He conjectured that Ramanujan expanded $f(-q^{1/5})$ and then arranged the terms in the expansion in groups. Watson used the Jacobi triple product identity (2.1), Euler's identity (2.2), Jacobi's identity and the quintuple product identity [1, pp. 39, 82] to prove them. Another nice proof was found by Berndt [1, p. 84]. Recently, Berndt, Huang, Sohn and Son [5] found proofs of certain refinements of these identities, which lead to different proofs of (1.1) and (1.2). Berndt, Chan and Zhang [3], [4] employed these identities in their evaluation of $F(q)$ for several values of q , including all the values that Ramanujan claimed in his lost notebook.

On page 207 in his lost notebook [8], Ramanujan listed three identities

$$(1.3) \quad P - Q = 1 + \frac{f(-q^{1/5}, -\lambda q^{2/5})}{q^{1/5} f(-\lambda^{10} q^5, -\lambda^{15} q^{10})},$$

$$(1.4) \quad PQ = 1 - \frac{f(-\lambda, -\lambda^4 q^3) f(-\lambda^2 q, -\lambda^3 q^2)}{f^2(-\lambda^{10} q^5, -\lambda^{15} q^{10})},$$

and

$$(1.5) \quad P^5 - Q^5 = 1 + 5PQ + 5P^2Q^2 + \frac{f(-q, -\lambda^5 q^2) f^5(-\lambda^2 q, -\lambda^3 q^2)}{q f^6(-\lambda^{10} q^5, -\lambda^{15} q^{10})},$$

without specifying the functions P and Q . In this article, the functions P and Q are determined and the identities which are remarkable generalizations of (1.1) and (1.2) are proved. In addition, a useful formula called the Ramanujan product is presented in two distinct forms.

Unexpectedly, it turns out that these claims are evidence that Watson's conjecture is amazingly accurate. Ramanujan indeed had known the simplest way of deriving the identities.

2. PRELIMINARY RESULTS

As customary, set

$$(a; q)_\infty := \prod_{i=0}^{\infty} (1 - aq^i), \quad |q| < 1.$$

We shall frequently use the following Jacobi triple product identity [1, p. 35].

Theorem 2.1. For $|ab| < 1$, we have

$$(2.1) \quad f(a, b) = (-a; ab)_\infty (-b; ab)_\infty (ab; ab)_\infty.$$

By applying (2.1) and simplification, we find that

$$(2.2) \quad f(-q) = (q; q)_\infty,$$

which is Euler's pentagonal number theorem.

The following identities are written in Ramanujan's notebooks [1, pp. 48, 34].

Theorem 2.2. Let $U_n = a^{n(n+1)/2}b^{n(n-1)/2}$ and $V_n = a^{n(n-1)/2}b^{n(n+1)/2}$ for each integer n . Then

$$(2.3) \quad f(a, b) = \sum_{r=0}^{n-1} U_r f\left(\frac{U_{n+r}}{U_r}, \frac{V_{n-r}}{U_r}\right).$$

Theorem 2.3. For $|ab| < 1$,

$$(2.4) \quad f(a, b) = a^{n(n+1)/2}b^{n(n-1)/2} f(a(ab)^n, b(ab)^{-n}).$$

On page 207 in his lost notebook, Ramanujan recorded the quintuple product identity in the following form [1, p. 82].

Theorem 2.4. For $|\lambda x^3| < 1$,

$$(2.5) \quad f(-\lambda^2 x^3, -\lambda x^6) + x f(-\lambda, -\lambda^2 x^9) = \frac{f(-x^2, -\lambda x) f(-\lambda x^3)}{f(-x, -\lambda x^2)}.$$

The Rogers-Ramanujan continued fraction can be expressed in terms of the quotient of theta functions.

Theorem 2.5. For $|q| < 1$,

$$(2.6) \quad F(q) = \frac{f(-q, -q^4)}{f(-q^2, -q^3)}, \quad |q| < 1.$$

The proof follows from the famous Rogers-Ramanujan identities [1, p. 79] which were established for the first time by Rogers in 1894 [9].

3. THE RAMANUJAN PRODUCT

Special cases of Theorem 3.1 can be found in Berndt’s books [1, pp. 264, 307, 346, 348], [2, pp. 142, 145, 188, 192]. We shall use instances of the following general product formula in the proof of (1.5).

Theorem 3.1. For $|ab| < 1$, arbitrary integers j, k with $(j, k) \not\equiv (0, 0) \pmod{p}$, and a prime $p > 2$, let $\zeta := \exp(2\pi i/p)$, and s be a solution of

$$(j + k)x + j \equiv 0 \pmod{p} \quad \text{and} \quad 0 \leq x < p.$$

Then

$$(3.1) \quad \prod_{n=1}^p f(\zeta^{jn} a, \zeta^{kn} b) = \begin{cases} \frac{f^p(a^{s+1} b^s, a^{p-s-1} b^{p-s}) f(a^p, b^p)}{f(a^{p(s+1)} b^{ps}, a^{p(p-s-1)} b^{p(p-s)})}, & \text{if } j + k \not\equiv 0 \pmod{p}, \\ f^p(-ab) \frac{f(a^p, b^p)}{f(-a^p b^p)}, & \text{if } j + k \equiv 0 \pmod{p}. \end{cases}$$

Proof. Let

$$C := \prod_{n=1}^p f(-\zeta^{jn} a, -\zeta^{kn} b).$$

By the Jacobi triple product identity (2.1),

$$C = \prod_{n=1}^p (\zeta^{jn} a; \zeta^{(j+k)n} ab)_\infty (\zeta^{kn} b; \zeta^{(j+k)n} ab)_\infty (\zeta^{(j+k)n} ab; \zeta^{(j+k)n} ab)_\infty.$$

Set

$$C_1 := \prod_{\ell=1}^p (\zeta^{j\ell} a; \zeta^{(j+k)\ell} ab)_\infty,$$

$$C_2 := \prod_{\ell=1}^p (\zeta^{k\ell} b; \zeta^{(j+k)\ell} ab)_\infty,$$

and

$$C_3 := \prod_{\ell=1}^p (\zeta^{(j+k)\ell} ab; \zeta^{(j+k)\ell} ab)_\infty.$$

If $j + k \not\equiv 0 \pmod{p}$, then

$$\begin{aligned} C_1 &= \prod_{\substack{n=0 \\ n \equiv s \pmod{p}}}^\infty (1 - a(ab)^n)^p \prod_{\substack{n=0 \\ n \not\equiv s \pmod{p}}}^\infty (1 - a^p(ab)^{pn}) \\ &= \prod_{n=0}^\infty (1 - a(ab)^{pn+s})^p \prod_{n=0}^\infty (1 - a^p(ab)^{pn}) \bigg/ \prod_{\substack{n=0 \\ n \equiv s \pmod{p}}}^\infty (1 - a^p(ab)^{pn}) \\ &= (a^{s+1}b^s; a^p b^p)_\infty^p \frac{(a^p; a^p b^p)_\infty}{(a^{p(s+1)}b^{ps}; a^{p^2}b^{p^2})_\infty}. \end{aligned}$$

Similarly, as $p - s - 1$ is a solution of $(j + k)x + k \equiv 0 \pmod{p}$,

$$C_2 = (a^{p-s-1}b^{p-s}; a^p b^p)_\infty^p \frac{(b^p; a^p b^p)_\infty}{(a^{p(p-s-1)}b^{p(p-s)}; a^{p^2}b^{p^2})_\infty}$$

and

$$C_3 = (a^p b^p; a^p b^p)_\infty^p \frac{(a^p b^p; a^p b^p)_\infty}{(a^{p^2}b^{p^2}; a^{p^2}b^{p^2})_\infty}.$$

Hence, by the Jacobi triple product identity,

$$\begin{aligned} C &= C_1 C_2 C_3 \\ &= \{(a^{s+1}b^s; a^p b^p)_\infty (a^{p-s-1}b^{p-s}; a^p b^p)_\infty (a^p b^p; a^p b^p)_\infty\}^p \\ &\quad \times \frac{(a^p; a^p b^p)_\infty (b^p; a^p b^p)_\infty (a^p b^p; a^p b^p)_\infty}{(a^{p(s+1)}b^{ps}; a^{p^2}b^{p^2})_\infty (a^{p(p-s-1)}b^{p(p-s)}; a^{p^2}b^{p^2})_\infty (a^{p^2}b^{p^2}; a^{p^2}b^{p^2})_\infty} \\ &= f^p(-a^{s+1}b^s, -a^{p-s-1}b^{p-s}) \frac{f(-a^p, -b^p)}{f(-a^{p(s+1)}b^{ps}, -a^{p(p-s-1)}b^{p(p-s)})}. \end{aligned}$$

If $j + k \equiv 0 \pmod{p}$,

$$C_1 = \prod_{n=0}^\infty (1 - a^p(ab)^{pn}) = (a^p; a^p b^p)_\infty.$$

Similarly,

$$C_2 = (b^p; a^p b^p)_\infty,$$

and by (2.2),

$$C_3 = (ab; ab)_\infty^p = f^p(-ab).$$

Hence, by the Jacobi triple product identity, we deduce that

$$C = C_1 C_2 C_3 = f^p(-ab)(a^p; a^p b^p)_\infty (b^p; a^p b^p)_\infty = f^p(-ab) \frac{f(-a^p, -b^p)}{f(-a^p b^p)},$$

and so the proof is complete after replacing $(-a, -b)$ by (a, b) . □

As Ramanujan’s general theta function $f(a, b)$ has the same generality as the classical theta functions $\theta_i(z, q)$, for $i = 1, 2, 3, 4$, and they are convertible to each other directly, the Ramanujan product formula (3.1) can be expressed in terms of the classical theta functions. The theta function defined by

$$\vartheta(z, q) := \theta_3(z, q) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2inz}, \quad |q| < 1,$$

shall be used in the following theorem.

Theorem 3.2. *For $0 < |q| < 1$ and $z \in \mathbb{C}$, let j, k, p and s be defined as in Theorem 3.1. Then*

$$\prod_{n=1}^p \vartheta\left(z + \frac{\pi n(j-k)}{2p}, e^{\pi i(j+k)n/p} q\right) = \begin{cases} \frac{\vartheta^p\left(z + \frac{i}{2}(p-2s-1)\log q, q^p\right) \vartheta(pz, q^p)}{\vartheta\left(pz + \frac{i}{2}p(p-2s-1)\log q, q^{p^2}\right)}, & \text{if } j+k \not\equiv 0 \pmod{p}, \\ \vartheta^p\left(\frac{i}{2}\log q, q^3\right) \vartheta(pz, q^p) / \vartheta\left(\frac{i}{2}p\log q, q^{3p}\right), & \text{if } j+k \equiv 0 \pmod{p}. \end{cases}$$

Proof. Let $a := qe^{2iz}$ and $b := qe^{-2iz}$. If $A = Qe^{2iZ}$ and $B = Qe^{-2iZ}$, equivalently, if $Z = \frac{i}{4}\log(B/A)$ and $Q = \sqrt{AB}$, then by definitions,

$$f(A, B) = \vartheta(Z, Q),$$

which are well-defined if $0 < |Q| = |\sqrt{AB}| < 1$. Let $\zeta := \exp(2\pi i/p)$. If $A = -\zeta^{jn} a$ and $B = -\zeta^{kn} b$, then by the above relations,

$$Z = z + \frac{\pi n(j-k)}{2p} \quad \text{and} \quad Q = e^{\pi i(j+k)n/p} q.$$

Thus,

$$f(-\zeta^{jn} a, -\zeta^{kn} b) = \vartheta\left(z + \frac{\pi n(j-k)}{2p}, e^{\pi i(j+k)n/p} q\right).$$

Similarly,

$$\begin{aligned} f^p(-a^{s+1} b^s, -a^{p-s-1} b^{p-s}) &= \vartheta^p\left(z + \frac{i}{2}(p-2s-1)\log q, q^p\right), \\ f(-a^p, -b^p) &= \vartheta(pz, q^p), \\ f(-a^{p(s+1)} b^{ps}, -a^{p(p-s-1)} b^{p(p-s)}) &= \vartheta\left(pz + \frac{i}{2}p(p-2s-1)\log q, q^{p^2}\right), \\ f^p(-ab) &= \vartheta^p\left(\frac{i}{2}\log q, q^3\right) \end{aligned}$$

and

$$f(-a^p b^p) = \vartheta\left(\frac{i}{2}p \log q, q^{3p}\right).$$

Using each of these equalities in (3.1), we achieve the desired result. □

4. MAIN IDENTITIES

Theorem 4.1. *If*

$$(4.1) \quad P = \frac{f(-\lambda^{10}q^7, -\lambda^{15}q^8) + \lambda q f(-\lambda^5q^2, -\lambda^{20}q^{13})}{q^{1/5}f(-\lambda^{10}q^5, -\lambda^{15}q^{10})}$$

and

$$(4.2) \quad Q = \frac{\lambda f(-\lambda^5q^4, -\lambda^{20}q^{11}) - \lambda^3 q f(-q, -\lambda^{25}q^{14})}{q^{-1/5}f(-\lambda^{10}q^5, -\lambda^{15}q^{10})},$$

then (1.3), (1.4) and (1.5) hold.

Proofs. In (2.3), let $a = -q^{1/5}$, $b = -\lambda q^{2/5}$, and $n = 5$, and then employ (2.4) so as to obtain (1.3).

By (4.1) and (4.2), the identity (1.4) is equivalent to the identity,

$$(4.3) \quad \begin{aligned} S &:= f(-\lambda, -\lambda^4q^3)f(-\lambda^2q, -\lambda^3q^2) \\ &= f(-\lambda^{10}q^5, -\lambda^{15}q^{10})f(-\lambda^{10}q^5, -\lambda^{15}q^{10}) \\ &\quad - \lambda f(-\lambda^5q^4, -\lambda^{20}q^{11})f(-\lambda^{10}q^7, -\lambda^{15}q^8) \\ &\quad - \lambda^2 q f(-\lambda^5q^4, -\lambda^{20}q^{11})f(-\lambda^5q^2, -\lambda^{20}q^{13}) \\ &\quad + \lambda^3 q f(-q, -\lambda^{25}q^{14})f(-\lambda^{10}q^7, -\lambda^{15}q^8) \\ &\quad + \lambda^4 q^2 f(-q, -\lambda^{25}q^{14})f(-\lambda^5q^2, -\lambda^{20}q^{13}). \end{aligned}$$

Let

$$S = \sum_{u=-\infty}^{\infty} \sum_{v=-\infty}^{\infty} h(u, v),$$

where

$$h(u, v) := (-1)^{u+v} \lambda^{\frac{1}{2}(5u^2+5v^2-u-3v)} q^{\frac{1}{2}(3u^2+3v^2-u-3v)}.$$

Since

$$5u = 2(2u + v) + (u - 2v) \equiv 0 \pmod{5},$$

$u - 2v \equiv -2k \pmod{5}$ if $2u + v \equiv k \pmod{5}$. Write

$$(4.4) \quad S = S_0 + S_1 + S_2 + S_3 + S_4,$$

where S_k denotes the sum for $2u + v \equiv k \pmod{5}$, $0 \leq k \leq 4$. If $2u + v \equiv 0$, then $u - 2v \equiv 0$. Let $2u + v = 5m$ and $u - 2v = -5n$. Then $u = 2m - n$, $v = m + 2n$, and

$$h(u, v) = h(2m - n, m + 2n) = (-1)^{(3m+n)} \lambda^{\frac{5}{2}(5m^2+5n^2-m-n)} q^{\frac{5}{2}(3m^2+3n^2-m-n)}.$$

Therefore,

$$\begin{aligned}
 S_0 &= \sum_{\substack{u,v \\ 2u+v \equiv 0}} h(u,v) \\
 &= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} h(2m-n, m+2n) \\
 &= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} (-1)^{(3m+n)} \lambda^{\frac{5}{2}(5m^2+5n^2-m-n)} q^{\frac{5}{2}(3m^2+3n^2-m-n)} \\
 &= \sum_{m=-\infty}^{\infty} (-1)^m (\lambda^{25} q^{15})^{m^2/2} (\lambda^{-5} q^{-5})^{m/2} \\
 &\quad \times \sum_{n=-\infty}^{\infty} (-1)^n (\lambda^{25} q^{15})^{n^2/2} (\lambda^{-5} q^{-5})^{n/2} \\
 (4.5) \quad &= f(-\lambda^{10} q^5, -\lambda^{15} q^{10}) f(-\lambda^{10} q^5, -\lambda^{15} q^{10}).
 \end{aligned}$$

Similarly,

$$(4.6) \quad S_1 = -\lambda f(-\lambda^5 q^4, -\lambda^{20} q^{11}) f(-\lambda^{10} q^7, -\lambda^{15} q^8),$$

$$(4.7) \quad S_2 = -\lambda^2 q f(-\lambda^5 q^4, -\lambda^{20} q^{11}) f(-\lambda^5 q^2, -\lambda^{20} q^{13}),$$

$$(4.8) \quad S_3 = \lambda^3 q f(-q, -\lambda^{25} q^{14}) f(-\lambda^{10} q^7, -\lambda^{15} q^8),$$

and

$$(4.9) \quad S_4 = \lambda^4 q^2 f(-q, -\lambda^{25} q^{14}) f(-\lambda^5 q^2, -\lambda^{20} q^{13}).$$

Substituting (4.5)–(4.9) in (4.4) and then using (4.3), we complete the proof of (1.4).

Let $\zeta = \exp(2\pi i/5)$. In (1.3), replace $q^{1/5}$ by $\zeta^n q^{1/5}$ for $n = 1, 2, 3, 4, 5$, and then multiply the five identities. Then we find that

$$(4.10) \quad \prod_{n=1}^5 \left(\frac{P}{\zeta^n} - \zeta^n Q - 1 \right) = \frac{1}{q f^5(-\lambda^{10} q^5, -\lambda^{15} q^{10})} \prod_{n=1}^5 f(-\zeta^n q^{1/5}, -\zeta^{2n} \lambda q^{2/5}).$$

Simplifying the left side of (4.10) yields

$$P^5 - Q^5 - 1 - 5PQ - 5P^2Q^2.$$

In Theorem 3.1, let $j = 1, k = 2, s = 3, a = -q^{1/5}, b = -\lambda q^{2/5}$ and $p = 5$. Then

$$(4.11) \quad \prod_{n=1}^5 f(-\zeta^n q^{1/5}, -\zeta^{2n} \lambda q^{2/5}) = \frac{f(-q, -\lambda^5 q^2) f^5(-\lambda^2 q, -\lambda^3 q^2)}{f(-\lambda^{10} q^5, -\lambda^{15} q^{10})}.$$

Using (4.11) in (4.10), we finish the proof of (1.5). □

Now we shall show that (1.1) and (1.2) are special cases of (1.3) and (1.5).

Proofs of (1.1) and (1.2). Let $\lambda = 1$ in (1.3) and (1.5). Then by applying the quintuple identity (2.5) with $(x, \lambda) = (q, q^2)$ and (q^2, q^{-1}) , respectively, we see that, by (2.4) and (2.6),

$$P = \frac{f(-q^7, -q^8) + q f(-q^2, -q^{13})}{q^{1/5} f(-q^5)} = \frac{f(-q^2, -q^3)}{q^{1/5} f(-q, -q^4)} = \frac{1}{q^{1/5} F(q)},$$

and

$$Q = \frac{f(-q^4, -q^{11}) - qf(-q, -q^{14})}{q^{-1/5}f(-q^5)} = \frac{q^{1/5}f(-q, -q^4)}{f(-q^2, -q^3)} = q^{1/5}F(q).$$

Since $PQ = 1$, (1.3) and (1.5) become (1.1) and (1.2). \square

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