PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 126, Number 11, November 1998, Pages 3371–3376
S 0002-9939(98)04305-6

HOMOGENEOUS IDEALS IN WICK $*$-ALGEBRAS

DANIIL PROSKURIN

(Communicated by Palle E. T. Jorgensen)

Abstract. The necessary and sufficient condition for the family of homogeneous elements to determine a Wick ideal is presented. The structure of homogeneous Wick ideals with degree higher than 2 is discussed. For the braided operator T a formula to calculate the largest cubic ideal when the quadratic one is known is obtained. Irreducible $*$-representations of the μ-CAR algebra are classified.

Introduction

In the study of $*$-representations of Wick algebras, it is useful to know the structure of Wick ideals, and especially, as examples show [1], of homogeneous Wick ideals. For example, in any bounded representation of μ-CCR or q_{ij}-CCR, $|q_{ij}| = 1$ as $i \neq j$, any quadratic ideal vanishes.

In the paper [1], the main attention was concentrated on quadratic ideals, and there was presented a necessary and sufficient condition imposed on the system of elements to generate a quadratic ideal.

In this note this criterion is extended to the general case (Sec. 2). In the case when the operator T (see Sec. 1) satisfies the braid relation, and $-1 \leq T \leq 1$, we investigate the structure of homogeneous Wick ideals of higher degrees. In particular, we improve the theorem on strict positivity of the Fock representation, and derive a formula, which allows one to calculate a cubic ideal providing that a quadratic one is known (Sec. 3). In Sec. 4 these results are illustrated in the example of μ-CAR algebra.

1. Preliminaries

Let $I = \{1, \dots, d\}$, and $T_{ij}^{kl} \in \mathbb{C}$, i, j, k, $l \in I$, be such that $T_{ij}^{kl} = \bar{T}_{ji}^{lk}$. The Wick algebra with coefficients $\{T_{ij}^{kl}\}$ (see [1]) is a $*$-algebra generated by the elements a_i, a_i^* and the defining relations

$$a_i^* a_j = \delta_{ij} 1 + \sum_{k,l=1}^{d} T_{ij}^{kl} a_l a_k^*.$$

Denote by $\mathcal{H} = \langle e_1, \dots, e_d \rangle$ the finite-dimensional space over $\mathbb{C}$, and by $\mathcal{H}^*$ its formal dual. $\mathcal{T}(\mathcal{H}, \mathcal{H}^*)$ will denote the tensor algebra over $\mathcal{H}$, $\mathcal{H}^*$. Then $\mathcal{W}$ can be

Received by the editors November 21, 1996 and, in revised form, December 10, 1996.
1991 *Mathematics Subject Classification.* Primary 81R50, 47A62, 46L05.
This work was partially supported by the CRDF, grant no. 292.

canonically realized as

$$\mathcal{T}(\mathcal{H},\mathcal{H}^*)\Big/\Big\langle e_i^*\otimes e_j-\delta_{ij}1-\sum T_{ij}^{kl}e_l\otimes e_k^*\Big\rangle.$$

In this realization, the subalgebra, generated by $\{a_i\}$ is identified with $\mathcal{T}(\mathcal{H})$.

It is obvious that any element of $\mathcal{W}$ can be uniquely represented as a polynomial in the noncommuting variables a_i, a_i^*, where in each monomial, variables a_i are placed to the left from a_j^*. Such monomials are called Wick ordered ones, and they form a basis in $\mathcal{W}$.

When studying properties of $\mathcal{W}$, one can find useful the following operators (see [1]):

$$\begin{aligned}
&T\colon \mathcal{H}\otimes\mathcal{H}\mapsto\mathcal{H}\otimes\mathcal{H}, \quad Te_k\otimes e_l=\sum_{i,j}T_{ik}^{lj}e_i\otimes e_j,\\
&T_i\colon \mathcal{H}^{\otimes n}\mapsto\mathcal{H}^{\otimes n}, \quad T_i=\underbrace{1\otimes\cdots\otimes 1}_{i-1}\otimes T\otimes\underbrace{1\otimes\cdots\otimes 1}_{n-i-1},\\
&R_n\colon \mathcal{H}^{\otimes n}\mapsto\mathcal{H}^{\otimes n}, \quad R_n=1+T_1+T_1T_2+\cdots+T_1T_2\cdots T_{n-1},\\
&P_n\colon \mathcal{H}^{\otimes n}\mapsto\mathcal{H}^{\otimes n}, \quad P_2=R_2,\ P_{n+1}=(1\otimes P_n)R_{n+1}.
\end{aligned}$$

In what follows, we will use the commutation rule in $\mathcal{W}$ between e_i^* and X, where $X\in\mathcal{H}^{\otimes n}$.

Proposition 1. *Let $X\in\mathcal{H}^{\otimes n}$; then*

$$e_i^*\otimes X=\mu(e_i^*)R_nX+\mu(e_i^*)\sum_{k=1}^{d}T_1T_2\cdots T_n(X\otimes e_k)e_k^*, \tag{1}$$

where $\mu(e_i^*)\colon \mathcal{T}(\mathcal{H})\mapsto\mathcal{T}(\mathcal{H})$ is defined as follows:

$$\mu(e_i^*)1=0, \quad \mu(e_i^*)e_{i_1}\otimes\cdots\otimes e_{i_n}=\delta_{ii_1}e_{i_2}\otimes\cdots\otimes e_{i_n}.$$

Proof. It follows from the defining relations that $e_i^*\otimes X\in\mathcal{H}^{\otimes n-1}\oplus\mathcal{H}^{\otimes n}\otimes\mathcal{H}^*$. The fact that the term $e_i^*\otimes X$ from $\mathcal{H}^{\otimes n-1}$ is equal to the first term in the formula (1) is essentially contained in [1], Lemma 2.1.1.

Now we prove that the term belonging to $\mathcal{H}^{\otimes n}\otimes\mathcal{H}^*$ is equal to the second term. It is obvious that one can assume $X=e_{i_1}\otimes e_{i_2}\otimes\cdots\otimes e_{i_n}$. Denote by F_i the component of $e_i^*\otimes X$ which belongs to $\mathcal{H}^{\otimes n}\otimes\mathcal{H}^*$. Consider "deriving function" $F=\sum_{i=1}^d e_i\otimes F_i$. It is evident that $F_i=\mu(e_i^*)F$. The relations imply

$$F=\sum_i\sum_{k_1,l_1}\cdots\sum_{k_n,l_n}T_{ii_1}^{k_1l_1}T_{k_1i_2}^{k_2l_2}\cdots T_{k_{n-1}i_n}^{k_nl_n}e_i\otimes e_{l_1}\otimes\cdots\otimes e_{l_n}\otimes e_{k_n}^*.$$

Taking into account the definition of T_i and changing summation order, we get

$$F=\sum_{k_n}T_1\cdots T_n(e_{i_1}\otimes\cdots\otimes e_{i_n}\otimes e_{k_n})\otimes e_{k_n}^*,$$

which completes the proof. □

2. Homogeneous ideals. General case

A Wick ideal (see [1]) is a two-sided ideal $I\subset\mathcal{T}(\mathcal{H})$, such that $\mathcal{T}(\mathcal{H}^*)\subset I\mathcal{T}(\mathcal{H}^*)$. If I is generated by a set $I_0\subset\mathcal{H}^{\otimes n}$, then I is called a homogeneous Wick ideal of degree n.

The following statement is a generalization of the fact established in [1] for $n=2$.

Proposition 2. *Let $P\colon \mathcal{H}^{\otimes n} \mapsto \mathcal{H}^{\otimes n}$ be a projection. Then $I_n = \langle P\mathcal{H}^{\otimes n}\rangle$ is a Wick ideal if and only if*
1. $R_n P = 0$,
2. $[1 \otimes (1-P)]T_1 T_2 \cdots T_n [P \otimes 1] = 0$.

Moreover, if T satisfies the braid condition $T_1T_2T_1 = T_2T_1T_2$ and P is a projection on $\ker R_n$, then the condition 2 holds automatically.

Proof. It follows from [1], Lemma 3.1.1 that P generates a homogeneous ideal if and only if $\forall i = 1, \dots, d$, $\forall X \in \mathcal{H}^{\otimes n}$, the relation

$$e_i^* \otimes PX \in P(\mathcal{H}^{\otimes n}) \otimes \mathcal{H}^*$$

holds.

Then we have

$$e_i^* \otimes PX = \mu(e_i^*)\Big(R_n PX + \sum_{k=1}^{d} T_1 \cdots T_n (PX \otimes e_k) \otimes e_k^*\Big).$$

Since $\mu(e_i^*)R_nPX \in \mathcal{H}^{\otimes(n-1)}$, we have $\mu(e_i^*)R_nPX = 0$ $\forall i, X$, which implies $R_nP = 0$. Further, $\mu(e_i^*)\sum T_1\cdots T_n(PX\otimes e_k)\otimes e_k^* \in P(\mathcal{H}^{\otimes n})\otimes\mathcal{H}^*$ if and only if $\forall k$ $\mu(e_i^*)T_1\cdots T_n(PX\otimes e_k) \in P(\mathcal{H}^{\otimes n})$, i.e., if

$$(1-P)\mu(e_i^*)T_1\cdots T_n(P\otimes 1)(X\otimes e_k) = 0,$$
$$\mu(e_i^*)(1\otimes(1-P))T_1\cdots T_n(P\otimes 1)(X\otimes e_k) = 0.$$

Since the equality holds $\forall i, k, X$, we have

$$[1\otimes(1-P)]T_1\cdots T_n[P\otimes 1] = 0.$$

Now, let T satisfy the braid condition. Then it is easy to check that $\forall k = 1, \dots, n-1$ we have $T_1T_2\cdots T_nT_k = T_{k+1}T_1T_2\cdots T_n$, which implies

$$T_1T_2\cdots T_n(R_n\otimes 1) = (1\otimes R_n)T_1T_2\cdots T_n.$$

Therefore, if $X \in \ker R_n$, then $\forall k$ we have

$$(1\otimes R_n)T_1\cdots T_n(X\otimes e_k) = T_1\cdots T_n(R_n\otimes 1)(X\otimes e_k) = 0.$$

The proof is done. □

3. Homogeneous ideals. Braided case

In this section, we assume that T satisfies the braid relation, and $-1 \le T \le 1$. Under these conditions, the maximal quadratic ideal I_2 is generated by $\ker(1+T)$. It is obvious that if $-1 < T \le 1$, then $I_2 = \{0\}$. It appears that in this case $\forall n \ge 2$, $I_n = \{0\}$.

Proposition 3. *If $-1 < T \le 1$, and $T_1T_2T_1 = T_2T_1T_2$, then $\ker R_n = \{0\}$ for all $n \ge 2$.*

Remark 1. This statement is an elaboration of Theorem 2.3 in [2] for the case S_n.

If $-1 \le T < 1$, then one can prove that $I_n \subset I_2$. For $n = 3$, a more precise statement holds.

Theorem 1. *If $T_1T_2T_1 = T_2T_1T_2$ and $-1 \le T \le 1$, then*

$$\ker R_3 = (1 - T_1T_2)(\ker R_2 \otimes \mathcal{H});$$

in particular, $I_3 \subset I_2$.

Proof. We recall that $R_2 = 1 + T$; then $\ker R_2 \otimes \mathcal{H} = \ker(1 + T_1)$, $\mathcal{H} \otimes \ker R_2 = \ker(1 + T_2)$. Consider operator $P_3 = (1 \otimes R_2)R_3$; then it is obvious that $\ker R_3 \subset \ker P_3$. We prove that $\ker P_3 = \ker(1 + T_1) + \ker(1 + T_2)$. To do it, we need the following formulas:

$$P_3 = (1 + T_2)(1 + T_1 + T_1T_2),$$
$$P_3 = (1 + T_1)(1 + T_2 + T_2T_1),$$
$$R_3 = 1 + T_1 + T_1T_2,\ \hat{R}_3 = 1 + T_2 + T_2T_1.$$

These formulas demonstrate that R_3, $\hat{R}_3\colon \ker P_3 \mapsto \ker P_3$, and R_3 maps $\ker P_3$ on $\ker(1 + T_2)$, $\hat{R}_3$ projects on $\ker(1 + T_1)$. We denote by R the restriction of the operator $R_3 + \hat{R}_3$ to $\ker P_3$. Since $R = R^*$, we have $\ker P_3 = \operatorname{im} R \oplus \ker R$, and moreover $\operatorname{im} R \subset \ker(1 + T_1) + \ker(1 + T_2)$. Let $Y \in \ker\ R$. Since $R_3 + \hat{R}_3 = P_3 + 1 - T_1T_2T_1$ and $Y \in \ker\ P_3$, we have $(1 - T_1T_2T_1)Y = 0$. The latter equality holds if and only if the following equalities hold:

$$(1 - T_1^2)Y + T_1(1 - T_2)T_1Y = 0,$$
$$(1 - T_2^2)Y + T_2(1 - T_1)T_2Y = 0.$$

Taking into account the condition $-1 \leq T \leq 1$, we get

$$(1 - T_1^2)Y = 0, \quad (1 - T_2^2)Y = 0, \quad T_2T_1Y = T_1Y, \quad T_1T_2Y = T_2Y.$$

The condition $P_3Y = 0$ takes the form $(1 + T_1 + T_2)Y = 0$. Now we can easily check that

$$(1 - T_1)Y \in \ker(1 + T_1), \quad (2 + T_1Y) \in \ker(1 + T_2),$$

which implies that $Y \in \ker(1 + T_1) + \ker(1 + T_2)$.

Therefore, $\ker\ R \subset \ker(1 + T_1) + \ker(1 + T_2)$, and thus $\ker P_3 = \ker R_2 \otimes \mathcal{H} + \mathcal{H} \otimes \ker R_2$.

Since the restriction of R_3 to $\ker P_3$ is a projection, and $\ker R_3 \subset \ker P_3$, we have $\ker R_3 = (1 - R_3)(\ker P_3)$. It is easy to check that $(1 - R_3)(\ker P_3) = (1 - T_1T_2)(\ker R_2 \otimes \mathcal{H})$. □

4. $*$-Representations of μ-CAR algebra

Here we consider $*$-representations of the μ-CAR algebra by the Hilbert space operators. This algebra was introduced in [1] as the algebra generated by the elements a_i, a_i^*, $i = 1, \ldots, d$, which satisfy the following relations:

$$a_i^*a_i = 1 - a_ia_i^* - (1 - \mu^2)\sum_{k<i} a_ka_k^*,$$
$$a_i^*a_j = -\mu a_ja_i^*, \qquad 0 < \mu < 1.$$

It is easy to see that any $*$-representation of μ-CAR is bounded.

To classify $*$-representations of any Wick algebra it is useful to investigate the Wick ideal structure of this algebra, and especially to describe quadratic Wick ideals (see [1]). For μ-CAR the largest quadratic ideal

$$I_2 = \langle a_i^2,\ i = 1, \ldots, d;\ a_ja_i + \mu a_ia_j,\ i < j\ \rangle$$

is very large; we consider a smaller quadratic ideal

$$\hat{I}_2 = \langle a_i^2,\ i = 1, \ldots, d-1;\ a_ja_i + \mu a_ia_j,\ i < j\ \rangle.$$

The main result of this note is the following theorem:

Theorem 2. *The ideal $\hat{I}_2$ vanishes in any representation of the μ-CAR algebra.*

Proof. Let us denote $A = a_1^2$, $B = a_2 a_1 + \mu a_1 a_2$. It follows from the basic relations that

$$A^* A = A A^*,$$
$$A^* a_k = \mu^2 a_k A^*, \qquad k > 1.$$

By the Fuglede-Phutnam theorem we have that $A a_k = \mu^2 a_k A$, $k > 1$. (It is obvious that $A a_1 = a_1 A$.) It is easy to see, that these relations imply that in any irreducible representation either $A = 0$ or $\ker A = \{0\}$. Suppose now that $\ker A = \{0\}$. Then we have

$$B^* B = \mu^2 B B^* + (1 - \mu^4)(1 + \mu^2) A A^*,$$
$$A^* B = \mu^2 B A^*, \quad AB = \mu^2 BA.$$

Since $AA^* > 0$, we have $B^* B > 0$. Let π be an irreducible representation of μ-CAR. Consider a polar decomposition $\pi(b^*) = WT$; here $T \geq 0$ and W is coisometry. Using the unitary reduction we can represent $\pi(A)$, T, W^* in the following form:

$$\pi(A) = \mathrm{diag}(\lambda \mu^{2n} x 1,\ n = 0, 1, 2, \dots),$$
$$T = \mathrm{diag}(T_n,\ n = 0, 1, 2, \dots),$$
$$T_0 = 0,$$
$$T_n = x(1 + \mu^2)\mu^{n-1}(1 - \mu^{2n})^{\frac{1}{2}}, \qquad n \geq 1,$$

W^* is multiple of the unilateral shift. Since $a_1 A = A a_1$, $a_1^* A = A a_1^*$, we have

$$a_1 = \mathrm{diag}(b_i,\ i = 1, 2, \dots),$$

$B = TW^*$, and in the matrix form:

$$B = \begin{pmatrix} 0 & & & \\ T_1 & 0 & & \\ & T_2 & 0 & \\ & & \ddots & \ddots \end{pmatrix}.$$

The relation $a_1^* B = \mu B a_1^*$ takes the form:

$$b_{n+1}^* T_n = \mu T_n b_n^*.$$

If $x \neq 0$, then $b_k = \mu^{k-1} b_1$, and from $a_1^* a_1 = 1 - a_1 a_1^*$ we have:

$$b_1^* b_1 = 1 - b_1 b_1^*,$$
$$\mu^2 b_1 b_1^* = 1 - \mu^2 b_1 b_1^*.$$

The latter equations are compatible only in the case $\mu^2 = 1$. This implies that $x = 0$, $B = 0$, $A = 0$.

Let us denote $B_k = a_k a_1 + \mu a_1 a_k$, $k > 2$; then from the basic relations we have:

$$B_k^* B_k = \mu^2 B_k B_k^* + \mu^2 (1 - \mu^2) \sum_{1 < i < k} B_i B_i^* + (1 + \mu^2)(1 - \mu^4) A A^*.$$

It follows from the boundedness of B_k that applying induction we obtain $B_k = 0$, $k = 3, \dots, d$. Hence we have $a_k a_1 + \mu a_1 a_k = 0$, and in the irreducible representation

$$a_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes 1$$

$$a_k = \begin{pmatrix} 1 & 0 \\ 0 & -\mu \end{pmatrix} \otimes \hat{a}_k, \quad k > 1.$$

where $\{\hat{a}_k, k > 1\}$ satisfy μ-CAR with $d - 1$ generators.

It is obvious that $a_j^2 = 0$ if and only if $\hat{a}_j^2 = 0$ and $a_j a_i + \mu a_i a_j = 0$ if and only if its relation holds for $\hat{a}_j$, $\hat{a}_i$. Then the induction on d completes the proof. $\square$

The author expresses his gratitude to Prof. Yu. S. Samoĭlenko and Prof. V. Ostrovskyĭ.

References

1. P. E. T. Jørgensen, L. M. Schmitt and R. F. Werner. Positive representations of general commutation relations allowing Wick ordering. *J. Funct. Anal.* **134**, no. 1 (1995), 33–99. MR **96h:**81033
2. M. Bożejko and R. Speicher. Completely positive maps on Coxeter groups, deformed commutation relations, and operator spaces. *Mat. Ann.* **300** (1994), 97–120. MR **95g:**46105

Department of Mathematics, Kiev University, Kiev, Ukraine
E-mail address: prosk@imath.kiev.ua