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STIELTJES MOMENT SEQUENCES AND POSITIVE DEFINITE MATRIX SEQUENCES

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ABSTRACT. For a certain constant $\delta > 0$ (a little less than 1/4), every function $f\colon \mathbb{N}_0 \to]0,\infty[$ satisfying $f(n)^2 \le \delta f(n-1)f(n+1)$, $n \in \mathbb{N}$, is a Stieltjes indeterminate Stieltjes moment sequence. For every indeterminate moment sequence $f\colon \mathbb{N}_0 \to \mathbb{R}$ there is a positive definite matrix sequence (a_n) which is not of positive type and which satisfies $\operatorname{tr}(a_{n+2}) = f(n)$, $n \in \mathbb{N}_0$. For a certain constant $\varepsilon > 0$ (a little greater than 1/6), for every function $\varphi\colon \mathbb{N}_0 \to]0,\infty[$ satisfying $\varphi(n)^2 \le \varepsilon\varphi(n-1)\varphi(n+1)$, $n \in \mathbb{N}$, there is a convolution semigroup $(\mu_t)_{t\ge 0}$ of measures on $\mathbb{R}_+$, with moments of all orders, such that $\varphi(n) = \int x^n \, d\mu_1(x)$, $n \in \mathbb{N}_0$, and for every such convolution semigroup (μ_t) the measure μ_t is Stieltjes indeterminate for all $t > 0$.

1. Introduction

It was shown by Boas [4] that every function $f\colon \mathbb{N}_0 \to]0,\infty[$ of sufficiently rapid growth is a Stieltjes indeterminate Stieltjes moment sequence. More precisely, he proved that every function $f\colon \mathbb{N}_0 \to]0,\infty[$ satisfying

$$f(0) \ge 1, \qquad f(n) \ge \big(nf(n-1)\big)^n \quad \text{for } n \in \mathbb{N} \tag{1}$$

is a Stieltjes moment sequence, and inferred that if f satisfies (1) and if furthermore $f(2) \ge \big(2f(1)+2\big)^2$ then the Stieltjes moment sequence $n \mapsto f(2n)$ is Stieltjes indeterminate. We shall show, for a certain absolute constant δ (a little less than 1/4), that every function $f\colon \mathbb{N}_0 \to]0,\infty[$ satisfying

$$f(n)^2 \le \delta f(n-1)f(n+1), \qquad n \in \mathbb{N}, \tag{2}$$

is a Stieltjes indeterminate Stieltjes moment sequence. The fact that every such f is a moment sequence was shown in [3].

It was shown in [2] by an example that there is a positive definite sequence (a_n) in $\mathbb{M}_2$ (the 2×2 complex matrices) which is not of positive type. (Hence there is a positive linear mapping of $\mathbb{C}[x]$ into $\mathbb{M}_2$ which is not completely positive.) In that example,

$$a_{2n} = \begin{pmatrix} 2^{(n+2)!} & 0 \\ 0 & 2^{(n+2)!} \end{pmatrix}, \qquad n \ge 2.$$

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We shall show that the same can be achieved with the more modest growth

$$\operatorname{tr}(a_{2n+2}) = \Gamma\big(c(2n+1)\big)$$

for any $c > 1$.

Finally, for a certain absolute constant ε (a little greater than $1/6$), we shall show that whenever $\varphi\colon \mathbb{N}_0 \to]0,\infty[$ is such that

$$\varphi(n)^2 \le \varepsilon\varphi(n-1)\varphi(n+1), \qquad n \in \mathbb{N}, \tag{3}$$

then there is a convolution semigroup $(\mu_t)_{t\ge 0}$ of measures on $\mathbb{R}_+$, with moments of all orders, such that

$$\varphi(n) = \int x^n \, d\mu_1(x), \qquad n \in \mathbb{N}_0,$$

and for every such semigroup (μ_t) the measure μ_t is Stieltjes indeterminate for all $t > 0$. This applies to the moment sequence φ of the measure

$$\frac{1}{\sigma\sqrt{2\pi}} e^{-(\log x)^2/2\sigma^2} 1_{]0,\infty[}(x)\frac{dx}{x}, \tag{4}$$

provided that $\sigma > 4/3$.

2. Moment sequences and matrix sequences

A sequence $f\colon \mathbb{N}_0 \to \mathbb{R}$ is a *moment sequence* if

$$f(n) = \int x^n \, d\mu(x), \qquad n \in \mathbb{N}_0, \tag{5}$$

for some measure μ on $\mathbb{R}$ having moments of all orders; a *Stieltjes moment sequence* if (5) holds for some measure μ on $\mathbb{R}_+$. A moment sequence f is *determinate* if there is only one measure μ on $\mathbb{R}$ satisfying (5); otherwise, *indeterminate*. A Stieltjes moment sequence f is *Stieltjes determinate* if there is only one measure μ on $\mathbb{R}_+$ satisfying (5); otherwise, *Stieltjes indeterminate*. A sequence $f\colon \mathbb{N}_0 \to \mathbb{R}$ is a moment sequence if and only if f is *positive definite* in the sense that for each $n \in \mathbb{N}_0$ the matrix $\big(f(j+k)\big)_{j,k=0}^{n}$ is positive semidefinite. A sequence $f\colon \mathbb{N}_0 \to \mathbb{R}$ is a Stieltjes moment sequence if and only if f and the shifted sequence $n \mapsto f(n+1)\colon \mathbb{N}_0 \to \mathbb{R}$ are both positive definite.

Definition 1. We denote by δ the positive number satisfying $\sum_{n=1}^{\infty} \delta^{n^2} = 1/4$.

Clearly $\delta < 1/4$. Below we shall obtain the estimate $\delta > 15/61$. Since (for $0 < x < 1$) $x + x^4 < \sum_{n=1}^{\infty} x^{n^2} < x + x^4 + \sum_{n=3}^{\infty} x^{7n-12} = x + x^4 + x^9/(1-x^7)$, one can show $x < \delta < y$ by verifying $x + x^4 + x^9/(1-x^7) < 1/4 < y + y^4$. In this way, using a calculator, one gets $0.246315 < \delta < 0.246319$.

Theorem 1. *Every function* $f\colon \mathbb{N}_0 \to]0,\infty[$ *satisfying* (2) *is a Stieltjes indeterminate Stieltjes moment sequence.*

Proof. We shall show here that every such function f is a Stieltjes moment sequence. This is used in the proof of Theorem 2 below, from which it then follows that every such f is Stieltjes indeterminate.

If f satisfies (2), the shifted sequence $n \mapsto f(n+1)$ satisfies the corresponding set of inequalities. It therefore suffices to show that every such f is positive definite. The positive definiteness of such f was obtained in [3] as a corollary of a difficult theorem on positive definite kernels. We present here a simplified proof.

Suppose $f\colon \mathbb{N}_0 \to]0,\infty[$ satisfies (2). We shall show that f is strictly positive definite in the sense that for each $n \in \mathbb{N}_0$ the matrix $\big(f(j+k)\big)_{j,k=0}^{n}$ is positive definite, that is, the determinants

$$D_n = \begin{vmatrix} f(0) & \dots & f(n) \\ \vdots & & \vdots \\ f(n) & \dots & f(2n) \end{vmatrix}$$

are positive. The inequalities (2) are equivalent to saying that the function $\psi\colon \mathbb{N}_0 \to]0,\infty[$ defined by $\psi(n) = f(n)\delta^{n^2/2}$ is log convex. It follows that if $s \in \mathbb{N}_0$ and $j,k \in \{0,\dots,s\}$ then $\psi(j+s)\psi(k+s) \le \psi(j+k)\psi(2s)$, that is,

$$\frac{f(j+s)f(k+s)}{f(j+k)f(2s)} \le \delta^{(s-j)(s-k)}. \tag{6}$$

Keeping n fixed, by backwards induction on m we shall define $u_{j,k}(m)$ for $m = n, n-1, \dots, 0$ and $j,k \in \{0,\dots,m\}$ such that

$$u_{j,k}(n) = 1 \qquad \text{for all } j,k, \tag{7}$$

$$\frac{1}{2} \le 1 - 2\sum_{s=m+1}^{\infty} \delta^{(s-j)(s-k)} \le u_{j,k}(m) \le 1 \qquad \text{for all } m,j,k, \tag{8}$$

$$u_{j,k}(m) = u_{j,k}(m+1) - \frac{u_{j,m+1}(m+1)u_{m+1,k}(m+1)}{u_{m+1,m+1}(m+1)} \times \frac{f(j+m+1)f(m+1+k)}{f(j+k)f(2m+2)}$$

$$\text{for } m < n \text{ and all } j,k. \tag{9}$$

The inequality to the extreme left in (8) always holds for $j,k \in \{0,\dots,m\}$ since

$$\sum_{s=m+1}^{\infty} \delta^{(s-j)(s-k)} \le \sum_{s=m+1}^{\infty} \delta^{(s-m)^2} = \sum_{p=1}^{\infty} \delta^{p^2} = \frac{1}{4}.$$

Clearly (8) is fulfilled for $m = n$. Let $m < n$ and suppose that $u_{j,k}(m+1)$ is defined for all j and k and satisfies the condition (8^*) obtained by taking $m+1$ instead of m in (8). By (8^*) we have $u_{j,k}(m+1) > 0$ for all j and k, which shows that $u_{j,k}(m)$ is well defined by (9) and satisfies $u_{j,k}(m) \le u_{j,k}(m+1) \le 1$ (by (8^*)). By (6), applied to $s = m+1$, and by (8^*),

$$u_{j,k}(m) \ge 1 - 2\sum_{s=m+2}^{\infty} \delta^{(s-j)(s-k)} - 2\delta^{(m+1-j)(m+1-k)} = 1 - 2\sum_{s=m+1}^{\infty} \delta^{(s-j)(s-k)},$$

completing the definition of the $u_{j,k}(m)$ and the proof that they have the properties stated. We now define

$$E_m = \det\Big(\big(u_{j,k}(m)f(j+k)\big)_{j,k=0}^{m}\Big), \qquad m = 0,\dots,n.$$

Here $E_0 = u_{0,0}(0)f(0) > 0$. Supposing that $m < n$ and that $E_m > 0$, using row operations to get zeroes in the last column of E_{m+1} except in the last entry, we get $E_{m+1} = u_{m+1,m+1}(m+1)f(2m+2)E_m > 0$. In particular, $D_n = E_n > 0$. $\square$

It is shown in [3] that for $x > 1/4$, a function $f\colon \mathbb{N}_0 \to]0,\infty[$ may satisfy the inequalities obtained by replacing δ by x in (2), yet not be positive definite.

Remark. Let us call a Stieltjes moment sequence $f\colon \mathbb{N}_0 \to]0,\infty[$ *log convex stable* if fg is a Stieltjes moment sequence for every log convex function $g\colon \mathbb{N}_0 \to]0,\infty[$. We have seen that $n \mapsto \gamma^{-n^2/2}$ is log convex stable if $\gamma \le \delta$, but not if $\gamma > 1/4$. Suppose f is log convex stable. It can be shown (using ideas from the Remark following Corollary 1 in [3]) that the kernel

$$\begin{pmatrix} f(0) & f(1) & & \\ f(1) & f(2) & f(3) & \\ & f(3) & f(4) & \ddots \\ & & \ddots & \ddots \end{pmatrix}$$

(with zeroes everywhere except in the main diagonal and the two adjacent ones) is positive definite. (Similarly with $f(n+1)$ instead of $f(n)$.) It follows that the numbers

$$\delta_n = \frac{f(2n+1)^2}{f(2n)f(2n+2)}, \qquad n \in \mathbb{N}_0,$$

have the property that the numbers D_n defined by

$$D_{-1} = D_0 = 1, \qquad D_n = D_{n-1} - \delta_{n-1} D_{n-2} \quad (n \in \mathbb{N})$$

are all positive. This implies that there are limits to how many times in a row, and by how much, the δ_n can exceed $1/4$. In fact, it can be shown that if k consecutive δ_n are $\ge x > 1/4$ then

$$k + 1 < \frac{\pi}{\arctan\sqrt{4x-1}}.$$

A sequence $(a_n)_{n\in\mathbb{N}_0}$ in $\mathbb{M}_2$ is *positive definite* if for each $\xi \in \mathbb{C}^2$ the scalar sequence $(\langle a_n\xi,\xi\rangle)_{n\in\mathbb{N}_0}$ is positive definite. The sequence (a_n) is of *positive type* if $\sum_{j,k=0}^{n} \langle a_{j+k}\xi_j,\xi_k\rangle \ge 0$ for all $n \in \mathbb{N}_0$ and all $\xi_0,\dots,\xi_n \in \mathbb{C}^2$.

Theorem 2. *If $f\colon \mathbb{N}_0 \to]0,\infty[$ satisfies (2), there is a sequence $(a_n)_{n\in\mathbb{N}_0}$ in $\mathbb{M}_2$ such that the sequences (a_n) and (a_{n+1}) are positive definite, (a_n) is not of positive type, and $\operatorname{tr}(a_{n+2}) = f(n)$ for all $n \in \mathbb{N}_0$. It follows that f is Stieltjes indeterminate.*

Proof. Put $\alpha = f(0)^2/\delta f(1)$, choose $g > \alpha$, choose $c > g^2/\delta f(0)$, choose $\varphi > \sqrt{cf(0)}$, let $d = \alpha + \varphi^2/(g-\alpha)$, define $h > 0$ by $(g + \varphi/h)^2 = \delta c f(0)$, and put $b = (d + \varphi h)^2/\delta f(0)$. Now define (a_n) by

$$a_0 = \begin{pmatrix} b & 0 \\ 0 & c \end{pmatrix}, \qquad a_1 = \begin{pmatrix} d & \varphi \\ \varphi & g \end{pmatrix}, \qquad a_n = \begin{pmatrix} f(n-2) & 0 \\ 0 & f(n-2) \end{pmatrix}$$

for $n \ge 2$. To show that the sequences (a_n) and (a_{n+1}) are positive definite, that is, $(\langle a_n\xi,\xi\rangle)$ is a Stieltjes moment sequence for each $\xi \in \mathbb{C}^2$, by that part of Theorem 1 which was proved above it suffices to show

$$\langle a_n\xi,\xi\rangle^2 \le \delta \langle a_{n-1}\xi,\xi\rangle\langle a_{n+1}\xi,\xi\rangle$$

for all $\xi \in \mathbb{C}^2$ and $n \ge 1$. For $n \ge 3$, this inequality follows from (2). For $n = 2$, what we have to show is that the smallest eigenvalue of a_1 is at least α. But that eigenvalue is

$$\frac{d+g}{2} - \sqrt{\left(\frac{d-g}{2}\right)^2 + \varphi^2} = \alpha.$$

For $n = 1$ and $\xi = (x, y) \in \mathbb{C}^2$ with $\|\xi\| = 1$, using $2|\operatorname{Re}(x\overline{y})| \le h|x|^2 + |y|^2/h$,

$$\begin{aligned}
&\delta\langle a_0\xi,\xi\rangle\langle a_2\xi,\xi\rangle - \langle a_1\xi,\xi\rangle^2 \\
&\quad= \delta f(0)(b|x|^2 + c|y|^2)(|x|^2 + |y|^2) - \big(d|x|^2 + g|y|^2 + 2\varphi \operatorname{Re}(x\overline{y})\big)^2 \\
&\quad\ge \delta f(0)(b|x|^2 + c|y|^2)(|x|^2 + |y|^2) - \big((d + \varphi h)|x|^2 + (g + \varphi/h)|y|^2\big)^2 \\
&\quad= \big(\delta bf(0) - (d + \varphi h)^2\big)|x|^4 + \big(\delta cf(0) - (g + \varphi/h)^2\big)|y|^4 \\
&\qquad\qquad + \big(\delta(b + c)f(0) - 2(d + \varphi h)(g + \varphi/h)\big)|x|^2|y|^2 \\
&\quad\ge \big(\delta bf(0) - (d + \varphi h)^2\big)|x|^4 + \big(\delta cf(0) - (g + \varphi/h)^2\big)|y|^4 \\
&\qquad\qquad + \big(\delta(b + c)f(0) - (d + \varphi h)^2 - (g + \varphi/h)^2\big)|x|^2|y|^2 \\
&\quad= 0.
\end{aligned}$$

To see that (a_n) is not of positive type, note that with $\xi_0 = (0, \varphi)$ and $\xi_1 = (-c, 0)$ we have

$$\sum_{j,k=0}^{1} \langle a_{j+k}\xi_j, \xi_k\rangle = c\varphi^2 - 2c\varphi^2 + c^2 f(0) = -c\big(\varphi^2 - cf(0)\big) < 0. \tag{10}$$

If for all $\xi, \eta \in \mathbb{C}^2$ the Stieltjes moment sequence $(\langle a_n\xi,\xi\rangle + \langle a_n\eta,\eta\rangle)_{n\in\mathbb{N}_0}$ were Stieltjes determinate, as in the proof of [2], Theorem 2, it would follow that (a_n) were the moment sequence of a measure on $\mathbb{R}_+$ with positive semidefinite matrix values, contradicting the fact that (a_n) is not of positive type. Hence there exist $\xi, \eta \in \mathbb{C}^2$ such that $(\langle a_n\xi,\xi\rangle + \langle a_n\eta,\eta\rangle)$ is Stieltjes indeterminate. Now there exist $\zeta_1, \dots, \zeta_m \in \mathbb{C}^2$ such that

$$\langle a\xi,\xi\rangle + \langle a\eta,\eta\rangle + \sum_{k=1}^{m} \langle a\zeta_k, \zeta_k\rangle = C \operatorname{tr}(a)$$

for some $C > 0$ and all $a \in \mathbb{M}_2$. The sequence

$$n \mapsto C \operatorname{tr}(a_n) - \langle a_n\xi,\xi\rangle - \langle a_n\eta,\eta\rangle = \sum_{k=1}^{m} \langle a_n\zeta_k, \zeta_k\rangle$$

is clearly a Stieltjes moment sequence, which shows that the sequence $\big(\operatorname{tr}(a_n)\big)$ is Stieltjes indeterminate. Hence so is the sequence $\big(\operatorname{tr}(a_{n+2})\big)$. Finally note that we have $\operatorname{tr}(a_{n+2}) = 2f(n)$, but we can get $\operatorname{tr}(a_n) = f(n)$, without affecting the other statements of the theorem, by dividing each a_n by 2. □

As far as the growth of positive definite matrix sequences, not of positive type, is concerned, Theorem 2 is not the best possible. Theorem 2 allows us to conclude the existence of a positive definite sequence (a_n), not of positive type, such that

$$\operatorname{tr}(a_{n+2}) = \delta^{-n^2/2},$$

the right side being a little worse than 2^{n^2}. Something better can be derived from the following result.

Theorem 3. *If $f\colon \mathbb{N}_0 \to \mathbb{R}$ is an indeterminate moment sequence, there is a positive definite sequence $(a_n)_{n\in\mathbb{N}_0}$ in $\mathbb{M}_2$, which is not of positive type, such that $\operatorname{tr}(a_{n+2}) = f(n)$ for all $n \in \mathbb{N}_0$.*

Proof. With no restriction, assume $f(0) = 1$. The N-extremal measures σ_t, $t \in \mathbb{R}$, representing f satisfy

$$\int \frac{d\sigma_t(x)}{x - z} = -\frac{A(z)t - C(z)}{B(z)t - D(z)}, \qquad z \in \mathbb{C} \setminus \mathbb{R}, \tag{11}$$

for certain entire functions A, B, C, D with $A(0) = D(0) = 0$ and $-B(0) = C(0) = 1$ (cf. [1]). Letting z tend to 0 through imaginary values, we get

$$\int \frac{d\sigma_t(x)}{x} = -\frac{1}{t}, \qquad t \neq 0.$$

Differentiating (11) with respect to z yields

$$\int \frac{d\sigma_t(x)}{(x-z)^2} = -\frac{\big(A'(z)t - C'(z)\big)\big(B(z)t - D(z)\big) - \big(A(z)t - C(z)\big)\big(B'(z)t - D'(z)\big)}{\big(B(z)t - D(z)\big)^2},$$

and letting z tend to 0 through imaginary values, we get

$$\int \frac{d\sigma_t(x)}{x^2} = A'(0) - \big(B'(0) + C'(0)\big)\frac{1}{t} + D'(0)\frac{1}{t^2}.$$

Hence there is some $\beta > 0$ such that

$$\int \frac{d\sigma_{-t}(x)}{x^2} \leq \frac{\beta}{t^2} \qquad \text{for } 0 < t \leq 1.$$

Choose $g > 1$, choose $c > \beta g^2$, choose $\varphi > \sqrt{cf(0)}$, define $h > 0$ by $(g + \varphi/h)^2 = c/\beta$, and let $d = 1 + \varphi^2/(g-1)$ and $b = \beta(d + \varphi h)^2$. Now define (a_n) by

$$a_0 = \begin{pmatrix} b & 0 \\ 0 & c \end{pmatrix}, \qquad a_1 = \begin{pmatrix} d & \varphi \\ \varphi & g \end{pmatrix}, \qquad a_n = \begin{pmatrix} f(n-2) & 0 \\ 0 & f(n-2) \end{pmatrix}$$

for $n \geq 2$. Suppose $\xi \in \mathbb{C}^2$ with $\|\xi\| = 1$. For $n \geq 2$ we have

$$\langle a_n \xi, \xi \rangle = f(n-2) = \int x^n \frac{d\sigma_{-t}(x)}{x^2}, \qquad 0 < t \leq 1.$$

The smallest eigenvalue of a_1 is

$$\frac{d+g}{2} - \sqrt{\Big(\frac{d-g}{2}\Big)^2 + \varphi^2} = 1.$$

Hence $\langle a_1 \xi, \xi \rangle \geq 1$, and with $t = 1/\langle a_1 \xi, \xi \rangle$ we have

$$\langle a_1 \xi, \xi \rangle = \frac{1}{t} = \int \frac{d\sigma_{-t}(x)}{x} = \int x \frac{d\sigma_{-t}(x)}{x^2}.$$

Finally, with $\xi = (x,y)$, using $2|\operatorname{Re}(x\overline{y})| \le h|x|^2 + |y|^2/h$,

$$\begin{aligned}
&\langle a_0\xi,\xi\rangle - \beta\langle a_1\xi,\xi\rangle^2 \\
&\quad = (b|x|^2 + c|y|^2)(|x|^2+|y|^2) - \beta\big(d|x|^2 + g|y|^2 + 2\varphi \operatorname{Re}(x\overline{y})\big)^2 \\
&\quad \ge (b|x|^2 + c|y|^2)(|x|^2+|y|^2) - \beta\big((d+\varphi h)|x|^2 + (g+\varphi/h)|y|^2\big)^2 \\
&\quad = \big(b - \beta(d+\varphi h)^2\big)|x|^4 + \big(c - \beta(g+\varphi/h)^2\big)|y|^4 \\
&\qquad\qquad + \big(b + c - 2\beta(d+\varphi h)(g+\varphi/h)\big)|x|^2|y|^2 \\
&\quad \ge \big(b - \beta(d+\varphi h)^2\big)|x|^4 + \big(c - \beta(g+\varphi/h)^2\big)|y|^4 \\
&\qquad\qquad + \big(b + c - \beta(d+\varphi h)^2 - \beta(g+\varphi/h)^2\big)|x|^2|y|^2 \\
&\quad = 0,
\end{aligned}$$

that is,

$$\langle a_0\xi,\xi\rangle \ge \frac{\beta}{t^2} \ge \int \frac{d\sigma_{-t}(x)}{x^2}.$$

All of this shows that $(\langle a_n\xi,\xi\rangle)$ is a moment sequence, represented by the measure $x^{-2}\sigma_{-t} + k\varepsilon_0$ for some $k \ge 0$. That (a_n) is not of positive type, follows by a computation like (10). Finally, divide (a_n) by 2 to get $\operatorname{tr}(a_{n+2}) = f(n)$. □

For $c > 1$, the function $f\colon \mathbb{N}_0 \to \mathbb{R}$ defined by $f(2n) = \Gamma\big(c(2n+1)\big)$, $f(2n+1) = 0$, is the moment sequence of the measure with density $\phi(x) = e^{-|x|^{1/c}}/2c$. Since $\int_{-\infty}^{\infty}\big(\log\phi(x)/(1+x^2)\big)dx > -\infty$, by the criterion of Krein ([1], p. 87) it follows that f is indeterminate. Hence there is a positive definite sequence (a_n) in $\mathbb{M}_2$, not of positive type, such that $\operatorname{tr}(a_{2n+2}) = \Gamma\big(c(2n+1)\big)$ for all n.

3. Convolution semigroups

If μ is a measure on $\mathbb{R}$ with moments of all orders, we denote by $s\mu\colon \mathbb{N}_0 \to \mathbb{R}$ the moment sequence of μ, i.e.,

$$s\mu(n) = \int x^n\, d\mu(x), \qquad n \in \mathbb{N}_0.$$

If μ and ν are measures on $\mathbb{R}$ with moments of all orders, the convolution $\mu * \nu$ again has moments of all orders, and a simple computation shows

$$s(\mu*\nu)(n) = \sum_{k=0}^{n}\binom{n}{k} s\mu(k)s\nu(n-k).$$

We introduce a binary operation $\circ$ in $\mathbb{R}^{\mathbb{N}_0}$ by

$$\varphi\circ\psi(n) = \sum_{k=0}^{n}\binom{n}{k}\varphi(k)\psi(n-k),$$

and the preceding formula takes the form

$$s(\mu*\nu) = s\mu \circ s\nu.$$

Note that the indicator $1_{\{0\}}$ is a neutral element for $\circ$. For $\varphi \in \mathbb{R}^{\mathbb{N}_0}$ and $q \in \mathbb{N}_0$ we write $\varphi^{\circ q} = \overbrace{\varphi\circ\cdots\circ\varphi}^{q}$ (equal to $1_{\{0\}}$, by definition, in case $q = 0$), and note that

$$\varphi^{\circ q}(n) = n! \sum_{n_1+\cdots+n_q=n} \prod_{i=1}^{q} \frac{\varphi(n_i)}{n_i!}.$$

For $\varphi \in \mathbb{R}^{\mathbb{N}_0}$ with $\varphi(0) = 1$ and for $t \in \mathbb{R}_+$ we define $\varphi^{\circ t} \in \mathbb{R}^{\mathbb{N}_0}$ by

$$\varphi^{\circ t}(n) = \sum_{m=0}^{\infty} \binom{t}{m} (\varphi - 1_{\{0\}})^{\circ m}(n), \qquad n \in \mathbb{N}_0,$$

noting that the right side has only finitely many terms because of

$$\operatorname{supp}\bigl((\varphi - 1_{\{0\}})^{\circ m}\bigr) \subset \{m, m+1, \dots\}.$$

This latter fact also shows that $t \mapsto \varphi^{\circ t}$ is continuous with respect to the topology of pointwise convergence on $\mathbb{R}^{\mathbb{N}_0}$. (We always consider $\mathbb{R}^{\mathbb{N}_0}$ with this topology.) If t happens to equal some integer q then $\varphi^{\circ t}$ as newly defined is the same as $\varphi^{\circ q}$ in the previous sense, by the binomial formula.

From the identity $(1+z)^{r+t} = (1+z)^r(1+z)^t$, inserting $(1+z)^{r+t} = \sum_{m=0}^{\infty} \binom{r+t}{m} z^m$ ($|z| < 1$) and similarly for the exponents r and t, we get the identity

$$\binom{r+t}{m} = \sum_{p=0}^{m} \binom{r}{p} \binom{t}{m-p},$$

and it follows that

$$\varphi^{\circ(r+t)} = \varphi^{\circ r} \circ \varphi^{\circ t}, \qquad r, t \in \mathbb{R}_+. \tag{12}$$

For $q \in \mathbb{N}$, the function $\varphi^{\circ 1/q}$ is the unique $\psi \in \mathbb{R}^{\mathbb{N}_0}$ such that $\psi(0) = 1$ and $\psi^{\circ q} = \varphi$. To see this, first note that repeated application of (12) yields $(\varphi^{\circ 1/q})^{\circ q} = \varphi$. Furthermore, suppose $\psi, \omega \in \mathbb{R}^{\mathbb{N}_0}$ are such that $\psi(0) = \omega(0) = 1$ and $\psi^{\circ q} = \omega^{\circ q}$, that is,

$$\sum_{n_1+\cdots+n_q=n} \prod_{i=1}^{q} \frac{\psi(n_i)}{n_i!} = \sum_{n_1+\cdots+n_q=n} \prod_{i=1}^{q} \frac{\omega(n_i)}{n_i!}$$

for $n \in \mathbb{N}_0$. If $n \in \mathbb{N}$ and if $\psi(m) = \omega(m)$ for $m < n$, the terms with $n_i < n$ for all i cancel, and we are left with the terms with $n_i = n$ for one i and $n_i = 0$ for all other i, and the equation simplifies to $\psi(n) = \omega(n)$.

Lemma 1. *If $(\mu_t)_{t\geq 0}$ is a convolution semigroup of probability measures on $\mathbb{R}_+$ with moments of all orders, and if $\varphi = s\mu_1$, then $s\mu_t = \varphi^{\circ t}$ for all $t \in \mathbb{R}_+$.*

Proof. For $q \in \mathbb{N}$ we have $(s\mu_{1/q})^{\circ q} = s\mu_1 = \varphi = (\varphi^{\circ 1/q})^{\circ q}$, hence $s\mu_{1/q} = \varphi^{\circ 1/q}$ by uniqueness. It follows that $s\mu_t = \varphi^{\circ t}$ for all $t \in \mathbb{Q}_+$. Let us show that $(s\mu_t)_{t\geq 0}$ is continuous at 0. Firstly, $s\mu_{1/q} = \varphi^{\circ 1/q} \to 1_{\{0\}}$ as $\mathbb{N} \ni q \to \infty$. Secondly, if $m \in \mathbb{N}$, $q \in \mathbb{N}$, and $t \leq 1/q$ then

$$\begin{aligned} s\mu_{1/q}(m) &= s\mu_{1/q-t} \circ s\mu_t(m) = \sum_{k=0}^{m} \binom{m}{k} s\mu_{1/q-t}(k) s\mu_t(m-k) \\ &\geq s\mu_{1/q-t}(0) s\mu_t(m) = s\mu_t(m), \end{aligned}$$

which shows $s\mu_t \to 1_{\{0\}}$ as $t \to 0$. Now for arbitrary $t > 0$ there is a sequence (r_n) of rationals converging to t from the left, and then

$$\begin{aligned} s\mu_t &= \lim_{n\to\infty} s\mu_{r_n} \circ s\mu_{t-r_n} = \Bigl(\lim_{n\to\infty} s\mu_{r_n}\Bigr) \circ \Bigl(\lim_{n\to\infty} s\mu_{t-r_n}\Bigr) \\ &= \Bigl(\lim_{n\to\infty} \varphi^{\circ r_n}\Bigr) \circ 1_{\{0\}} = \varphi^{\circ t} \end{aligned}$$

since $\circ$ is continuous. ∎

Definition 2. We denote by ε the number in $]0,1[$ satisfying $\varepsilon/(1-\varepsilon)^2 = \delta$.

Since $(1/5)/(1-1/5)^2 = 5/16 > 1/4 > \delta$ then $\varepsilon < 1/5$. Below we shall find the estimate $\varepsilon > 10/59$. Due to our earlier estimates of δ, one can show $x < \varepsilon < y$ by verifying $x/(1-x)^2 < 0.246315$ and $y/(1-y)^2 > 0.246319$. Using a calculator, one gets $0.169777 < \varepsilon < 0.169780$.

Lemma 2. *If* $\varphi\colon \mathbb{N}_0 \to]0,\infty[$ *satisfies* (3) *then* $\varphi^{\circ t}$ *is a Stieltjes indeterminate Stieltjes moment sequence for each* $t > 0$.

Proof. With no loss of generality, assume $\varphi(0) = 1$. For fixed $n \in \mathbb{N}$ we have

$$\varphi^{\circ t}(n) = n! \sum_{m=1}^{n} \binom{t}{m} S_m$$

where

$$S_m = \sum_{n_1+\cdots+n_m=n, n_i\geq 1 \forall i} \prod_{i=1}^{m} \frac{\varphi(n_i)}{n_i!}.$$

Now

$$\sum_{k=1}^{n-1} \binom{n}{k} \frac{\varphi(k)\varphi(n-k)}{\varphi(n)} \leq (2^n-2)\varepsilon^{n-1} \leq 2\varepsilon < 1. \tag{13}$$

To get the first inequality in (13) note that the function $\psi\colon \mathbb{N}_0 \to]0,\infty[$ defined by $\psi(j) = \varepsilon^{j^2/2}\varphi(j)$ is log convex according to (3), so for $1 \leq k \leq n-1$ we have $\psi(k)\psi(n-k) \leq \psi(0)\psi(n)$, that is, $\varphi(k)\varphi(n-k)/\varphi(n) \leq \varepsilon^{k(n-k)}$, whence

$$\sum_{k=1}^{n-1} \binom{n}{k} \frac{\varphi(k)\varphi(n-k)}{\varphi(n)} \leq \sum_{k=1}^{n-1} \binom{n}{k} \varepsilon^{k(n-k)} \leq (2^n-2)\varepsilon^{n-1}.$$

In the second inequality in (13) we have strict inequality for $n = 1$ and equality for $n = 2$, and for $n \geq 3$ we have

$$(2^n-2)\varepsilon^{n-1} \leq 2^n\varepsilon^{n-1} = 8(2\varepsilon)^{n-3}\varepsilon^2 \leq 8\varepsilon^2 < 2\varepsilon$$

since $\varepsilon < 1/4$. This proves (13).

For $m \in \mathbb{N}$, by (13) applied to $n-p$ instead of n,

$$\begin{aligned}
S_{m+1} &= \sum_{p=m-1}^{n-2} \sum_{n_1+\cdots+n_{m-1}=p, n_i\geq 1\forall i} \left(\prod_{i=1}^{m-1} \frac{\varphi(n_i)}{n_i!}\right) \sum_{k=1}^{n-p-1} \frac{\varphi(k)\varphi(n-p-k)}{k!(n-p-k)!} \\
&\leq 2\varepsilon \sum_{p=m-1}^{n-2} \sum_{n_1+\cdots+n_{m-1}=p, n_i\geq 1\forall i} \left(\prod_{i=1}^{m-1} \frac{\varphi(n_i)}{n_i!}\right) \frac{\varphi(n-p)}{(n-p)!} \\
&\leq 2\varepsilon \sum_{p=m-1}^{n-1} \sum_{n_1+\cdots+n_{m-1}=p, n_i\geq 1\forall i} \left(\prod_{i=1}^{m-1} \frac{\varphi(n_i)}{n_i!}\right) \frac{\varphi(n-p)}{(n-p)!} \\
&= 2\varepsilon S_m.
\end{aligned}$$

For $0 < t < 1$, therefore, the series $\sum_{m=1}^{n} \binom{t}{m} S_m$ is alternating with absolutely decreasing terms, and so

$$\begin{aligned}
(1-\varepsilon)t\varphi(n) &= n!(1-\varepsilon)tS_1 \leq n!\big(1-(1-t)\varepsilon\big)tS_1 \\
&\leq n!\Big(tS_1 - \frac{t(1-t)}{2}S_2\Big) \leq \varphi^{\circ t}(n) \leq n!tS_1 = t\varphi(n).
\end{aligned}$$

It follows that

$$\frac{\varphi^{\circ t}(n)^2}{\varphi^{\circ t}(n-1)\varphi^{\circ t}(n+1)} \le \frac{\varphi(n)^2}{(1-\varepsilon)^2\varphi(n-1)\varphi(n+1)} \le \frac{\varepsilon}{(1-\varepsilon)^2} = \delta,$$

showing that $\varphi^{\circ t}$ is a Stieltjes indeterminate Stieltjes moment sequence for $0 < t < 1$. For $t = 1$ we have the same conclusion directly from (3) since $\varepsilon < \delta$, and for arbitrary $t > 0$ we get the desired conclusion from $\varphi^{\circ t} = \varphi^{\circ r} \circ \varphi^{\circ p}$ where $r \in]0,1]$ and $p \in \mathbb{N}_0$ are so chosen that $t = r + p$. □

Theorem 4. *If $\varphi\colon \mathbb{N}_0 \to]0,\infty[$ satisfies* (3), *there is a convolution semigroup $(\mu_t)_{t\ge 0}$ of measures on $\mathbb{R}_+$ with moments of all orders such that*

$$\int x^n \, d\mu_1(x) = \varphi(n), \qquad n \in \mathbb{N}_0,$$

and for every such semigroup (μ_t) the measure μ_t is Stieltjes indeterminate for each $t > 0$.

Proof. With no loss of generality, assume $\varphi(0) = 1$. For $t > 0$, by Lemma 2, the function $\varphi^{\circ t}$ is, in particular, a Stieltjes moment sequence, so we can choose a measure λ_t on $\mathbb{R}_+$, with moments of all orders, such that $s\lambda_t = \varphi^{\circ t}$. For $t \to 0$ we have $s\lambda_t = \varphi^{\circ t} \to 1_{\{0\}} = s\varepsilon_0$, so $\lambda_t \to \varepsilon_0$ by the method of moments. Now let (t_i) be a universal subnet of the identical net on $(]0,\infty[, \ge)$. For $t > 0$ we have, denoting by $\lfloor t/t_i \rfloor$ the greatest integer not exceeding t/t_i,

$$s(\lambda_{t_i}^{*\lfloor t/t_i\rfloor}) = s(\lambda_{t_i})^{\circ\lfloor t/t_i\rfloor} = (\varphi^{\circ t_i})^{\circ\lfloor t/t_i\rfloor} = \varphi^{\circ\lfloor t/t_i\rfloor t_i} \to \varphi^{\circ t},$$

so by the method of moments, $\lambda_{t_i}^{*\lfloor t/t_i\rfloor} \to \mu_t$ for some measure μ_t on $\mathbb{R}_+$, with moments of all orders, such that $s\mu_t = \varphi^{\circ t}$. For $r, t > 0$, for each i there is some $m_i \in \{0,1\}$ such that $\lfloor (r+t)/t_i\rfloor = \lfloor r/t_i\rfloor + \lfloor t/t_i\rfloor + m_i$, and it follows that

$$\mu_{r+t} = \lim \lambda_{t_i}^{*\lfloor (r+t)/t_i\rfloor} = \lim \lambda_{t_i}^{*\lfloor r/t_i\rfloor} * \lambda_{t_i}^{*\lfloor t/t_i\rfloor} * \lambda_{t_i}^{*m_i} = \mu_r * \mu_t.$$

This shows the existence of a convolution semigroup as stated. Now for an arbitrary such semigroup (μ_t), by Lemma 1 we have $s\mu_t = \varphi^{\circ t}$ for $t > 0$, and the Stieltjes indeterminacy follows from Lemma 2. □

If φ is the moment sequence of the measure (4) with $\sigma > 4/3$ then Theorem 4 applies. To see this, note that $\varphi(n) = e^{\sigma^2 n^2/2}$. Now

$$\sum_{n=1}^{\infty}\Big(\frac{15}{61}\Big)^{n^2} \le \sum_{n=1}^{\infty}\Big(\frac{15}{61}\Big)^{3n-2} = \frac{15\cdot 61^2}{61^3 - 15^3} = \frac{55815}{223606} < \frac{1}{4},$$

so $\delta > 15/61$. Furthermore,

$$\frac{10/59}{(1-10/59)^2} = \frac{10\cdot 59}{(59-10)^2} = \frac{590}{2401} < \frac{15}{61} < \delta,$$

so $\varepsilon > 10/59$. But

$$e^{1/9} > 1 + \frac{1}{9} + \frac{1}{2}\Big(\frac{1}{9}\Big)^2 + \frac{1}{6}\Big(\frac{1}{9}\Big)^3 = \frac{2444}{2187} > \frac{485}{434},$$

so

$$\frac{\varphi(n)^2}{\varphi(n-1)\varphi(n+1)} = e^{-\sigma^2} < e^{-16/9}$$
$$< \Big(\frac{434}{485}\Big)^{16} < \Big(\frac{213}{266}\Big)^{8} < \Big(\frac{84}{131}\Big)^{4} < \Big(\frac{44}{107}\Big)^{2} < \frac{10}{59} < \varepsilon.$$

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