

## POLYNOMIAL APPROXIMATION WITH VARYING WEIGHTS ON COMPACT SETS OF THE COMPLEX PLANE

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ABSTRACT. For a compact set  $E$  with connected complement, let  $A(E)$  be the uniform algebra of functions continuous on  $E$  and analytic interior to  $E$ . We describe  $A(E, W)$ , the set of uniform limits on  $E$  of sequences of the weighted polynomials  $\{W^n(z)P_n(z)\}_{n=0}^\infty$ , as  $n \rightarrow \infty$ , where  $W \in A(E)$  is a nonvanishing weight on  $E$ . If  $E$  has empty interior, then  $A(E, W)$  is completely characterized by a zero set  $Z_W \subset E$ . However, if  $E$  is a closure of Jordan domain, the description of  $A(E, W)$  also involves an inner function.

In both cases, we exhibit the role of the support of a certain extremal measure, which is the solution of a weighted logarithmic energy problem, played in the descriptions of  $A(E, W)$ .

### 1. INTRODUCTION

Let  $E$  be a compact set in the complex plane  $\mathbb{C}$  with the connected complement  $\overline{\mathbb{C}} \setminus E$ . We denote the uniform algebra of continuous on  $E$  and analytic in the interior of  $E$  functions by  $A(E)$  (see, e.g., [5, p. 25]). Clearly, the corresponding uniform norm for any  $f \in A(E)$  is defined by

$$(1.1) \quad \|f\|_E := \max_{z \in E} |f(z)|.$$

Consider a weight function  $W \in A(E)$  such that  $W(z) \neq 0$  for any  $z \in E$ , and define the *weighted polynomials*  $W^n(z)P_n(z)$ , where  $P_n(z)$  is an algebraic polynomial in  $z$  with complex coefficients, with  $\deg P_n \leq n$ . Note that the power of weight varies with the degree of polynomial. We are interested in a description of the function set  $A(E, W)$ , consisting of the *uniform limits on  $E$*  of sequences of the weighted polynomials  $\{W^n(z)P_n(z)\}_{n=0}^\infty$ , as  $n \rightarrow \infty$ . It is well known that if  $W(z) \equiv 1$  on  $E$  then  $A(E, 1) = A(E)$  by Mergelyan's theorem [5, p. 48]. In general, we have that  $A(E, W) \subset A(E)$ .

Our problem originated in the work of Lorentz [12] on incomplete polynomials on the real line. Surveys of results in this area, dealing with weighted approximation on the real line, can be found in [22] and [18, Ch. VI]. The most recent developments are in [8]–[10].

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The questions of density of the weighted polynomials in the set of analytic functions in a domain have been considered in [4], [15] and [16]. In particular, [16] contains a necessary and sufficient condition such that any analytic in a bounded open set function is uniformly approximable by the weighted polynomials  $W^n(z)P_n(z)$  on compact subsets. However, the description of  $A(E, W)$  seems to be much more complicated, in that no general necessary and sufficient condition is known (in terms of the weight  $W(z)$ ), even for the real interval case, i.e., for  $E = [a, b] \subset \mathbb{R}$ .

We shall approach the above mentioned problems on  $A(E, W)$ , using ideas of the theories of uniform algebras and of weighted potentials.

The second section of this paper deals with certain general inclusions of  $A(E, W)$  and some weighted potential background. We state the results on  $A(E, W)$  for  $E$  having empty interior in Section 3. The corresponding results for  $E$  being the closure of the unit disk  $D$  or, more generally, of a Jordan domain  $G$ , are developed in Section 4. Section 5 contains all proofs. We conclude with remarks and open problems in Section 6.

## 2. INCLUSION OF $A(E, W)$ AS A CLOSED IDEAL AND WEIGHTED POTENTIALS

We start with

**Proposition 2.1.**  *$A(E, W)$ , endowed with norm (1.1), is a closed function algebra (not necessarily containing constants and separating points).*

We have already remarked that  $A(E, W) \subset A(E)$ . To make this inclusion more precise, let us introduce the algebra  $[W(z), zW(z)]$  generated by the two functions  $W(z)$  and  $zW(z)$ , which is the uniform closure of all polynomials in  $W(z)$  and  $zW(z)$  (with constant terms included) on  $E$ . Clearly,  $[W(z), zW(z)] \subset A(E)$ . Furthermore, since any weighted polynomial  $W^n(z)P_n(z)$  is an element of  $[W(z), zW(z)]$ , then  $A(E, W) \subset [W(z), zW(z)]$ . Thus, we arrive at the following

**Proposition 2.2.**  $A(E, W) \subset [W(z), zW(z)] \subset A(E)$ .

The next fact is rather simple but important.

**Proposition 2.3.**  $A(E, W)$  is a closed ideal of  $[W(z), zW(z)]$ .

It turns out that in many cases  $[W(z), zW(z)] = A(E)$ , so that  $A(E, W)$  becomes a closed ideal of  $A(E)$  by Proposition 2.3. This situation is the most interesting for us, because then we can employ characterizations of the closed ideals of  $A(E)$  for some types of the compact  $E$ . This is done in Sections 3 and 4.

**Proposition 2.4.**  $[W(z), zW(z)] = A(E)$  iff  $1/W(z) \in [W(z), zW(z)]$ .

Unfortunately, we do not know any effectively verifiable necessary and sufficient condition on the weight  $W(z)$ , so that the equality  $[W(z), zW(z)] = A(E)$  is valid. Nevertheless, a number of sufficient conditions can be given, guaranteeing that the two algebras  $[W(z), zW(z)]$  and  $A(E)$  coincide.

**Proposition 2.5.** *Each of the following conditions implies that  $[W(z), zW(z)] = A(E)$ :*

- (a) *The point  $\zeta = 0$  belongs to the unbounded component of  $\overline{\mathbb{C}} \setminus W(E)$ ;*
- (b)  *$E$  is the closure of a Jordan domain or a Jordan arc, and  $W(z)$  is one-to-one on  $E$ ;*
- (c)  *$E$  is a Jordan arc and  $W(z)$  is of bounded variation on  $E$ ;*

- (d)  $E$  is a Jordan arc and  $W(z)$  is locally one-to-one on  $E$ ;
- (e)  $E = \overline{G}$ , where  $G$  is a Jordan domain bounded by an analytic curve, and  $W'(z) \in A(\overline{G})$ .

Other sufficient conditions, implying the conclusion of Proposition 2.5, can be found in [1], [6], [21, §30] and [20]. Obviously, if the interior of  $E$  is empty, then  $A(E) = C(E)$  by definition, where  $C(E)$  is the algebra of all continuous functions on  $E$ .

A very important tool for analyzing the behavior of the weighted polynomials  $W^n P_n$  is the theory of logarithmic potentials with external fields (cf. [18]). Assuming that  $E$  has positive logarithmic capacity (cf. [23, p. 55]), then

$$(2.1) \quad w(z) := \begin{cases} |W(z)|, & z \in E, \\ 0, & z \notin E, \end{cases}$$

is an *admissible weight* for the *weighted logarithmic energy problem* on  $E$  considered in Section I.1 of [18]. This enables us to use certain results of [18], which we summarize below for the convenience of the reader. Recall that the logarithmic potential of a compactly supported Borel measure  $\mu$  is given by (cf. [23, p. 53])

$$(2.2) \quad U^\mu(z) := \int \log \frac{1}{|z - t|} d\mu(t).$$

**Proposition 2.6.** *There exists a positive unit Borel measure  $\mu_w$ , with support  $S_w := \text{supp } \mu_w \subset \partial E$ , such that for any polynomial  $P_n(z)$ ,  $\deg P_n \leq n$ , we have*

$$(2.3) \quad |W^n(z)P_n(z)| \leq \|W^n P_n\|_{S_w} \exp(n(F_w - U^{\mu_w}(z) + \log |W(z)|)), \quad z \in E,$$

where  $F_w$  is a constant.

Furthermore, the inequality

$$(2.4) \quad U^{\mu_w}(z) - \log |W(z)| \geq F_w$$

holds quasi-everywhere on  $E$ , and

$$(2.5) \quad U^{\mu_w}(z) - \log |W(z)| \leq F_w, \text{ for any } z \in S_w.$$

By saying quasi-everywhere (q.e.), we mean that a property holds everywhere, with the exception of a set of zero logarithmic capacity. The measure  $\mu_w$  is the solution of a weighted energy problem, corresponding to the weight  $w(z)$  of (2.1) (see Section I.1 of [18]).

It follows from (2.3) and (2.4) that the norm of a weighted polynomial  $W^n P_n$  essentially “lives” on  $S_w$ , i.e.,

$$(2.6) \quad |W^n(z)P_n(z)| \leq \|W^n P_n\|_{S_w}$$

for quasi-every  $z \in E$ . We can replace, in fact, the words “quasi-everywhere on  $E$ ” in (2.4) and (2.6) by “everywhere on  $E$ ”, by requiring a certain regularity for the set  $E$ . In particular, the following is valid (see Corollary III.2.6 of [18]).

**Proposition 2.7.** *Suppose that for every point  $z_0 \in E$ , the set  $\{z : |z - z_0| < \delta, z \in E\}$  has positive capacity for any  $\delta > 0$ . Then*

$$(2.7) \quad \|W^n P_n\|_E = \|W^n P_n\|_{S_w}$$

for any polynomial  $P_n$ ,  $\deg P_n \leq n$ .

Consequently, the Shilov boundary (cf. [21, pp. 36–39]) of  $A(E, W)$  is contained in  $S_w$ .

One may consult Section III.2 of [18] on further details, regarding the supremum norm of weighted polynomials.

### 3. WEIGHTED APPROXIMATION ON SETS WITH EMPTY INTERIOR

Let  $E$  be a compact set with connected complement and empty interior. Obviously,  $A(E) = C(E)$  in this case. We characterize  $A(E, W)$  in terms of a certain zero set.

**Theorem 3.1.** *Suppose that  $E$  has a connected complement and an empty interior, and that  $W \in C(E)$  is a nonvanishing weight on  $E$ . Assume that  $[W(z), zW(z)] = C(E)$ .*

*Then, there exists a closed set  $Z_W \subset E$  such that*

$$f \in A(E, W) \text{ if and only if } f \in C(E) \text{ and } f|_{Z_W} \equiv 0.$$

It is clear that  $A(E, W) = C(E)$  if and only if the set  $Z_W$  is empty. This is true, for example, for  $W(z) \equiv 1$  on  $E$ .

Theorem 3.1 generalizes a recent result of Kuijlaars (see Theorem 3 of [10]), related to polynomial approximation with varying weights on the real line. However, it has a new part even in the latter case, allowing us to consider the *complex valued weights*  $W(z)$  on subsets of the real line.

A description of the set  $Z_W$  in terms of the weight  $W(z)$  is unknown in general. We can only show that  $Z_W$  must contain the complement of  $S_w$  (see Proposition 2.6) in  $E$ .

**Theorem 3.2.** *Let  $E$  be an arbitrary compact set with the connected complement  $\overline{\mathbb{C}} \setminus E$  and let  $W \in A(E)$  be a nonvanishing weight on  $E$ . Suppose that for every point  $z_0 \in E$ , the set  $\{z : |z - z_0| < \delta, z \in E\}$  has positive logarithmic capacity for any  $\delta > 0$ . Assume further that  $\overline{\mathbb{C}} \setminus S_w$  is connected and  $[W(z), zW(z)] = C(S_w)$  on  $S_w$ .*

*If  $f \in A(E, W)$ , then  $f(z) = 0$  for any  $z \in \overline{E \setminus S_w}$ . In particular, if  $E$  has empty interior, then  $\overline{E \setminus S_w} \subset Z_W$ .*

The proof of Theorem 3.2 is based on an idea of Kuijlaars (see Theorem 2 and its proof in [10]).

If  $E$  is a compact subset of the real line and the weight  $W(z)$  is real valued, then condition (a) of Proposition 2.5 is clearly satisfied, so that  $[W(z), zW(z)] = C(E)$ . Therefore, the conclusion of Theorem 3.1 is valid, and coincides with that of Theorem 3 of [10]. Furthermore, if for any point in  $E$ , the intersection of its arbitrary neighborhood with  $E$  has positive logarithmic capacity, then  $\overline{E \setminus S_w} \subset Z_W$ . Since  $[W(z), zW(z)] = C(S_w)$  on  $S_w$  by Proposition 2.5(a), Theorem 3.2 essentially reduces to Theorem 2 of [10] in this case, which in turn contains an earlier result of Theorem 4.1 of [22].

We mention two examples here just for illustrative purposes. A number of additional examples, with their complete discussions, is in [22], [18, Ch. VI], and [8]-[9].

**Example 3.3** (Incomplete Polynomials). Let  $E = [0, 1]$  and  $W(x) = x^{\theta/(1-\theta)}$ , where  $0 < \theta < 1$ . It is known that  $S_w = [\theta^2, 1]$  and that  $A([0, 1], x^{\theta/(1-\theta)})$  consists of all continuous functions on  $[0, 1]$ , vanishing on  $[0, \theta^2]$  (see [19] and [18, Sect. VI.1]). Thus, the set  $Z_W$  of Theorem 3.1 is just  $[0, \theta^2] = \overline{E \setminus S_w}$  in this case.

However, it is not always true that  $Z_W = \overline{E \setminus S_w}$ , as the next example shows.

**Example 3.4** (Exponential Weights). Let  $E = [-2, 2]$  and  $W(x) = \exp(-\gamma_\alpha|x|^\alpha)$ , where  $\alpha > 0$  and

$$\gamma_\alpha = \frac{\Gamma(\alpha/2) \Gamma(1/2)}{2\Gamma(\alpha/2 + 1/2)}.$$

It is known that  $S_w = [-1, 1]$  and that

$$(3.1) \quad \begin{cases} A([-2, 2], \exp(-\gamma_\alpha|x|^\alpha)) = \\ \{f \in C([-2, 2]) : f|_{[-2, -1] \cup [1, 2]} \equiv 0\} & \text{for } \alpha \geq 1, \\ \{f \in C([-2, 2]) : f|_{[-2, -1] \cup [1, 2] \cup \{0\}} \equiv 0\} & \text{for } 0 < \alpha < 1, \end{cases}$$

where the range  $\alpha > 1$  was studied in [13], and the case of the remaining interval  $0 < \alpha \leq 1$  was covered by the results of [14]. In our context, (3.1) means that  $Z_W = [-2, -1] \cup [1, 2] = \overline{E \setminus S_w}$  for  $\alpha \geq 1$  and that  $Z_W = [-2, -1] \cup [1, 2] \cup \{0\} = \overline{E \setminus S_w} \cup \{0\}$  for  $0 < \alpha < 1$ .

4. WEIGHTED APPROXIMATION ON THE UNIT DISK AND ON JORDAN DOMAINS

The first result of this section is a consequence of the well-known description of closed ideals of  $A(\overline{D})$ , where  $D$  is the unit disk, due to Beurling (unpublished) and Rudin [17] (see also [7, pp. 82-87] for a discussion). Recall that  $g$  is an *inner function* if it is analytic in  $D$ , with  $\|g\|_{\overline{D}} \leq 1$ , and  $|g(e^{i\theta})| = 1$  almost everywhere on the unit circle (cf. [7, p. 62]). By the factorization theorem, every inner function can be uniquely expressed in the form

$$(4.1) \quad g(z) = B(z)S(z), \quad z \in D,$$

where  $B(z)$  is a Blaschke product and  $S(z)$  is a *singular function*, i.e.,

$$(4.2) \quad S(z) := \exp\left(-\int \frac{e^{i\theta} + z}{e^{i\theta} - z} d\nu_s(\theta)\right), \quad z \in D,$$

with  $\nu_s$  being a positive measure on the unit circle, singular with respect to  $d\theta$  (see [7, pp. 63-67]).

**Theorem 4.1.** *Let a nonvanishing weight  $W \in A(\overline{D})$  be such that  $[W(z), zW(z)] = A(\overline{D})$ . Assume that  $A(\overline{D}, W)$  contains a not identically zero function.*

*Then there exist a closed set  $H_W \subset \partial D$  of Lebesgue measure zero and an inner function  $g_W$  satisfying*

- (i) *every accumulation point of the zeros of its Blaschke product is in  $H_W$ ,*
- (ii) *the measure  $\nu_s$  of its singular function is supported on  $H_W$ ;*

*such that*

$$f \in A(\overline{D}, W) \text{ if and only if } f = g_W h, \text{ where } h \in A(\overline{D}) \text{ and } h|_{H_W} \equiv 0.$$

The case of a Jordan domain  $G$  can be reduced to that of the unit disk, using a canonical conformal mapping  $\phi : G \rightarrow D$  and its inverse  $\psi := \phi^{-1}$ . We state the corresponding result for completeness.

**Theorem 4.2.** *Let a nonvanishing weight  $W \in A(\overline{G})$ , where  $G$  is a Jordan domain, be such that  $[W(z), zW(z)] = A(\overline{G})$ . Assume that  $A(\overline{G}, W)$  contains a not identically zero function.*

Then there exist a closed set  $H_W \subset \partial D$  and an inner function  $g_W$ , as in Theorem 4.1, such that

$$f \in A(\overline{G}, W) \text{ if and only if } f = (g_W \circ \phi)h,$$

where  $h \in A(\overline{G})$  and  $h|_{\psi(H_W)} \equiv 0$ .

It follows from Theorems 4.1 and 4.2 that  $A(\overline{D}, W) = A(\overline{D})$  or  $A(\overline{G}, W) = A(\overline{G})$  if and only if  $g_W \equiv 1$  and  $H_W$  is empty.

**Example 4.3** (Exponential weight on the Szegő domain). Let  $W(z) = e^{-z}$  and  $G$  be the Szegő domain

$$(4.3) \quad \{z : |ze^{1-z}| < 1 \text{ and } |z| < 1\},$$

which is bounded by a piecewise analytic curve (the *Szegő curve*) with the only corner point  $z = 1$  (see [15] for more information and the graph of  $G$ ). We remark that the function  $\phi(z) = ze^{1-z}$  maps  $G$  conformally onto  $D$ . It follows from Proposition 3.1 of [15] that  $f_0(z) := (z - 1)e^{-z}$  belongs to  $A(\overline{G}, e^{-z})$ . Thus, either  $H_W = \{1\}$  or  $H_W$  is empty. Clearly,  $g_W \equiv 1$  and we obtain from Theorem 4.2 that if  $f \in A(\overline{G})$  and  $f(1) = 0$ , then  $f \in A(\overline{G}, e^{-z})$ . This implies the result of Theorem 3.2 of [15], in particular. It was also conjectured in [15] that  $H_W = \{1\}$  (in the present notation), but this problem remains open.

Our next goal is to exhibit the role of the set  $S_w$  (see Proposition 2.6) in the case of weighted approximation on Jordan domains. Since  $W \in A(\overline{G})$  is analytic in  $G$ , then  $S_w \subset \partial G$  by Theorem IV.1.10(a) of [18] and (2.1). The following result shows that  $S_w = \partial G$  is necessary for nontrivial weighted approximation on  $\overline{G}$ .

**Theorem 4.4.** *Let  $G$  be a Jordan domain and let  $W \in A(\overline{G})$  be a nonvanishing weight. Assume that  $S_w$  is a proper subset of  $\partial G$  and that  $[W(z), zW(z)] = C(S_w)$  on  $S_w$ . Then  $A(\overline{G}, W)$  contains the identically zero function only.*

We construct below a specific example of the weight  $W(z)$ , satisfying the conditions of Theorem 4.4.

**Example 4.5.** Let  $G$  be a Jordan domain, as before, and let  $\gamma$  be a proper closed subarc of  $\partial G$ . Define the conformal mapping  $\Phi : \overline{\mathbb{C}} \setminus \gamma \rightarrow \{w : |w| > 1\}$ , normalized by  $\Phi(\infty) = \infty$  and  $\lim_{z \rightarrow \infty} \Phi(z)/z > 0$ . We extend  $\Phi$  to  $\gamma$ , using boundary limits from inside of  $G$ , so that  $\Phi \in A(\overline{G})$  is one-to-one on  $\overline{G}$ . Thus, we can consider approximation by weighted polynomials on  $\overline{G}$ , with  $W(z) = 1/\Phi(z)$ ,  $z \in \overline{G}$ .

Denote the Green function of the domain  $\overline{\mathbb{C}} \setminus \gamma$ , with pole at  $\infty$ , by  $g(z, \infty)$  [23, p. 14], and the classical equilibrium distribution of  $\gamma$  (in the sense of logarithmic potential theory) by  $\mu_\gamma$  [23, p. 55]. Then,

$$g(z, \infty) = \log \frac{1}{\text{cap}(\gamma)} - U^{\mu_\gamma}(z), \quad z \in \mathbb{C},$$

where  $\text{cap}(\gamma)$  is the logarithmic capacity of  $\gamma$  (see [18, Sect. I.4]). Furthermore, since  $g(z, \infty) = \log |\Phi(z)|$ ,  $z \in \mathbb{C}$  (cf. [23, p. 18]), we obtain

$$(4.4) \quad U^{\mu_\gamma}(z) - \log |1/\Phi(z)| = \log \frac{1}{\text{cap}(\gamma)}, \quad z \in \mathbb{C}.$$

It follows from (4.4) and Theorem I.3.3 of [18] that  $\mu_\gamma$  is the solution of a weighted energy problem (cf. [18, Sec. I.1]) for the weight  $w(z) = 1/|\Phi(z)|$  on  $\overline{G}$ . Hence, we have shown that  $S_w = \text{supp } \mu_\gamma = \gamma$ .

Observe that the weighted polynomials  $W^n(z)P_n(z)$ , with  $W(z) = 1/\Phi(z)$ ,  $z \in \overline{G}$ , can approximate an arbitrary analytic function in  $G$  on compact subsets of  $G$  by Theorem 1.1 of [16] and (4.4). On the other hand, Proposition 2.5(b) and Theorem 4.4 indicates that they can only approximate the identically zero function uniformly on  $\overline{G}$ .

## 5. PROOFS

*Proof of Proposition 2.1.* We have to show that  $A(E, W)$  is closed under addition, multiplication by constants and by functions of  $A(E, W)$ , and under uniform limits. Suppose that  $W^n P_n \rightarrow f \in A(E, W)$  and  $W^n Q_n \rightarrow g \in A(E, W)$  uniformly on  $E$ , as  $n \rightarrow \infty$ . Then  $W^n(P_n + Q_n) \rightarrow (f + g)$ , as  $n \rightarrow \infty$ , so that  $(f + g) \in A(E, W)$ . If  $\alpha \in \mathbb{C}$  then  $W^n \alpha P_n \rightarrow \alpha f$ , as  $n \rightarrow \infty$ , and  $\alpha f \in A(E, W)$ . Observe that

$$\begin{aligned} \|fg - W^{2n} P_n Q_n\|_E &\leq \|fg - fW^n Q_n\|_E + \|fW^n Q_n - W^{2n} P_n Q_n\|_E \leq \\ &\|f\|_E \|g - W^n Q_n\|_E + \|W^n Q_n\|_E \|f - W^n P_n\|_E \rightarrow 0, \end{aligned}$$

as  $n \rightarrow \infty$ , i.e.,  $fg \in A(E, W)$ . Applying the standard diagonalization argument, we see that  $A(E, W)$  is closed in norm (1.1).  $\square$

*Proof of Proposition 2.3.* Assume that  $f \in A(E, W)$  and  $W^n P_n \rightarrow f$  uniformly on  $E$ , as  $n \rightarrow \infty$ . Then, for any pair of nonnegative integers  $k$  and  $\ell$  such that  $k \geq \ell$ , we have

$$\begin{aligned} \|f(z)W^k(z)z^\ell - W^{n+k}(z)z^\ell P_n(z)\|_E &\leq \\ \|W^k(z)z^\ell\|_E \|f - W^n P_n\|_E &\rightarrow 0, \quad \text{as } n \rightarrow \infty, \end{aligned}$$

which gives that  $f(z)W^k(z)z^\ell \in A(E, W)$ . Since  $A(E, W)$  is closed under addition and multiplication by constants (by Proposition 2.1), then the product of  $f$  and any polynomial in  $W(z)$  and  $zW(z)$  belongs to  $A(E, W)$ . Thus, if  $g \in [W(z), zW(z)]$  then  $fg \in A(E, W)$  follows immediately, because  $A(E, W)$  is closed in the uniform norm on  $E$  (cf. Proposition 2.1). The proof is now complete in view of Propositions 2.1 and 2.2.  $\square$

*Proof of Proposition 2.4.* Obviously, if  $[W(z), zW(z)] = A(E)$  then  $1/W(z) \in A(E) = [W(z), zW(z)]$ .

Assume that  $1/W(z) \in [W(z), zW(z)]$ . It follows that  $z \in [W(z), zW(z)]$  and, consequently, every polynomial in  $z$  is in  $[W(z), zW(z)]$ . Since  $[W(z), zW(z)]$  is uniformly closed on  $E$  by definition, then  $A(E) \subset [W(z), zW(z)]$  by Mergelyan's theorem [5, p. 48]. Thus, Proposition 2.2 implies at once that  $A(E) = [W(z), zW(z)]$ .  $\square$

*Proof of Proposition 2.5.* First, we remark that  $W(z)$  and  $zW(z)$  together separate points of any set  $E$ .

(a) Observe that  $W(E)$ , the image of  $E$  in  $\zeta$ -plane under the mapping  $\zeta = W(z)$ , is compact. By assumption, function  $1/\zeta$  is analytic on the polynomial convex hull of  $W(E)$  and can be uniformly approximated there by polynomials in  $\zeta$  (by Mergelyan's theorem). Returning to  $z$ -plane, we obtain that  $1/W(z)$  is uniformly approximable on  $E$  by polynomials in  $W(z)$ . It follows that  $[W(z), zW(z)] = A(E)$  by Proposition 2.4.

(b) The mapping  $\zeta = W(z)$  can be continued as a homeomorphism between  $z$ -plane and  $\zeta$ -plane (cf. [11, p. 535]). Since  $W(z)$  doesn't vanish on  $E$ , then  $\zeta = 0$  belongs to the domain  $\overline{\mathbb{C}} \setminus W(E) = W(\overline{\mathbb{C}} \setminus E)$ , which contains  $\zeta = \infty$ . Hence, (b) follows from (a).

(c) If  $E = [0, 1]$  then (c) is a direct consequence of Theorem 2 of [2]. For  $E$  being a Jordan arc, we consider a homeomorphic parametrization of  $E$  by  $\tau : [0, 1] \rightarrow E$ . Since  $W \circ \tau(x)$  is of bounded variation on  $[0, 1]$ , we have, as before, that  $[W \circ \tau(x), \tau(x)(W \circ \tau)(x)] = C([0, 1])$ . Clearly,  $\tau$  induces an isometric isomorphism between  $C([0, 1])$  and  $C(E)$ . Thus, the result follows after returning to  $E$  with the help of  $\tau^{-1}$ .

(d) is implied by Theorem 1 of [1] for  $E = [0, 1]$ . The case of a Jordan arc can be reduced to that of the interval as in the proof of (c).

(e) First, assume that  $E = \overline{D}$ . Then (e) follows at once from [24, p. 135] (see also [3]). It is well known that the conformal mapping  $\phi : G \rightarrow D$  extends as a diffeomorphism between  $\overline{G}$  and  $\overline{D}$  (with nonvanishing derivatives of  $\phi$  and  $\psi := \phi^{-1}$ ), because  $G$  is bounded by an analytic Jordan curve. Using  $\phi$ , the result for  $E = \overline{G}$  is a consequence of [24, p. 135] too.  $\square$

*Proof of Proposition 2.6.* Since  $W(z)$  is a continuous nonvanishing function on  $E$  and  $w(z)$  of (2.1) is so too, then the existence of  $\mu_w$  and inequalities (2.4)-(2.5) follow from Theorem I.1.3 of [18]. Moreover,  $W(z)$  is analytic in the interior of  $E$ , which implies that  $S_w \subset \partial E$  by Theorem IV.1.10(a) of [18] and (2.1). The inequality (2.3) is a direct consequence of Theorem III.2.1 of [18].  $\square$

*Proof of Theorem 3.1.* We have that  $[W(z), zW(z)] = C(E)$  by the assumption of the theorem. Thus,  $A(E, W)$  is a closed ideal of  $C(E)$  (cf. Proposition 2.3), which is known to be described by its zero set (see [21, p. 32]).  $\square$

*Proof of Theorem 3.2.* We essentially follow the proof of Theorem 2 of [10]. Suppose that there exist  $f_0 \in A(E, W)$  and  $z_0 \in E \setminus S_w$  such that  $f_0(z_0) \neq 0$  and  $W^n P_n \rightarrow f_0$  uniformly on  $E$ , as  $n \rightarrow \infty$ .

It is clear that  $f_0|_{S_w} \in A(S_w, W)$ . Recall that  $S_w \subset \partial E$  by Proposition 2.6, i.e.,  $S_w$  has empty interior. Applying Theorem 3.1, with  $E$  replaced by  $S_w$ , we obtain that  $A(S_w, W)$  is described by the zero set  $Z_W^* \subset S_w$ . Observe that multiplying  $A(S_w, W)$  by  $(z - z_0)W(z)$ , we obtain a closed ideal of  $[W(z), zW(z)] = C(S_w)$  (cf. Proposition 2.3), which consists of all functions, uniformly approximable on  $S_w$  by the weighted polynomials  $W^n(z)Q_n(z)$  such that  $Q_n(z_0) = 0$ , as  $n \rightarrow \infty$ . On the other hand, the zero set of the ideal  $(z - z_0)W(z)A(S_w, W)$  coincides with that of  $A(S_w, W)$ . It follows that  $(z - z_0)W(z)A(S_w, W) = A(S_w, W)$  (see [21, p. 32]) and that  $f_0|_{S_w} \in (z - z_0)W(z)A(S_w, W)$ .

Thus, there exists a sequence of the weighted polynomials  $\{W^n Q_n\}_{n=0}^\infty$ , with  $Q_n(z_0) = 0$ , uniformly convergent to  $f_0$  on  $S_w$ , as  $n \rightarrow \infty$ . Since  $W^n(z)(P_n(z) - Q_n(z))$  converges to zero uniformly on  $S_w$  and converges to  $f_0(z_0) \neq 0$  for  $z = z_0 \in E \setminus S_w$ , as  $n \rightarrow \infty$ , then we obtain a direct contradiction with (2.7) for some sufficiently large  $n$ .

Consequently, if  $f \in A(E, W)$  then  $f(z) = 0$  for any  $z \in E \setminus S_w$ . Furthermore, the same is true for any  $z \in \overline{E \setminus S_w}$  by the continuity of  $f(z)$ .  $\square$

*Proof of Theorem 4.1.* Since  $[W(z), zW(z)] = A(\overline{D})$  by the assumption of the theorem, then  $A(\overline{D}, W)$  is a closed ideal of  $A(\overline{D})$  by Proposition 2.3. The result now follows from the description of nontrivial closed ideals of the disk algebra (see [17] and [7, pp. 82-87]).  $\square$

*Proof of Theorem 4.2.* First, we obtain that  $A(\overline{G}, W)$  is a closed ideal of  $A(\overline{G})$  (cf. Proposition 2.3). It is well known that the conformal mapping  $\phi$  extends to a

homeomorphism between  $\overline{G}$  and  $\overline{D}$ , defining an isometric isomorphism between the algebras  $A(\overline{G})$  and  $A(\overline{D})$ , and their closed ideals. Thus, we apply the result of [17] to the isomorphic image of  $A(\overline{G}, W)$  in  $A(\overline{D})$ , as in the proof of Theorem 4.1, and return to  $A(\overline{G}, W)$  with the help of the inverse conformal mapping  $\psi$ .  $\square$

*Proof of Theorem 4.4.* Since  $G$  is a Jordan domain, then the set  $\{z : |z - z_0| < \delta, z \in \overline{G}\}$  has positive logarithmic capacity for any  $z_0 \in \overline{G}$  and  $\delta > 0$ . It is clear that  $S_w$  is contained in some Jordan arc, as a proper closed subset of  $\partial G$ , so that  $\overline{\mathbb{C}} \setminus S_w$  is connected. Observe that all conditions of Theorem 3.2 are satisfied in this case, which yields that any function  $f \in A(\overline{G}, W)$  must vanish on  $\overline{(\overline{G} \setminus S_w)} = \overline{G}$ .  $\square$

## 6. FURTHER REMARKS

One of the main assumptions of the theorems in Sections 3 and 4 is that  $[W(z), zW(z)] = A(E)$ . This equality is valid for a wide classes of sets  $E$  and weights  $W \in A(E)$ , as described in Proposition 2.5. On the other hand, it is clear that our results may be extended further. The main ingredients of such extensions are:

- (i) a proof of the equality  $[W(z), zW(z)] = A(E)$ ;
- (ii) a description of the closed ideals of  $A(E)$ .

Nevertheless, we believe that a problem of much greater interest and difficulty is to uncover a more explicit relation between the zero set  $Z_W$  (or  $H_W$  and the inner function  $g_W$ ) and the weight  $W(z)$ . A considerable progress has been achieved on this problem for real valued weights on the real line (see [22], [18, Ch. VI] and [8]-[9]), but a general description of the set  $Z_W$  through  $W(z)$  is unknown even in the latter case.

## REFERENCES

- [1] H. Alexander, *Polynomial approximation and analytic structure*, Duke Math. J. **38** (1971), 123–135. MR **44**:477
- [2] H. Alexander, *Polynomial approximation and hulls in sets of finite linear measure in  $\mathbb{C}^n$* , Amer. J. Math. **93** (1971), 65–74. MR **44**:1841
- [3] R. G. Blumenthal, *Holomorphically closed algebras of analytic functions*, Math. Scand. **34** (1974), 84–90. MR **52**:1323
- [4] P. B. Borwein and W. Chen, *Incomplete rational approximation in the complex plane*, Constr. Approx. **11** (1995), 85–106. MR **95k**:41024
- [5] T. W. Gamelin, *Uniform Algebras*, Chelsea Publ. Co., New York, 1984.
- [6] T. W. Gamelin, *Polynomial approximation on thin sets*, Symposium on Several Complex Variables (R. M. Brooks, ed.), pp. 50–78, Lecture Notes in Math., vol. 184, Springer-Verlag, Berlin, 1971. MR **45**:9145
- [7] K. Hoffman, *Banach spaces of analytic functions*, Prentice-Hall, Englewood Cliffs, 1962. MR **24**:A2844
- [8] A. B. J. Kuijlaars, *The role of the endpoint in weighted polynomial approximation with varying weights*, Constr. Approx. **12** (1996), 287–301.
- [9] A. B. J. Kuijlaars, *Weighted approximation with varying weights: the case of a power type singularity*, J. Math. Anal. Appl. **204** (1996), 409–418. CMP 97:04
- [10] A. B. J. Kuijlaars, *A note on weighted polynomial approximation with varying weights*, J. Approx. Theory **87** (1996), 112–115. CMP 97:01
- [11] K. Kuratowski, *Topology*, vol. II, Academic Press, New York, 1968.
- [12] G. G. Lorentz, *Approximation by incomplete polynomials (problems and results)*, Padé and Rational Approximations: Theory and Applications (E. B. Saff and R. S. Varga, eds.), pp. 289–302, Academic Press, New York, 1977. MR **57**:6956
- [13] D. S. Lubinsky and E. B. Saff, *Uniform and mean approximation by certain weighted polynomials, with applications*, Constr. Approx. **4** (1988), 21–64. MR **88j**:41049

- [14] D. S. Lubinsky and V. Totik, *Weighted polynomial approximation with Freud weights*, Constr. Approx. **10** (1994), 301–315. MR **95i**:41007
- [15] I. E. Pritsker and R. S. Varga, *The Szegő curve, zero distribution and weighted approximation*, Trans. Amer. Math. Soc. **349** (1997), 4085–4105. CMP 96:17
- [16] I. E. Pritsker and R. S. Varga, *Weighted polynomial approximation in the complex plane*, Electron. Res. Announc. Amer. Math. Soc. **3** (1997), 38–44. CMP 97:11
- [17] W. Rudin, *The closed ideals in an algebra of analytic functions*, Can. J. Math. **9** (1957), 426–434. MR **19**:641c
- [18] E. B. Saff and V. Totik, *Logarithmic Potentials with External Fields*, Springer-Verlag, Heidelberg, 1997.
- [19] E. B. Saff and R. S. Varga, *On incomplete polynomials*, Numerische Methoden der Approximationstheorie (L. Collatz, G. Meinardus and H. Werner, eds.), ISNM 42, pp. 281–298, Birkhäuser-Verlag, Basel, 1978. MR **80d**:41008
- [20] N. Sibony and J. Wermer, *Generators for  $A(\Omega)$* , Trans. Amer. Math. Soc., **194** (1974), 103–114. MR **54**:7856
- [21] E. L. Stout, *The Theory of Uniform Algebras*, Bogden and Quigley, Inc., Belmont, 1971. MR **54**:11066
- [22] V. Totik, *Weighted Approximation with Varying Weights*, Lecture Notes in Math., vol. 1569, Springer-Verlag, Heidelberg, 1994. MR **96f**:41002
- [23] M. Tsuji, *Potential Theory in Modern Function Theory*, Maruzen, Tokyo, 1959. MR **22**:5712
- [24] J. Wermer, *Banach Algebras and Several Complex Variables*, Springer-Verlag, New York, 1976. MR **52**:15021

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