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CONSTRUCTING ULTRAWEAKLY CONTINUOUS FUNCTIONALS ON $\mathcal{B}(H)$

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Abstract. In this paper we give a constructive characterisation of ultraweakly continuous linear functionals on the space of bounded linear operators on a separable Hilbert space.

Let H be a separable complex Hilbert space, with orthonormal basis $(e_n)_{n=1}^{\infty}$, and $\mathcal{B}(H)$ the set of bounded linear operators on H. The **weak operator norm** associated with the orthonormal basis (e_n) is defined on $\mathcal{B}(H)$ by

$$\|T\|_w \equiv \sum_{j,k=1}^{\infty} 2^{-j-k} \langle Te_j, e_k \rangle .$$

Weak operator norms associated with different orthonormal bases of H give rise to equivalent metrics on the unit ball

$$\mathcal{B}_1(H) \equiv \{T \in \mathcal{B}(H) : \forall x \in H \ (\|Tx\| \leq \|x\|)\} .$$

Moreover, $\mathcal{B}_1(H)$ is totally bounded with respect to the weak operator norm, but the completeness of $\mathcal{B}_1(H)$ with respect to that norm is an essentially nonconstructive property; see [2].

In this paper we discuss, within Bishop's constructive mathematics [1], the characterisation of those linear functionals on $\mathcal{B}(H)$ that are uniformly continuous on $\mathcal{B}_1(H)$ with respect to some, and therefore each, weak operator norm. Classically, these are precisely the linear functionals on $\mathcal{B}(H)$ that are continuous with respect to the ultraweak operator topology [10]; for this reason, we shall refer to them as **ultraweakly continuous linear functionals** on $\mathcal{B}(H)$. Since $\mathcal{B}_1(H)$ is weak operator totally bounded, an ultraweakly continuous linear functional f on $\mathcal{B}(H)$ is **normable**, in the sense that its **norm**,

$$\|f\| \equiv \sup\{|f(x)| : x \in H, \ \|x\| \leq 1\} ,$$

exists ([1], Ch. 4, (4.3)).

The classical characterisation of ultraweakly continuous linear functionals on $\mathcal{B}(H)$ is usually proved using the Riesz Representation Theorem and the Hahn-Banach Theorem (see [7]); unfortunately, in order to apply each of these theorems constructively we need additional hypotheses about the computability of certain suprema and infima that cannot be verified in the present case. Another approach,

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taken by Kadison and Ringrose ([11], Section 7.1), is set in the more general context of von Neumann algebra theory and requires the theory of comparison of projections; as presently developed, the latter theory depends on nonconstructive applications of Zorn's lemma. A third proof, which is similar in spirit to ours, is found in [12] and uses a nonconstructive version of the spectral decomposition of a compact selfadjoint operator (cf. [4]).

Thus there are significant obstacles to be overcome in obtaining the desired constructive characterisation. In order to show how these obstacles can be surmounted, we assume familiarity with, or access to, Chapters 4 and 7 of [1]. In addition, we will need the following background definitions and facts, the proofs of which are found in either [3] or standard references such as [13], [10], and [11].

Let $\mathcal{A}(H)$ be the set of elements of $\mathcal{B}(H)$ that have adjoints[1]. An element A of $\mathcal{A}(H)$ is **positive** if it is selfadjoint and $\langle Ax, x\rangle \geq 0$ for each $x \in H$; we then write $A \geq 0$. If $A \in \mathcal{A}(H)$, then $A^*A \geq 0$; the positive square root of A^*A is written $|A|$. An element U of $\mathcal{A}(H)$ is a **partial isometry** if there exists a projection P, called the **initial projection** of U, such that $\|UPx\| = \|x\|$ and $U(I-P)x = 0$ for each $x \in H$. U is a partial isometry if and only if U^* exists and U^*U is a projection, in which case U^*U is the initial projection of U, and U^* is a partial isometry with initial projection UU^*.

Let $A \in \mathcal{A}(H)$. We say that A is a **Hilbert-Schmidt operator** if $\sum_{n=1}^{\infty} \|Ae_n\|^2$ converges, in which case the sum of this series is independent of the orthonormal basis (e_n) and we write

$$\|A\|_2 \equiv (\sum_{n=1}^{\infty} \|Ae_n\|^2)^{1/2}.$$

If A is a Hilbert-Schmidt operator, then so is A^*; if also $B \in \mathcal{A}(H)$, and $c > 0$ is a bound for B, then AB and BA are Hilbert-Schmidt operators, $\|AB\|_2 \leq c\,\|A\|_2$, and $\|BA\|_2 \leq c\,\|A\|_2$.

We say that $A \in \mathcal{A}(H)$ is of **trace class** if

$$\|A\|_1 \equiv \sum_{n=1}^{\infty} \langle |A|\, e_n, e_n\rangle$$

converges. In that case, $\|A\|_1 = \sum_{n=1}^{\infty} \left\| |A|^{1/2}\, e_n \right\|^2$, so $|A|^{1/2}$ is a Hilbert-Schmidt operator and the **trace class norm** $\|A\|_1$ is independent of the orthonormal basis (e_n); moreover, A is a compact operator. The set of trace class operators on H is a Banach space with respect to the trace class norm.

If A is of trace class, then the **trace** of A,

$$\mathrm{Tr}\,(A) \equiv \sum_{n=1}^{\infty} \langle Ae_n, e_n\rangle$$

exists and is independent of the orthonormal basis (e_n). If also $B \in \mathcal{A}(H)$, then AB and BA are of trace class, and $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.

Now, the argument used in [3] to show that BA is of trace class for all $B \in \mathcal{A}(H)$ does not actually need B to have an adjoint, but it does require that BA have an adjoint. It follows that BA is of trace class for all $B \in \mathcal{B}(H)$: for, as A is compact,

[1]The proposition *every bounded operator on l^2 has an adjoint* is essentially nonconstructive; see Brouwerian Example 3 of [9].

so is BA, which is therefore uniformly approximable by finite-rank operators and hence has an adjoint. The linear functional f_A defined on $\mathcal{B}(H)$ by

$$f_A(B) \equiv \mathrm{Tr}(BA)$$

is uniformly continuous on $\mathcal{B}_1(H)$ with respect to the weak operator norm. As $\mathcal{B}_1(H)$ is weak operator totally bounded, f_A is normable; in fact, $\|f_A\| = \|A\|_1$.

If A and B are Hilbert-Schmidt operators, then AB and BA are of trace class, and $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.

Our first result was proved in [3] (Theorem 1.1).

Proposition 1. Approximate polar decomposition: *If $A \in \mathcal{A}(H)$ and $\varepsilon > 0$, then there exists a partial isometry U such that ε is a bound for both $A - U|A|$ and $|A| - U^*A$.* □

For more on the constructive theory of polar decompositions and related matters, see [6].

Lemma 1. *Let A be an operator of trace class, and let $\varepsilon > 0$. Then there exists a partial isometry U such that $\|A - U|A|\|_1 < \varepsilon$.*

Proof. Choose a positive integer N such that

$$\|A\|_1^{1/2}\left(\sum_{n=N+1}^{\infty}\left\||A|^{1/2}e_n\right\|^2\right)^{1/2} < \varepsilon/4.$$

By Proposition 1, there exists a partial isometry U such that $\varepsilon/2N$ is a bound for $A - U|A|$. Likewise, given $t > 0$, we can find a partial isometry V such that t is a bound for

$$|(A - U|A|)| - V^*(A - U|A|).$$

For $q > p$ we then have

$$\begin{aligned}
&\sum_{n=p+1}^{q}\langle|(A-U|A|)|\,e_n, e_n\rangle \\
=\ &\sum_{n=p+1}^{q}\langle(|(A-U|A|)| - V^*(A-U|A|))\,e_n, e_n\rangle \\
&\qquad + \sum_{n=p+1}^{q}\langle V^*Ae_n, e_n\rangle - \sum_{n=p+1}^{q}\langle V^*U|A|\,e_n, e_n\rangle \\
\le\ & t(q-p) + \sum_{n=p+1}^{q}\langle|V^*A|\,e_n, e_n\rangle + \sum_{n=p+1}^{q}\langle|(V^*U|A|)|\,e_n, e_n\rangle \\
\le\ & t(q-p) + \|A\|_1^{1/2}\sum_{n=p+1}^{q}\left(\left\||A|^{1/2}e_n\right\|^2\right)^{1/2} \\
&+ \|A\|_1^{1/2}\sum_{n=p+1}^{q}\left(\left\||A|^{1/2}e_n\right\|^2\right)^{1/2},
\end{aligned}$$

the last step following from (1.4) of [3]. Since $t > 0$ is arbitrary, it follows from the completeness of $\mathbf{C}$ that

$$\sum_{n=N+1}^{\infty} \langle |(A - U\,|A|)|\, e_n, e_n\rangle \le 2\,\|A\|_1^{1/2} \left(\sum_{n=N+1}^{\infty} \left\| |A|^{1/2}\, e_n \right\|^2 \right)^{1/2} < \varepsilon/2.$$

Hence

$$\begin{aligned} \|A - U\,|A|\|_1 &= \sum_{n=1}^{N} \langle |(A - U\,|A|)|\, e_n, e_n\rangle + \sum_{n=N+1}^{\infty} \langle |(A - U\,|A|)|\, e_n, e_n\rangle \\ &\le N\frac{\varepsilon}{2N} + \frac{\varepsilon}{2} \\ &= \varepsilon. \quad \square \end{aligned}$$

We denote by H_∞ the direct sum $\oplus_{n=1}^{\infty} H$ of a sequence of copies of the Hilbert space H. Taken with the usual scalar product, H_∞ is a Hilbert space.

The following theorem provides our constructive characterisation of ultraweakly continuous linear functionals.

Theorem. *The following are equivalent conditions on a linear functional f on $\mathcal{B}(H)$.*

(i) *f is ultraweakly continuous.*

(ii) *There exists an operator $A \in \mathcal{B}(H)$ of trace class such that $f(T) = \operatorname{Tr}(TA)$ for each $T \in \mathcal{B}(H)$.*

(iii) *There exist sequences (x_n) and (y_n) in H_∞ such that $f(T) = \sum_{n=1}^{\infty} \langle Tx_n, y_n\rangle$ for each $T \in \mathcal{B}(H)$.*

Proof. For all j and k let S_{jk} be the trace class operator defined by $S_{jk}x \equiv \langle x, e_j\rangle\, e_k$, and let P_j be the projection of H on $\operatorname{span}\{e_1, \ldots, e_j\}$. Then for each $T \in \mathcal{B}(H)$,

$$P_n T P_n = \sum_{j,k=1}^{n} \langle Te_j, e_k\rangle\, S_{jk}.$$

For each n let A_n be the trace class operator defined by

$$A_n \equiv \sum_{j,k=1}^{n} f(S_{kj}) S_{jk}.$$

Then for each $T \in \mathcal{B}(H)$, $f(P_n T P_n) = f_{A_n}(T)$ and therefore

$$(1) \qquad |f(T) - f_{A_n}(T)| \le |f\,(T\,(I - P_n))| + |f\,((I - P_n)\,TP_n)|\,.$$

Now assume that f is ultraweakly continuous. Given $\varepsilon > 0$, choose $\delta > 0$ such that if $T, T' \in \mathcal{B}_1(H)$ and $\|T - T'\|_w \le \delta$, then $|f(T) - f(T')| < \varepsilon/2$. Then choose N so that

$$\|T\|_w \le \sum_{j,k=1}^{N} 2^{-j-k}\, |\langle Te_j, e_k\rangle| + \delta$$

for all $T \in \mathcal{B}_1(H)$. For such T and all $n \ge N$ we have $\|T(I - P_n)\|_w \le \delta$ and $\|(I - P_n)TP_n\|_w \le \delta$, so $|f(T) - f_{A_n}(T)| < \varepsilon$, by (1); whence $\|f - f_{A_n}\| \le \varepsilon$. (Note that the linear functional $f - f_{A_n}$, being ultraweakly continuous, is normable.)

Since ε is arbitrary, we conclude that $\|f - f_{A_n}\| \to 0$ as $n \to \infty$. It follows from the relation

$$\|A_m - A_n\|_1 = \|f_{A_m} - f_{A_n}\| \leq \|f - f_{A_m}\| + \|f - f_{A_n}\|$$

that (A_n) is a Cauchy sequence with respect to the trace class norm. Since the set of all trace class operators is complete with respect to that norm, there exists a trace class operator A such that $\|A - A_n\|_1 \to 0$ as $n \to \infty$. Then

$$\begin{aligned} \|f - f_A\| &\leq \|f - f_{A_n}\| + \|f_A - f_{A_n}\| \\ &= \|f - f_{A_n}\| + \|A - A_n\|_1 \\ &\to 0 \text{ as } n \to \infty; \end{aligned}$$

whence $f = f_A$. This completes the proof that (i) $\Rightarrow$ (ii).

Now suppose that (ii) obtains, and for each n let $y_n \equiv |A|^{1/2} e_n$. Since A is of trace class, $|A|^{1/2}$ is a Hilbert-Schmidt operator, so $\sum_{n=1}^{\infty} \|y_n\|^2$ converges. We prove that for each $\varepsilon > 0$ there exists a partial isometry U such that

$$\left| f(T) - \sum_{n=1}^{\infty} \langle TUy_n, y_n \rangle \right| < \varepsilon \qquad (T \in \mathcal{A}_1(H)). \tag{2}$$

Using Lemma 1, construct a partial isometry U such that $\|A - U|A|\|_1 < \varepsilon$. For each $T \in \mathcal{A}_1(H)$, $|A|^{1/2} T |A|^{1/2}$ is a trace class operator. Moreover,

$$\begin{aligned} \operatorname{Tr}(TU|A|) &= \operatorname{Tr}\left(TU|A|^{1/2}|A|^{1/2}\right) \\ &= \operatorname{Tr}\left(|A|^{1/2} TU |A|^{1/2}\right) \\ &= \sum_{n=1}^{\infty} \left\langle |A|^{1/2} TU |A|^{1/2} e_n, e_n \right\rangle \\ &= \sum_{n=1}^{\infty} \left\langle TU |A|^{1/2} e_n, |A|^{1/2} e_n \right\rangle \\ &= \sum_{n=1}^{\infty} \langle TUy_n, y_n \rangle. \end{aligned}$$

Since $A - U|A|$ is of trace class, it follows from (1.4) of [3] that

$$\begin{aligned} \left| f(T) - \sum_{n=1}^{\infty} \langle TUy_n, y_n \rangle \right| &= \operatorname{Tr}(T(A - U|A|)) \\ &\leq \sum_{n=1}^{\infty} \langle |T(A - U|A|)| e_n, e_n \rangle \\ &\leq \|A - U|A|\|_1 \\ &< \varepsilon. \end{aligned}$$

This completes the proof of (2).

Now construct a sequence $(U_k)_{k=1}^{\infty}$ of partial isometries such that for each k,

$$\left| f(T) - \sum_{n=1}^{\infty} \langle TU_k y_n, y_n \rangle \right| < \tfrac{1}{k} \quad (T \in \mathcal{A}_1(H)). \tag{3}$$

Then for all j and k we have

$$\left| f(U_j^*) - \sum_{n=1}^{\infty} \left\langle U_j^* U_k y_n, y_n \right\rangle \right| < \tfrac{1}{k}$$

and

$$\left| f(U_j^*) - \sum_{n=1}^{\infty} \left\langle U_j^* U_j y_n, y_n \right\rangle \right| < \tfrac{1}{j},$$

so that

$$\left| \sum_{n=1}^{\infty} \left\langle U_j^* \left(U_j - U_k\right) y_n, y_n \right\rangle \right| < \tfrac{1}{j} + \tfrac{1}{k}.$$

Interchanging the roles of j and k, we have

$$\left| \sum_{n=1}^{\infty} \left\langle U_k^* \left(U_k - U_j\right) y_n, y_n \right\rangle \right| < \tfrac{1}{j} + \tfrac{1}{k}.$$

Hence

$$\begin{aligned}
\sum_{n=1}^{\infty} \left\| \left(U_j - U_k\right) y_n \right\|^2 &= \left| \sum_{n=1}^{\infty} \left\langle \left(U_j^* - U_k^*\right) \left(U_j - U_k\right) y_n, y_n \right\rangle \right| \\
&\le \left| \sum_{n=1}^{\infty} \left\langle U_j^* \left(U_j - U_k\right) y_n, y_n \right\rangle \right| + \left| \sum_{n=1}^{\infty} \left\langle U_k^* \left(U_k - U_j\right) y_n, y_n \right\rangle \right| \\
&< \tfrac{2}{j} + \tfrac{2}{k}.
\end{aligned}$$

Writing $\widetilde{x}_k \equiv (U_k y_n)_{n=1}^{\infty}$, we now see that $(\tilde{x}_k)_{k=1}^{\infty}$ is a Cauchy sequence, and therefore converges to a limit $(x_n)_{n=1}^{\infty}$, in the Hilbert space H_∞. It follows from (3) that

$$f(T) = \lim_{k \to \infty} \sum_{n=1}^{\infty} \langle T U_k y_n, y_n \rangle = \sum_{n=1}^{\infty} \langle T x_n, y_n \rangle$$

for all T in $\mathcal{A}_1(H)$, and therefore for all T in $\mathcal{A}(H)$. Finally, as $\mathcal{A}(H)$ is $\|\cdot\|_w$-dense in $\mathcal{B}(H)$, we see that (ii) $\Rightarrow$ (iii).

The proof that (iii) $\Rightarrow$ (i) is trivial. □

A famous theorem of Gleason [8], underpinning the mathematical foundations of quantum mechanics, says that if f is defined only on the set of projections in $\mathcal{B}(H)$, and if f is countably additive, in the sense that

$$f\left(\sum_{n=1}^{\infty} P_n\right) = \sum_{n=1}^{\infty} f(P_n)$$

whenever (P_n) is a sequence of pairwise orthogonal projections, then there exists a positive operator A of trace class such that $f(P) = \mathrm{Tr}(PA)$ for each projection P. In that case, f clearly extends to an ultraweakly continuous linear functional on $\mathcal{B}(H)$ such that $f(T) = \mathrm{Tr}(TA)$ for each $T \in \mathcal{B}(H)$. A constructive proof of Gleason's Theorem is given in [5].

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