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AN INDEX THEOREM FOR TOEPLITZ OPERATORS ON TOTALLY ORDERED GROUPS

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Abstract. We show that for every totally ordered group Γ and invertible function $f \in C(\widehat{\Gamma})$ which does not have a logarithm, there is a representation in which the Toeplitz operator T_f is a Breuer-Fredholm operator with nonzero index; this representation is the GNS-representation associated to a natural unbounded trace on the Toeplitz algebra $\mathcal{T}(\Gamma)$.

The Toeplitz algebra $\mathcal{T}(\Gamma)$ of a totally ordered abelian group Γ is the C^*-algebra of operators on the Hardy space $H^2(\widehat{\Gamma})$ generated by the compressions T_f of the multiplication operators M_f for $f \in C(\widehat{\Gamma})$. When $\Gamma = \mathbf{Z}$, the Toeplitz operator T_f is Fredholm if and only if $f \in C(\mathbf{T})^{-1}$, and then its Fredholm index is minus the winding number of f about 0. For other Γ, T_f is Fredholm if and only if it is invertible, and to get an interesting index theorem, one has to change one's concept of Fredholm operator.

Coburn, Douglas, Schaeffer and Singer [3] showed that, if Γ is a subgroup of $\mathbf{R}$, there is a representation π of $\mathcal{T}(\Gamma)$ such that $\pi(T_f)$ is a Breuer-Fredholm element of the II_∞-factor $\pi(\mathcal{T}(\Gamma))''$ whenever f is invertible in $C(\widehat{\Gamma})$, and gave a formula for the index. Subsequently Murphy proved a version of this index theorem for more general ordered groups [8], but only a restricted class of T_f with f invertible are Fredholm in his representation. Here we extend Murphy's result in two directions. First of all, we prove that for each totally ordered abelian group Γ and each invertible $f \in C(\widehat{\Gamma})$, there is a representation π of $\mathcal{T}(\Gamma)$ in which $\pi(T_f)$ is Breuer-Fredholm. Secondly, we show that these representations are the GNS-representations of certain unbounded traces on the Toeplitz algebra $\mathcal{T}(\Gamma)$, thus explaining more clearly how the representations used in [3], [8] are canonically associated to the Toeplitz algebra.

Murphy has shown elsewhere [9] that a trace τ on a C^*-algebra B naturally gives rise to an index theory of Fredholm elements of B. This is not obvious: since C^*-algebras need not contain projections, the obvious definitions of dimension as the trace of a projection and index as the difference of two dimensions are not available. His elegant solution, which deserves to be much better known, is to declare $b \in B$ to be Fredholm if it has a inverse c modulo the ideal $\mathcal{M}_\tau$ of elements of finite trace, and then define the index of b to be $\tau(bc - cb)$. We have couched our index theorem

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first in Murphy's C^*-algebraic context, and then apply the GNS-construction to convert to statements about the Breuer-Fredholm index.

We begin by constructing traces on the Toeplitz algebra $\mathcal{T}(\Gamma)$. We realise $\mathcal{T}(\Gamma)$ as a corner in a crossed product $B_\Gamma \times \Gamma$, construct invariant traces on B_Γ from Archimedean subquotients of Γ, and then use a theorem of Zeller-Meier to extend them to traces on $B_\Gamma \times \Gamma$. Our construction is based on an embedding of an order ideal as a subgroup of $\mathbf{R}$; in retrospect, we have merely used the language of traces to abbreviate Murphy's construction of an invariant measure in [8, pp.102–108]. Our index theorem is in §2; the idea is to associate to each invertible function f in $C(\widehat{\Gamma})$ an Archimedean subquotient of Γ, and then use the construction of §1 to produce traces on $\mathcal{T}(\Gamma)$ for which T_f has nonzero τ-index. The Breuer-Fredholm index version of our theorem is in §3, where we have also taken some care to relate our result to the original theorem of [3].

1. Traces on the Toeplitz algebra

Let Γ be a discrete totally ordered abelian group. For $x \in \Gamma$, we define $1_x \in \ell^\infty(\Gamma)$ by

$$1_x(y) = \begin{cases} 1 & \text{if } y \geq x, \\ 0 & \text{if } y < x. \end{cases}$$

Since $1_x 1_y = 1_{\max(x,y)}$, the closed span $B_\Gamma := \overline{\operatorname{sp}}\{1_x : x \in \Gamma\}$ is a C^*-subalgebra of $\ell^\infty(\Gamma)$, and the action of Γ by translation on $\ell^\infty(\Gamma)$ restricts to an action $\alpha : \Gamma \to \operatorname{Aut} B_\Gamma$ such that $\alpha_x(1_y) = 1_{x+y}$. If $\lambda : \Gamma \to U(\ell^2(\Gamma))$ is the left regular representation of Γ, and M is the action of $\ell^\infty(\Gamma)$ by multiplication operators, then $M \times \lambda$ is a faithful representation of $B_\Gamma \times_\alpha \Gamma$ which carries the corner associated to the projection $i_{B_\Gamma}(1_0)$ onto $\mathcal{T}(\Gamma) = C^*(M(1_0)\lambda_x M(1_0))$. This is Theorem 3.14 of [6]; alternatively, one can realise $\mathcal{T}(\Gamma)$ as a semigroup crossed product $B_{\Gamma^+} \times_\alpha \Gamma^+$ as in [2], and deduce from the general theory of [1] that $B_{\Gamma^+} \times_\alpha \Gamma^+ = i_{B_\Gamma}(1_0)\,(B_\Gamma \times_\alpha \Gamma)\,i_{B_\Gamma}(1_0)$.

To obtain traces on $\mathcal{T}(\Gamma)$, we first need to construct invariant traces on B_Γ. When Γ is a subgroup of $\mathbf{R}$, the linearisation of the map $1_x \mapsto \chi_{[x,\infty)}$ is an isometric isomorphism of $\operatorname{sp}\{1_x : x \in \Gamma\}$ onto a *-subalgebra of $L^\infty(\mathbf{R})$, which therefore extends to an embedding Φ of B_Γ as a C^*-subalgebra of $L^\infty(\mathbf{R})$. Thus we can define a trace σ on B_Γ by integrating with respect to Lebesgue measure: $\sigma(f) = \int_{\mathbf{R}} \Phi(f)\,dm$. More generally, we have:

Proposition 1. *Suppose I is an Archimedean ordered ideal of Γ, so that there is an order isomorphism ϕ of I into $\mathbf{R}$. For every such ϕ, there is an invariant semifinite lower semicontinuous trace σ on B_Γ such that $\sigma(1_x - 1_y) = \phi(y - x)$ whenever $y - x \in I$.*

Proof. Let $B_I^+ = \overline{\operatorname{sp}}\{1, 1_x : x \in I\}$ be the C^*-algebra obtained by adjoining an identity to B_I. The formula $\Phi(1_x) = \chi_{[\phi(x),\infty)}$ extends by linearity to an isometric homomorphism of $\operatorname{sp}\{1, 1_x : x \in I\}$ onto a $*$-subalgebra of $L^\infty(\mathbf{R})$, and hence to a homomorphism Φ on all of B_I^+. Next, choose a complete set of coset representatives $\{x_r\}$ for Γ/I. For each r, define $\Phi_r : B_\Gamma \to L^\infty(\mathbf{R})$ by

$$\Phi_r(f) := \Phi\big(\alpha_{x_r}^{-1}(f|_{x_r+I})\big) \quad \text{for } f \in B_\Gamma;$$

since every $f \in B_\Gamma$ has restriction $f|_I$ in B_I^+, and $\alpha_{x_r}^{-1}(f\big|_{x_r+I}) = \alpha_{x_r}^{-1}(f)|_I$, this definition of Φ_r makes sense. Now define σ on B_Γ by $\sigma(f) = \sum_r \int_{\mathbf{R}} \Phi_r(f)dm$. Note that, although $\Phi_r(f)$ depends on the choice of coset representatives $\{x_r\}$, the expression $\int_{\mathbf{R}} \Phi_r(f)\, dm$ does not, because Lebesgue measure is translation invariant. Similarly, the function σ is translation invariant: translating σ by $x \in \Gamma$ may move the cosets around, but $\{x_r + x\}$ is still a complete set of coset representatives of I in Γ. Because each $f \mapsto \int_{\mathbf{R}} \Phi_r(f)dm$ is semifinite, so is σ.

To check that σ is lower semicontinuous, let $\{f_n\}$ be an increasing sequence of positive functions in B_Γ converging to f. Then the sequence $s_n := \sum_r \Phi_r(f_n)$ is increasing with $s_n \to \sum_r \Phi_r(f)$. Thus by the Monotone Convergence Theorem, $\int s_n dm \to \int \sum_r \Phi_r(f)dm$, and hence $\sigma(f_n) \to \sigma(f)$ by two applications of Tonelli's Theorem.

Finally, if $y - x \in I$, then x and y are in the same coset $x_s + I$ for some $s \in \Gamma/I$. Therefore $\Phi_r(1_x - 1_y) = 0$ for $r \neq s$, and

$$\begin{aligned}\sigma(1_x - 1_y) &= \int_{\mathbf{R}} \Phi_s(1_x - 1_y)\, dm = \int_{\mathbf{R}} \Phi(1_{x-x_s} - 1_{y-x_s})\, dm \\ &= \int_{\mathbf{R}} \chi_{[\phi(x-x_s),\phi(y-x_s))}\, dm \\ &= \phi\left((y - x_s) - (x - x_s)\right) = \phi(y - x),\end{aligned}$$

as required. □

Corollary 2. *If I is an Archimedean order ideal in Γ and ϕ is an isomorphism of I into $\mathbf{R}$, then there is a semifinite lower semicontinuous trace τ on $\mathcal{T}(\Gamma)$ such that $\tau(T_xT_x^* - T_yT_y^*) = \phi(y - x)$ whenever $x, y \in \Gamma$ satisfy $y - x \in I$.*

Proof. Proposition 1 gives a semifinite lower semicontinuous trace σ on B_Γ. By [11, 9.3], the composition $\sigma \circ E$ of σ with the conditional expectation $E : B_\Gamma \times_\alpha \Gamma \to B_\Gamma$ is a lower semicontinuous trace on $B_\Gamma \times_\alpha \Gamma$. Let τ be the restriction of $\sigma \circ E$ to the corner $i_{B_\Gamma}(1_0)(B_\Gamma \times \Gamma)i_{B_\Gamma}(1_0)$. Then τ is a semifinite lower semicontinuous trace on $\mathcal{T}(\Gamma)$, and whenever $y - x \in I$, we have

$$\tau(T_xT_x^* - T_yT_y^*) = \sigma \circ E\big(i_{B_\Gamma}(1_x - 1_y)\big) = \sigma(1_x - 1_y) = \phi(y - x).$$

□

2. A C^*-algebraic index theorem

Suppose τ is a trace on a unital C^*-algebra B, and

$$\mathcal{M}_\tau := \operatorname{sp}\{b \in B^+ : \tau(b) < \infty\}$$

is the (non-closed) *-ideal of elements of finite trace. An element b of B is *Fredholm relative to* τ if there is an element $c \in B$ such that $1 - bc$ and $1 - cb$ belong to $\mathcal{M}_\tau$, and the index is then $\tau\text{-ind}(b) := \tau(bc - cb)$. Murphy has shown that most of the usual index theory of Fredholm operators carries over to this setting [9, §3]. (The possible exception is the vanishing of the index for normal elements of B, which is open.)

Theorem 3. *Suppose Γ is a totally ordered group, and $f \in C(\widehat{\Gamma})^{-1}$ does not have a logarithm in $C(\widehat{\Gamma})$. Then there is a semifinite lower semicontinuous trace τ on the Toeplitz algebra $\mathcal{T}(\Gamma)$ such that T_f is Fredholm relative to τ and $\tau\text{-}\operatorname{ind} T_f \neq 0$.*

Proof. Since Γ is totally ordered, it is torsion-free, and hence has connected dual $\widehat{\Gamma}$. Thus by a theorem of van Kampen (e.g. [7, Theorem 1.1]), there exist $x \in \Gamma$ and $g \in C(\widehat{\Gamma})$ such that $f = \epsilon_x e^g$ and $x \neq 0$. The set

$$I_x := \{y \in \Gamma : ny \leq |x| \text{ for all } n \in \mathbf{Z}\}$$

is an order ideal of Γ. The quotient Γ/I_x is itself a totally ordered group in which

$$z + I_x \geq 0 \iff z + y \geq 0 \text{ for some } y \in I_x;$$

let $p : \Gamma \to \Gamma/I_x$ be the quotient map. The image $p(x)$ of x is almost by definition a finite element of Γ/I_x, and is nonzero because $x \notin I_x$; in particular, the order ideal $F(\Gamma/I_x)$ of finite elements is nonzero. Since $F(\Gamma/I_x)$ is always Archimedean, there is an embedding ϕ of $F(\Gamma/I_x)$ in $\mathbf{R}$, and hence by Corollary 2 there is a trace τ_1 on $\mathcal{T}(\Gamma/I_x)$ such that $\tau_1(T_{p(x)}T_{p(x)}^* - 1) = -\phi(p(x))$. From the universal property of $\mathcal{T}(\Gamma)$, we deduce that there is a canonical map $q : \mathcal{T}(\Gamma) \to \mathcal{T}(\Gamma/I_x)$ satisfying $q(T_x) = T_{p(x)}$, and then $\tau := \tau_1 \circ q$ is a lower semicontinuous trace on $\mathcal{T}(\Gamma)$ satisfying $\tau(T_x T_x^* - 1) = -\phi(p(x))$.

To see that T_f is τ-Fredholm, suppose first that $x \geq 0$. Then $T_f = T_{e^g} T_x$. It is quite easy to see that T_{e^g} is invertible (for example, [7, p.4]), and hence it is trivially τ-Fredholm. Since $T_x^* T_x = 1$, the equation $\tau(1 - T_x T_x^*) = \phi(p(x))$ implies that T_x^* is a partial inverse for T_x relative to τ, and hence T_x is τ-Fredholm. We deduce that the product $T_f = T_{e^g} T_x$ is τ-Fredholm, with

$$(*) \qquad \tau\text{-ind}\, T_f = \tau\text{-ind}\, T_{e^g} + \tau\text{-ind}\, T_x = 0 - \phi(p(x)) = -\phi(p(x)),$$

which is nonzero because $x \notin I_x$ and ϕ is injective on Γ/I_x. If $x \leq 0$, we can apply the previous case to $T_f^* = T_{\overline{f}}$, and the result follows. □

Remark 4. Since the traces we constructed in §1 depend on the choice of an isomorphism ϕ of an abstract Archimedean ideal I into $\mathbf{R}$, and multiplying ϕ by any $c \in (0, \infty)$ gives another such isomorphism, the numerical value of the index is not significant. If, however, we start with a subgroup Γ of $\mathbf{R}$, then in some sense this choice has already been made, and our construction gives the trace σ on B_Γ corresponding to Lebesgue integration on $L^\infty(\mathbf{R}) \supset B_\Gamma$. Composing with the dual embedding of $\mathbf{R}$ in $\widehat{\Gamma}$ converts functions in $C(\widehat{\Gamma})$ to almost periodic functions on $\mathbf{R}$; if $f = \epsilon_x e^g$ for some $x \in \Gamma$, then the corresponding function F on $\mathbf{R}$ has the form $t \mapsto e^{ixt} e^{G(t)}$, where G is the almost periodic function $g|_{\mathbf{R}}$, and one can recover x as the mean motion

$$\lim_{t\to\infty} \frac{1}{2t}\big(\arg F(t) - \arg F(-t)\big) = \lim_{t\to\infty} \frac{1}{2t}\big(xt - iG(t) - (-xt - iG(-t))\big)$$

of the almost periodic function F.

The Toeplitz algebra of a subgroup Γ of $\mathbf{R}$ has a faithful representation as Wiener-Hopf operators on $L^2[0, \infty)$; the Toeplitz operator on $H^2(\widehat{\Gamma})$ with symbol $f \in C(\widehat{\Gamma})$ is carried into the Wiener-Hopf operator W_F with symbol $F := f|_{\mathbf{R}} \in AP(\mathbf{R})$. (It follows from a theorem of Douglas [5] that this is a faithful representation of $\mathcal{T}(\Gamma)$; alternatively, note that B_Γ^+ acts faithfully on $L^2[0, \infty)$ and apply the main theorem of [2].) Thus we can deduce from $(*)$ that there is a natural trace on the Wiener-Hopf C^*-algebra on $L^2[0, \infty)$ for which the index of W_F is minus the mean motion of F.

3. The Breuer-Fredholm index theorem

Proposition 5. *Suppose $\sigma : B \to [0,\infty]$ is a semifinite lower semicontinuous trace on a C^*-algebra B, and π_σ is the associated GNS-representation of B on H_σ. Then there is a faithful semifinite normal trace $\widetilde{\sigma}$ on $\pi_\sigma(B)'' \subset B(H_\sigma)$ such that $\widetilde{\sigma}(\pi_\sigma(b)) = \sigma(b)$ for $b \in B^+$.*

Proof. Let $\mathcal{N}_\sigma := \{b \in B : \sigma(b^*b) < \infty\}$. Then σ extends uniquely to a functional σ' on $\mathcal{M}_\sigma := \mathcal{N}_\sigma^2$ which agrees with σ on $\mathcal{M}_\sigma^+ = \mathcal{M}_\sigma \cap B^+$ [4, 6.1.2]. It follows from [4, §6.4 and 6.2] that if $N_\sigma := \{b \in B : \sigma(b^*b) = 0\}$, then $A := \mathcal{N}_\sigma/N_\sigma$ is a Hilbert algebra with respect to the inner product $(a + N_\sigma | b + N_\sigma) := \sigma'(b^*a)$. The Hilbert space completion H used in [4] is precisely H_σ, the operator $\pi_\sigma(b)$ is the operator U_b on H, and $U(A) = \pi_\sigma(B)''$. Thus [4, A60] says that there is a faithful semifinite normal trace $\widetilde{\sigma}$ on $\pi_\sigma(B)''$, and for each $b \in \mathcal{N}_\sigma$ we have

$$\widetilde{\sigma}(\pi_\sigma(b^*b)) = \widetilde{\sigma}(\pi_\sigma(b)^*\pi_\sigma(b)) = (b + N_\sigma | b + N_\sigma) = \sigma(b^*b) < \infty;$$

in particular this implies that $\pi_\sigma(\mathcal{N}_\sigma) \subset \mathcal{N}_{\widetilde{\sigma}}$. One further deduces from [4, A60] that the unique extension $\widetilde{\sigma}'$ of $\widetilde{\sigma}$ to $\mathcal{M}_{\widetilde{\sigma}}$ satisfies

$$\widetilde{\sigma}'(\pi_\sigma(b)^*\pi_\sigma(a)) = (a + N_\sigma | b + N_\sigma) = \sigma'(b^*a),$$

so that for every $x \in \mathcal{M}_\sigma$ we have $\sigma'(x) = \widetilde{\sigma}'(\pi_\sigma(x))$. It follows from [4, §6.6] that $\sigma = \widetilde{\sigma}' \circ \pi_\sigma$. ☐

Recall from [10] that if τ is a semifinite normal trace on a von Neumann algebra N, then an operator $T \in N$ is *Fredholm relative to* τ if the projection N_T on $\ker T$ has $\tau(N_T) < \infty$, and there is a projection $E \in N$ such that $\tau(E) < \infty$ and the range of $(1 - E)$ is contained in the range of T; the *Breuer-Fredholm* τ*-index* of T is then $\tau\text{-ind}\, T := \tau(N_T) - \tau(N_{T^*})$.

Theorem 6. *Let Γ be a totally ordered abelian group, and suppose that $f \in C(\widehat{\Gamma})$ does not have a logarithm in $C(\widehat{\Gamma})$. Then there are a representation π of the Toeplitz algebra $\mathcal{T}(\Gamma)$ and a trace μ on $\pi(\mathcal{T}(\Gamma))''$ such that $\pi(T_f) \in \pi(\mathcal{T}(\Gamma))''$ is Fredholm relative to μ, with Breuer-Fredholm index $\mu\text{-}\operatorname{ind}(\pi(T_f)) \neq 0$.*

Proof. We proceed as in Theorem 3, writing $f = \epsilon_x e^g$, etc. We then take π to be the GNS-representation π_τ associated to the trace τ in that theorem, and $\mu := \widetilde{\tau}$. Then $\pi_\tau(T_x)$ is an isometry with range projection $\pi_\tau(T_x T_x^*)$ satisfying

$$\widetilde{\tau}(1 - \pi_\tau(T_x T_x^*)) = \tau(1 - T_x T_x^*) = \phi(p(x));$$

since T_{e^g} is invertible, we deduce that $\pi_\tau(T_f)$ is Breuer-Fredholm with $\mu\text{-ind}\, \pi_\tau(T_f) = -\phi(p(x)) \neq 0$, as required. ☐

Remark 7. If Γ is a subgroup of $\mathbf{R}$, we can view B_Γ as a C^*-subalgebra of $L^\infty(\mathbf{R})$, and the trace σ on B_Γ is then given by $\sigma(f) := \int f\, dm$ (see Remark 4). Thus the trace $\tau := \sigma \circ E$ of Corollary 2 is given on positive elements of $C_c(\Gamma, B_\Gamma)$ by

$$\begin{aligned}\sigma \circ E(f^*f) &= \sigma(f^*f(e)) = \sigma\Big(\sum_s f^*(s)\alpha_s(f(s^{-1}))\Big)\\ &= \sum_s \sigma\big(\alpha_s\big(\overline{f(s^{-1})} f(s^{-1})\big)\big) = \sum_t \sigma\big(|f(t)|^2\big)\\ &= \sum_t \int_{\mathbf{R}} |f(t)|^2\, dm;\end{aligned}$$

we deduce that $H_{\sigma\circ E} = \ell^2(\Gamma, L^2(\mathbf{R}))$. Since the representation $\pi_{\sigma\circ E}$ of $C_c(\Gamma, B_\Gamma) \subset B_\Gamma \times \Gamma$ extends the action by left multiplication on $C_c(\Gamma, B_\Gamma) \subset \ell^2(\Gamma, L^2(\mathbf{R}))$, we can see by inspection that $\pi_{\sigma\circ E}$ is the integrated form of the covariant representation $(1 \otimes M, \lambda \otimes \lambda)$ of $(B_\Gamma, \Gamma, \alpha)$ on $\ell^2(\Gamma) \otimes L^2(\mathbf{R}) = \ell^2(\Gamma, L^2(\mathbf{R}))$. This is precisely the representation used in [3], and in view of Remark 4, our index theorem reduces in this case to that of [3].

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