

## UNIQUE DECOMPOSITION OF RIEMANNIAN MANIFOLDS

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(Communicated by Christopher Croke)

ABSTRACT. We prove an extension of de Rham's decomposition theorem to the non-simply connected case.

### 1. INTRODUCTION

A connected Riemannian manifold may allow more than one decomposition into a product of indecomposable factors: Euclidean space of dimension  $\geq 2$  splits orthogonally into a product of one-dimensional subspaces in many different ways. But by the classical theorem of de Rham ([dR], cf. also [KN], [M], [P]), this is essentially the only *simply* connected example with that property. The purpose of our note is to generalize this result to the non-simply connected case as well.

**Theorem.** *Any complete connected Riemannian manifold  $M$  decomposes into a Riemannian product*

$$(1) \quad M = M_0 \times M_1 \times \dots \times M_p$$

where  $M_0$  is a maximal factor isometric to euclidean space and each  $M_i$ ,  $i > 0$ , is indecomposable. This decomposition is unique up to the order of  $M_1, \dots, M_p$ .

We call a Riemannian manifold *indecomposable* if it is not isometric to a Riemannian product of lower dimensional manifolds. Any (holonomy) *irreducible* manifold is indecomposable and by de Rham's theorem also the converse is true for simply connected manifolds. But in general the two notions differ: A non-rectangular flat 2-torus is indecomposable but not irreducible. By decomposing a manifold further and further it is clear that any Riemannian manifold admits a decomposition into a product of indecomposable ones. Therefore, the only question is about uniqueness. We say that a product decomposition is *unique* if the corresponding foliations are uniquely determined.

If  $M$  is compact, there is no euclidean factor. Hence we get the following

**Corollary 1.** *Let  $M$  be a compact Riemannian manifold. Then  $M$  decomposes uniquely into a Riemannian product of indecomposable factors. Any isometry of  $M$  must preserve or interchange these factors. In particular, for any Riemannian product decomposition  $M = M_1 \times M_2$ , the identity component of the isometry group splits as  $I_0(M) = I_0(M_1) \times I_0(M_2)$ .*

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Received by the editors February 28, 1997.

1991 *Mathematics Subject Classification.* Primary 53C20; Secondary 53C12.

*Key words and phrases.* Riemannian products, indecomposable Riemannian manifolds, irreducible Riemannian manifolds, de Rham's theorem.

Another immediate consequence is a theorem due to Uesu [U] generalizing a previous result of Takagi [T]:

**Corollary 2.** *Let  $M$ ,  $N$  and  $B$  be complete connected Riemannian manifolds. If  $M \times B$  is isometric to  $N \times B$  then  $M$  is isometric to  $N$ .*

The main idea of the proof of our Theorem is to use a special (so-called “short”) set of generators of the fundamental group which is compatible to *any* Riemannian decomposition of  $M$ . The same generating set had been used by Gromov in order to estimate the number of generators of the fundamental group (cf. [G]).

## 2. PROOF OF THE THEOREM

By the remark above, we only have to show uniqueness. This will follow from a series of lemmas. We always denote by  $\tilde{M}$  the universal cover of  $M$  and by  $\Gamma$  its group of deck transformations. A decomposition of  $\tilde{M}$  into a product  $\tilde{M} = X_1 \times \dots \times X_k$  determines  $k$  foliations on  $\tilde{M}$  whose leaves through a point  $p \in \tilde{M}$  will be denoted by  $X_i(p)$ . We say that an isometry  $\phi$  of  $\tilde{M}$  acts *only on*  $X_i$  (or *trivially on all*  $X_j$ ,  $j \neq i$ ) if

$$\phi(x_1, \dots, x_i, \dots, x_k) = (x_1, \dots, \phi_i x_i, \dots, x_k)$$

for some isometry  $\phi_i$  of  $X_i$ . In the language of foliations this means that each leaf  $X_i(p)$  is  $\phi$ -invariant, i.e.  $\phi(p) \in X_i(p)$  for all  $p \in \tilde{M}$ .

**Lemma 1.** *The maximal euclidean factor  $M_0$  of  $M$  is uniquely determined.*

*Proof.* By de Rham’s theorem  $\tilde{M}$  splits uniquely into  $E \times N$  where  $E$  is euclidean and  $N$  has no euclidean factor. Furthermore  $\Gamma$  preserves this splitting, i.e. each  $\gamma \in \Gamma$  is of the form  $(\gamma_E, \gamma_N)$  where  $\gamma_E$  and  $\gamma_N$  are isometries of  $E$  and  $N$ , respectively. Now any euclidean factor of  $M$  corresponds to a factor  $E_1$  of  $E$  on which  $\Gamma$  acts trivially. If  $E = E_1 \times E_2$  as a Riemannian manifold, then  $E_i(x) = \vec{E}_i + x$  where  $\vec{E}_1 \oplus \vec{E}_2$  is an orthogonal splitting of the euclidean vector space  $\vec{E}$  acting simply transitively on the affine space  $E$  by translations. By the remark before Lemma 1,  $\gamma \in \Gamma$  acts trivially on  $E_1$  if

$$\gamma_E(x) \in E_2(x) = \vec{E}_2 + x$$

for all  $x \in E$  which in turn is equivalent to  $(\gamma_E x - x) \perp \vec{E}_1$ . Thus  $E_1$  is maximal if

$$\vec{E}_1 = \{\gamma_E x - x; x \in E, \gamma \in \Gamma\}^\perp \subset \vec{E}_1$$

but this is uniquely determined.  $\square$

**Lemma 2.** *Let  $\tilde{M} = \tilde{M}_1 \times \dots \times \tilde{M}_p = \tilde{M}'_1 \times \dots \times \tilde{M}'_q$  be two decompositions of  $\tilde{M}$ . Then there exists a decomposition  $\tilde{M} = \prod_{i,j} \tilde{M}_{ij} \times F$  of  $\tilde{M}$  where  $F$  is a euclidean factor and  $\tilde{M}_{ij}(p) = \tilde{M}_i(p) \cap \tilde{M}'_j(p)$ .*

*Proof.*  $\tilde{M}_i(p)$  and  $\tilde{M}'_j(p)$  are totally convex in the sense that any minimal geodesic of  $\tilde{M}$  joining two points in  $\tilde{M}_i(p)$  or  $\tilde{M}'_j(p)$  lies completely in  $\tilde{M}_i(p)$  or  $\tilde{M}'_j(p)$ , respectively. Therefore,  $\tilde{M}_{ij}(p)$  is a totally geodesic connected submanifold of  $\tilde{M}$ . The tangent spaces  $\mathcal{D}_{ij}(p) = T_p(\tilde{M}_{ij}(p)) = T_p(\tilde{M}_i) \cap T_p(\tilde{M}'_j)$  form a distribution  $\mathcal{D}_{ij}$  which is invariant under parallel translations. Therefore we get from de Rham’s theorem  $\tilde{M} = \prod_{i,j} \tilde{M}_{ij} \times F$  where  $F$  is some complementary factor. Since each

irreducible de Rham factor is contained in some  $\tilde{M}_i$  and in some  $\tilde{M}'_j$ , it is also contained in some  $\tilde{M}_{ij}$ . Thus the complementary factor  $F$  must be euclidean.  $\square$

Recall that a splitting  $\tilde{M} = \tilde{M}_1 \times \dots \times \tilde{M}_p$  of the universal cover is induced by a splitting  $M = M_1 \times \dots \times M_p$  of the manifold  $M$  itself if and only if the group  $\Gamma$  of deck transformations splits accordingly. This means that  $\Gamma$  has a set of generators each of which acts only on one of the factors  $\tilde{M}_i$ . We now show that there is even a set of generators which has this property for all splittings of  $M$  at the same time:

**Lemma 3.** *There exists a generating set  $\Sigma$  of  $\Gamma$  such that for any decomposition  $M = M_1 \times \dots \times M_p$  of  $M$ , each  $\sigma \in \Sigma$  acts only on one factor of the corresponding decomposition  $\tilde{M} = \tilde{M}_1 \times \dots \times \tilde{M}_p$ .*

*Proof.* Choose  $o \in \tilde{M}$  and let  $|\gamma| := \text{dist}(o, \gamma o)$  for each  $\gamma \in \Gamma$ . Let  $\Sigma = \{\sigma_1, \sigma_2, \dots\}$  be a *short generating set* in the sense of Gromov [G], i.e.  $\sigma_1$  is chosen with  $|\sigma_1| = \min\{|\gamma|; \gamma \in \Gamma \setminus \{1\}\}$  and  $\sigma_k$  inductively with  $|\sigma_k| = \min\{|\gamma|; \gamma \in \Gamma \setminus \Gamma_{k-1}\}$  where  $\Gamma_{k-1}$  denotes the subgroup generated by  $\sigma_1, \dots, \sigma_{k-1}$ . Each  $\sigma_k \in \Sigma$  (in fact each  $\sigma \in \Gamma$ ) can be written as  $\sigma_k = \gamma_1 \gamma_2 \dots \gamma_p$  where  $\gamma_i$  acts only on  $\tilde{M}_i$ . Hence

$$|\sigma_k|^2 = \sum_{i=1}^p |\gamma_i|^2 \geq |\gamma_j|^2$$

for all  $j$ . In case of a strict inequality we have  $\gamma_j \in \Gamma_{k-1}$  by the choice of  $\sigma_k$ . But this cannot happen for all  $j$  since  $\sigma_k \notin \Gamma_{k-1}$ . Thus there exists  $i \in \{1, \dots, p\}$  with  $|\sigma_k| = |\gamma_i|$  and  $|\gamma_j| = 0$  for all  $j \neq i$  which means  $\sigma_k = \gamma_i$ .  $\square$

*Proof of the Theorem.* By Lemma 1 we may assume that  $M$  contains no euclidean factor. Let  $M = M_1 \times \dots \times M_p = M'_1 \times \dots \times M'_q$  be two decompositions of  $M$  into indecomposable factors. According to Lemma 2 we get a decomposition  $\tilde{M} = \prod_{i,j} \tilde{M}_{ij} \times F$ . Now, if  $\sigma$  is any element of the special generating set of Lemma 3, there exist  $i \in \{1, \dots, p\}$  and  $j \in \{1, \dots, q\}$  such that the leaves  $\tilde{M}_i(p)$  and  $\tilde{M}'_j(p)$  and hence  $\tilde{M}_{ij}(p)$  are  $\sigma$ -invariant for all  $p \in \tilde{M}$ . In particular,  $\sigma$  and hence  $\Gamma$  act trivially on  $F$ . Since  $M$  has no euclidean factor,  $F$  must be trivial, i.e.  $\tilde{M} = \prod_{i,j} \tilde{M}_{ij}$ . Furthermore,  $\Gamma$  is generated by elements  $\sigma$  which act only on one of the factors  $\tilde{M}_{ij}$ . Thus, by the remark before Lemma 3 we get a corresponding decomposition  $M = \prod_{i,j} M_{ij}$  of  $M$  with  $M_{ij}(m) = M_i(m) \cap M'_j(m)$  for all  $m \in M$ . Since the  $M_i$  and  $M'_j$  are indecomposable, the theorem follows.  $\square$

## REFERENCES

- [dR] G. de Rham: *Sur la réductibilité d'un espace de Riemann*, Comm. Math. Helv. **26** (1952), 328 - 344 MR **14**:584a
- [G] M. Gromov: *Almost flat manifolds*, J. Diff. Geom. **13** (1978), 231 - 241 MR **80h**:53041
- [KN] S. Kobayashi, K. Nomizu: *Foundations of Differential Geometry*, vol. 1, Interscience, Wiley, New York 1963 MR **27**:2945
- [M] R. Maltz: *The de Rham product decomposition*, J. Diff. Geom. **7** (1972), 161 - 174 MR **48**:2930
- [P] R. Pantilie: *A simple proof of the de Rham decomposition theorem*, Bull. Math. Soc. Sc. Math. Roumanie **36 (84)** (1992), 341 - 343 MR **95m**:53068

- [T] H. Takagi: *Notes on the cancellation of Riemannian manifolds*, Tôhoku Math. J. **32** (1980), 411 - 417 MR **82g**:53048
- [U] K. Uesu: *Cancellation law for Riemannian direct products*, J. Math. Soc. Japan **36** (1984), 53 - 62 MR **85c**:53072

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