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THE BAIRE CATEGORY THEOREM AND THE EVASION NUMBER

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Abstract. In this paper we prove that $\mathfrak{e} \leq \mathsf{cov}(\mathcal{M})$ where $\mathfrak{e}$ is the evasion number defined by Blass. This answers negatively a question asked by Brendle and Shelah.

1. Introduction

Let $\mathsf{cov}(\mathcal{M})$ denote the smallest size of a family of meager sets whose union covers the real line. The combinatorial characterization for $\mathsf{cov}(\mathcal{M})$ has been studied by Miller and Bartoszyński, and the following result is established. For $g \in \omega^\omega$, let $\mathcal{S}^g = \prod_{n<\omega} [\omega]^{\leq g(n)}$. Each element of $\mathcal{S}^g$ is called a *slalom*.

Theorem 1.1 ([1, Lemma 2.4.2])**.** *The following cardinalities are the same:*

1. $\mathsf{cov}(\mathcal{M})$.
2. *the smallest size of* $F \subseteq \omega^\omega$ *such that for every* $h \in \omega^\omega$ *there exists* $f \in F$ *with* $f(n) \neq h(n)$ *for all but finitely many* $n < \omega$.
3. *the smallest cardinality* κ *satisfying the following: for every* $g \in \omega^\omega$ *there exists* $F \subseteq \omega^\omega$ *of size* κ *such that, for all* $\varphi \in \mathcal{S}^g$*, there exists* $f \in F$ *with* $f(n) \notin \varphi(n)$ *for all but finitely many* $n < \omega$. $\square$

Blass [2] introduced a combinatorial concept called 'predicting and evading', and using this he defined the following cardinal invariant. Let $\mathcal{P}$ be the collection of functions π from $\omega^{<\omega}$ to ω. Here we call each such π a *predictor*.

Definition 1.2 ([2])**.** The *evasion number* $\mathfrak{e}$ is the smallest size of $F \subseteq \omega^\omega$ such that, for every $\pi \in \mathcal{P}$ and $X \in [\omega]^\omega$, there exists $f \in F$ with $f(n) \neq \pi(f \restriction n)$ for infinitely many $n \in X$.

Brendle and Shelah [3], [4] studied the relations between $\mathfrak{e}$ and other cardinal invariants, and asked whether $\mathfrak{e} > \mathsf{cov}(\mathcal{M})$ is consistent [4, Subsection 3.4]. Here we give a negative answer to this question.

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2. THE MAIN RESULT

We introduce a different form of evasion number by modifying the original definition due to Blass.

Definition 2.1. $\mathfrak{e}^*$ is the smallest size of $F \subseteq \omega^\omega$ such that, for every $\pi \in \mathcal{P}$ there exists $f \in F$ with $f(n) \neq \pi(f \restriction n)$ for all but finitely many $n < \omega$.

Clearly $\mathfrak{e} \leq \mathfrak{e}^*$ holds, and it is easily seen from Theorem 1.1 that $\mathsf{cov}(\mathcal{M}) \leq \mathfrak{e}^*$.

We show that $\mathfrak{e}^*$ gives another combinatorial characterization for $\mathsf{cov}(\mathcal{M})$. We prove the following theorem by modifying the proof for $\mathfrak{e} \leq \mathfrak{d}$, which is due to Blass [2, Theorem 13].

Theorem 2.2. $\mathfrak{e}^* = \mathsf{cov}(\mathcal{M})$.

Proof. For a function $h \in \omega^{\omega\times\omega}$, define $x_h \in \omega^\omega$ recursively so that $x_h(n) = h(n, 1 + \max\{x_h(i) : i < n\})$. Next, for a predictor $\pi \in \mathcal{P}$, define a function φ_π from $\omega \times \omega$ to $[\omega]^{<\omega}$ by letting $\varphi_\pi(n,k) = \{\pi(t) : t \in k^n\}$. By identifying $\omega \times \omega$ with ω, we can regard φ_π as a slalom in $\mathcal{S}^g$ for a suitable $g \in \omega^\omega$ which does not depend on π.

Now we prove the following: for $n < \omega$, if $h(n,k) \notin \varphi_\pi(n,k)$ for all k, then $x_h(n) \neq \pi(x_h \restriction n)$. Suppose that $h(n,k) \notin \varphi_\pi(n,k)$ for all k. Let $k = 1 + \max\{x_h(i) : i < n\}$. Then $x_h \restriction n \in k^n$ and hence $\pi(x_h \restriction n) \in \varphi_\pi(n,k)$. On the other hand, $x_h(n) = h(n,k) \notin \varphi_\pi(n,k)$. Thus, $x_h(n) \neq \pi(x_h \restriction n)$.

By Theorem 1.1, we can choose $F \subseteq \omega^{\omega\times\omega}$ of size $\mathsf{cov}(\mathcal{M})$ so that, for each predictor $\pi \in \mathcal{P}$, there is $f \in F$ with $f(n,k) \notin \varphi_\pi(n,k)$ for all but finitely many $(n,k) \in \omega \times \omega$. Then the set $\{x_f : f \in F\}$ witnesses $\mathfrak{e}^* \leq \mathsf{cov}(\mathcal{M})$, and hence $\mathfrak{e}^* = \mathsf{cov}(\mathcal{M})$. □

Corollary 2.3. $\mathfrak{e} \leq \mathsf{cov}(\mathcal{M})$. □

Let $\mathsf{non}(\mathcal{M})$ denote the smallest size of a nonmeager set of reals. We can characterize $\mathsf{non}(\mathcal{M})$ in a dual fashion, using [1, Lemma 2.4.8] instead of Theorem 1.1.

Theorem 2.4. $\mathsf{non}(\mathcal{M})$ *is the smallest size of* $\Pi \subseteq \mathcal{P}$ *satisfying the following: for every* $f \in \omega^\omega$ *there exists* $\pi \in \Pi$ *such that* $f(n) = \pi(f \restriction n)$ *for infinitely many* $n < \omega$. □

3. A REMARK ON THE EVASION IDEAL

Brendle [3, Subsection 3.5] introduced the notion of the evasion ideal, that is, the σ-ideal generated by the sets of the form

$$\{f \in \omega^\omega : f(n) = \pi(f \restriction n) \text{ for all but finitely many } n \in X\}$$

for $\pi \in \mathcal{P}$ and $X \in [\omega]^\omega$. He considered the smallest size of a subset of ω^ω which does not belong to this ideal.

Definition 3.1 ([4, Definition 3.1])**.** $\mathfrak{e}(\omega)$, the uniformity of the evasion ideal, is the smallest size of $F \subseteq \omega^\omega$ satisfying the following: for any countable family of pairs $\{\langle \pi_i, X_i\rangle : i < \omega\} \subseteq \mathcal{P} \times [\omega]^\omega$ there is $f \in F$ such that for each $i < \omega$ we have $f(n) \neq \pi_i(f \restriction n)$ for infinitely many $n \in X_i$.

Clearly $\mathfrak{e} \leq \mathfrak{e}(\omega)$ holds. Brendle and Shelah asked whether $\mathfrak{e} = \mathfrak{e}(\omega)$ can be proved in ZFC, and they presented the following partial answer.

Theorem 3.2 ([4, Theorem 3.3])**.** $\mathfrak{e} \geq \min\{\mathfrak{e}(\omega), \mathsf{cov}(\mathcal{M})\}$. *Thus either* $\mathfrak{e} < \mathsf{cov}(\mathcal{M})$ *or* $\mathfrak{e}(\omega) \leq \mathsf{cov}(\mathcal{M})$ *implies* $\mathfrak{e} = \mathfrak{e}(\omega)$. ☐

We show that the latter assumption of the above theorem holds in ZFC.

Theorem 3.3. $\mathfrak{e}(\omega) \leq \mathfrak{e}^*$.

Proof. Fix $F \subseteq \omega^\omega$ of size less than $\mathfrak{e}(\omega)$ arbitrarily. Then we can choose a countable set of pairs $\{\langle \pi_i, X_i \rangle : i < \omega\} \subseteq \mathcal{P} \times [\omega]^\omega$ so that, for every $f \in F$, there is $i < \omega$ such that $f(n) = \pi_i(f \restriction n)$ for all but finitely many $n \in X_i$. By shrinking X_i's if necessary, we can assume that X_i's are pairwise disjoint. Now define $\pi \in \mathcal{P}$ as follows: for $t \in \omega^{<\omega}$ if $|t| \in X_i$ for some $i < \omega$ then $\pi(t) = \pi_i(t)$; otherwise $\pi(t)$ is arbitrary. Then for all $f \in F$ we have $f(n) = \pi(f \restriction n)$ for infinitely many $n < \omega$. ☐

Corollary 3.4. $\mathfrak{e} = \mathfrak{e}(\omega)$.

Proof. By Theorems 2.2, 3.2 and 3.3. ☐

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