PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 126, Number 10, October 1998, Pages 2855–2856
S 0002-9939(98)04564-X

NONCOMPLETE LINEAR SYSTEMS ON ELLIPTIC CURVES AND ABELIAN VARIETIES: ADDENDUM TO A PAPER BY CH. BIRKENHAKE

E. BALLICO

(Communicated by Ron Donagi)

ABSTRACT. Here we give a result on the postulation (i.e. the 2-normality) of nonlinearly normal embeddings of Abelian varieties. This result improves some of the results proved in a recent paper by Ch. Birkenhake.

In [Bi1], [Bi2] and [Bi3], Ch. II, Ch. Birkenhake considered the postulation and the minimal free resolution of noncomplete embeddings into $\mathbf{P}^N$, respectively of Abelian varieties, projective spaces and curves. In [Bi1], §1 and §2, a general set-up for the study of the minimal free resolution of noncomplete embeddings of algebraic varieties into $\mathbf{P}^N$ was given. A cursory reading of [Bi1] shows that a key point of the proofs was the reduction to the case of a product of elliptic curves and then to the case (via Künneth formula) to the case of an embedding of an elliptic curve. In [Bi1] only the good properties of general projections into $\mathbf{P}^{m-1}$ of a complete embedding into $\mathbf{P}^m$ of an elliptic curve were used. However, there are similar results for general projections into $\mathbf{P}^s$ with s much smaller than $m-1$ (see [BE1], [BE2], [BE3] and the statement here of 1.1). Here we want to show how to use these results to improve some of the results of [Bi1]. Our result is the following theorem which improves [Bi1], Cor. 4.3.

Theorem 0.1. *Suppose (X, L) is a general complex Abelian variety of type $(d_1, \dots, d_g)$. Fix an integer $n \geq 3$ such that $nd_g \geq 6$. Let w be the largest integer such that $(nd_g - w)(nd_g - w + 1) \geq 4nd_g$. Fix an integer c with $1 \leq c \leq wn^{g-1}$. Then the general vector subspace $V \subseteq H^0(X, L^{\otimes n})$ with $\operatorname{codim}(V) = c$ is 2-normal, i.e. the restriction map $S^2(V) \to H^0(X, L^{\otimes 2n})$ is surjective.*

For the postulation of general projections of Hirzebruch surfaces and Veronese embeddings of $\mathbf{P}^2$, see [BE4].

1. The proofs

As in [Bi1] we work in characteristic 0. Let $k \geq 1$ be an integer. Recall that with the terminology of [Bi1] a closed subscheme Z of $\mathbf{P}(V^*)$ is k-normal if the canonical map $S^k(V) \to H^0(Z, \boldsymbol{O}_Z(k))$ is surjective. The following result was proved in [BE1] and it is a very particular case of the particular case "$g = 1$" of [BE2] (projections into $\mathbf{P}^3$) and [BE3], Th. I (projections into $\mathbf{P}^N$, $N \geq 4$).

Received by the editors March 18, 1997.
1991 *Mathematics Subject Classification.* Primary 14C20, 14H52, 14K05.
The author was partially supported by MURST and GNSAGA of CNR (Italy).

Theorem 1.1. *Fix integers d, N with $d > N \geq 3$ and $(N+2)(N+1)/2 \geq 2d$. Fix an elliptic curve Y and $L \in \mathrm{Pic}^d(Y)$ and let $X \subset \mathbf{P}^{d-1}$ be the linearly normal embedding of Y determined by $H^0(Y, L)$. Then the general projection of X into $\mathbf{P}^N$ is k-normal for all integers $k \geq 2$.*

Remark 1.2. Note that since $h^0(\mathbf{P}^N, \boldsymbol{O}_{\mathbf{P}^N}(2)) = (N+2)(N+1)/2$ and $h^0(Y, L^{\otimes 2}) = 2d$, the statement of Theorem 1.1 is sharp.

Remark 1.3. Note that in the statement of Theorem 1.1 we do not need the assumption that L is a square, i.e. that d is even, made in [Bi1], Cor. 4.2. Hence in our improvement 0.1 of [Bi1], Cor. 4.3, we do not need the assumption that nd_g is even.

Proof of Theorem 0.1. Theorem 0.1 follows from the proof of [Bi1], Cor. 4.3, using 1.1 instead of [Bi1], Cor. 4.2. □

References

[BE1] E. Ballico, Ph. Ellia, *Sur la postulation des courbes de $\mathbf{P}^n$ et de leurs projections*, C. R. Acad. Sc. Paris **299** (1984), 237–240. MR **86a:**14029

[BE2] E. Ballico, Ph. Ellia, *On the projection of a general curve in $\mathbf{P}^3$*, Annali Mat. Pura e Applicata (4) **142** (1985), 15–48. MR **87g:**14026

[BE3] E. Ballico, Ph. Ellia, *On the postulation of a general projection of a curve in $\mathbf{P}^N$, $N \geq 4$*, Annali Mat. Pura e Applicata (4) **147** (1987), 267–301. MR **88i:**14026

[BE4] E. Ballico, Ph. Ellia, *On projections of ruled and Veronese surfaces*, J. of Algebra **121** (1989), 477–487. MR **90f:**14016

[Bi1] Ch. Birkenhake, *Linear systems on projective spaces*, Manuscripta Math. **88** (1995), 177–184. MR **96h:**14003

[Bi2] Ch. Birkenhake, *Noncomplete linear systems on abelian varieties*, Trans. Amer. Math. Soc. **348** (1995), 1885–1908. MR **97a:**14005

[Bi3] Ch. Birkenhake, *Nicht volloëindige Liniarsysteme*, Habilitaitionsschrift, Erlangen, 1994.

Department of Mathematics, University of Trento, 38050 Povo (TN), Italy

E-mail address: ballico@science.unitn.it