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MACKEY WEAK BARRELLEDNESS

STEPHEN A. SAXON AND L. M. SÁNCHEZ RUIZ

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Abstract. Answers to questions of Levin and Saxon (1971) and of Ferrando and López Pellicer (1991) complete a linear picture of weak barrelledness for Mackey spaces.

1. The Mackey picture

A space E (Hausdorff, locally convex with real or complex scalar field $\mathbb{K}$) is *inductive* (cf. [9]; in [3], *weakly barrelled*) if, given any seminorm p and an increasing covering sequence $\{E_n\}_n$ of subspaces such that each $p|_{E_n}$ is continuous, p itself is continuous. *Primitive* spaces (cf. [9, 12]) are defined by replacing the seminorm p by linear form f. Obviously, inductive $\Rightarrow$ primitive. But if E is primitive, its dual agrees with that of the inductive limit of any increasing covering sequence of subspaces, so that E, if Mackey, is inductive. That is, inductivity and primitivity, which comprise box 4) of [9], coincide in the class of Mackey spaces.

A space E is *dual locally complete* (dlc) if $(E', \sigma(E', E))$ is locally complete. Clearly, dlc $\Rightarrow$ primitive via Ruess' property (LC) (cf. [5, 8.1.29(i)], [8], [13]).

Precise knowledge of which conditions coincide for Mackey spaces facilitates "Reinventing weak barrelledness" [9], whose general two-dimensional picture gathers C-barrelled, c_0-barrelled, property (L) and dlc into box 2). Section 3 answers the Ferrando-López Pellicer question [2] to show that the four notions of box 2) coincide for Mackey spaces. The eleven distinct weak barrelledness conditions (cf. [4, 5, 9]) then become just seven linearly related ones:

Under the Mackey Topology

barrelled $\Rightarrow$ $\aleph_0$-barrelled $\Rightarrow$ ℓ^∞-barrelled
$\Rightarrow$ property (C) $\Rightarrow$ property (S) $\Rightarrow$ box 2) $\Rightarrow$ box 4).

The examples of [3, 10] show that, even for metrizable spaces, we cannot reverse the last arrow. Section 2 answers the Levin-Saxon question to show that the third arrow is irreversible, as are the remaining four via [4, 5, 13], so that the above scheme tells the entire Mackey space relational story.

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2. Two examples for Levin-Saxon

Recall [4, 5] that a space E is *ℓ^∞-barrelled*, respectively, has *property (C)*, if every $\sigma(E', E)$-bounded sequence $\{f_n\}_n$ is equicontinuous, respectively, has a weak adherence point $f \in E'$; i.e., each $\sigma(E', E)$-neighborhood of f contains f_n for infinitely many n. Levin and Saxon wondered [4, p.102] whether a Mackey space with property (C) must be ℓ^∞-barrelled. Ferrando and Sánchez Ruiz [3] gave a positive answer for the separable case, but we show the general answer is negative.

$\mathbb{K}^{\mathbb{K}}$ denotes the vector space of all functions from $\mathbb{K}$ into $\mathbb{K}$, and $\mathbb{K}^{(\mathbb{K})}$ those functions with finite support. Now $E = \mathbb{K}^{(\mathbb{K})}$ is $\mathfrak{c}$-dimensional and the algebraic dual E^* is identified with $\mathbb{K}^{\mathbb{K}}$ in the usual manner, so that the $\sigma(E^*, E)$ topology corresponds to the product topology on $\mathbb{K}^{\mathbb{K}}$. If $I = \{x \in \mathbb{K} : |x| \leq 1\}$, then, because separability is $\mathfrak{c}$-multiplicative [1, p.111], there is a sequence S dense in $I^{\mathbb{K}}$, and S is clearly $\sigma(E^*, E)$-bounded. The first uncountable ordinal ω_1 is the set of all smaller ordinals and has cardinality $\aleph_1$. Let $\mathcal{F}$ be the family of all $\mathfrak{c}$-dimensional subspaces of E^* containing S, and for each ordinal $\alpha \leq \omega_1$, let $\mathcal{F}_\alpha$ be the set of all maps T from α into $\mathcal{F}$ such that $\gamma < \beta < \alpha$ implies $T(\beta)$ contains a $\sigma(E^*, E)$-adherence point of each $\sigma(E^*, E)$-bounded sequence in $T(\gamma)$; necessarily, $T(\beta)$ contains $T(\gamma)$. While $\mathcal{F}_0$ is trivial, $\mathcal{F}_1$ is the set of maps from $1 = \{0\}$ into $\mathcal{F}$. We define a partial order $\preceq$ on $\bigcup_{\alpha \leq \omega_1} \mathcal{F}_\alpha$ by writing $T \preceq U$ to mean that $T \in \mathcal{F}_\alpha$ and $U \in \mathcal{F}_\beta$ for some $\alpha \leq \beta$ with $T = U|_\alpha$. Easily, each chain has an upper bound, and Zorn's Lemma provides a maximal element M contained in some $\mathcal{F}_{\mathfrak{m}}$.

Example 2.1. The subspace $E' = \bigcup_{\beta \in \mathfrak{m}} M(\beta)$ of E^* separates points of E, and the Mackey space $(E, \mu(E, E'))$ has property (C) but is not ℓ^∞-barrelled.

Proof. Obviously, $\mathfrak{m} \geq 1$ and $E' \in \mathcal{F}$. To choose the aggregate set $A \subset E^*$ we select a $\sigma(E^*, E)$-adherence point for each $\sigma(E', E)$-bounded sequence (Tychonoff's Theorem). The number of such sequences is $|E'|^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = \mathfrak{c}$, and thus $|A| = \mathfrak{c}$. Now $sp(A) \in \mathcal{F}$, and we may define $W : \mathfrak{m} + 1 \to \mathcal{F}$ by writing $W(\mathfrak{m}) = sp(A)$ and $W|_{\mathfrak{m}} = M$. It is apparent that $\mathfrak{m} = \omega_1$, for otherwise, $\mathfrak{m} + 1 \leq \omega_1$ implies $\mathcal{F}_{\mathfrak{m}+1}$ exists, and $W \in \mathcal{F}_{\mathfrak{m}+1}$ contradicts the maximality of M.

Since the set $I^{\mathbb{K}}$ separates points of E, so does the weakly dense subset S, and hence so does E'. Now suppose R is any $\sigma(E', E)$-bounded sequence. As the cofinality of $\mathfrak{m} = \omega_1$ is uncountable, there exists $\beta \in \mathfrak{m}$ such that $R \subset M(\beta)$, which implies that there is a weak adherence point of R in $M(\beta + 1) \subset E'$; i.e., the Mackey space E has property (C).

But E cannot be ℓ^∞-barrelled, for if S were equicontinuous, then its $\sigma(E^*, E)$-closure $I^{\mathbb{K}}$ would be contained in E', forcing $\dim(E') = 2^{\mathfrak{c}} \neq \mathfrak{c}$. □

Example 2.2. With E and E' as in the previous example, there is a subspace H of E^* with $E' \subset H$ such that $\dim(H/E') = \aleph_0$, and such that the Mackey space $(E, \mu(E, H))$ has property (C) but is not ℓ^∞-barrelled.

Proof. $F = (E, \sigma(E, E'))$ also has the duality invariant property (C), and has a base of $\mathfrak{c}$ neighborhoods of 0. Therefore $\operatorname{sub}(F) \leq \operatorname{dom}(F) \leq \mathfrak{c}^{\aleph_0} = \mathfrak{c} = \dim(F)$, so that by the proof of the main result in either [7] or [6], there exists $E' \subset H \subset E^*$, with $\dim(H/E') = \aleph_0$, satisfying the Tweddle-Yeomans Criterion. Hence $\sigma(E, H)$ preserves property (C) via a generalization [11] of the Tweddle-Catalán approach [14]. Duality invariance ensures property (C) for $(E, \mu(E, H))$.

Again, smallness of $\dim(H) = \mathfrak{c} + \aleph_0 = \mathfrak{c}$ denies ℓ^∞-barrelledness. □

3. A theorem for Ferrando-López Pellicer

The closed unit ball B of the Banach space ℓ^1 is closed in the product $I \times I \times \cdots$, which is compact (Tychonoff); thus we endow B with the topology of coordinatewise convergence to make it compact. We showed [8] that a space E is locally complete if and only if, given any bounded sequence $\{x_n\}_n \subset E$ and $(a_n)_n \in B$, the series $\sum_n a_n x_n$ converges in E. Since null sequences are bounded, such spaces satisfy the convergence hypothesis below.

Lemma 3.1. *Let $\{C_n\}_n$ be a sequence of nonempty compact absolutely convex subsets of the space E with the property that, given any neighborhood V of the origin, there exists some positive integer p such that $C_n \subset V$ for all $n \geq p$. If each series in the the set $C = \{\sum_n \lambda_n x_n : (\lambda_n)_n \in B,\ \text{each}\ \ x_n \in C_n\}$ converges in E, then C is compact and coincides with the closed absolutely convex hull $\overline{\Gamma}\left(\bigcup_n C_n\right)$ of the union.*

Proof. By Tychonoff's theorem, the product $P = B \times C_1 \times C_2 \times \cdots$ is compact. If the map $f : P \to E$ defined by

$$f\left((\lambda_n)_n, x_1, x_2, \ldots\right) = \sum_n \lambda_n x_n$$

is continuous, then $C = f(P)$ would be compact. A routine argument shows C is absolutely convex, and the obvious containments $\bigcup_n C_n \subset C \subset \overline{\Gamma}\left(\bigcup_n C_n\right)$ would then imply $C = \overline{\Gamma}\left(\bigcup_n C_n\right)$. Thus the continuity of f is the issue at hand.

Pick any $x = \left((\lambda_n)_n, x_1, x_2, \ldots\right) \in P$ and let V be a closed absolutely convex neighborhood of 0 in E. By hypothesis, there is a positive integer p such that $C_n \subset \frac{1}{6}V$ for $n > p$. Each C_n is bounded, so there exists $\epsilon > 0$ such that $|\alpha| \leq \epsilon$ implies $\alpha C_n \subset \frac{1}{3p}V$ for $n \leq p$. Let U be the neighborhood of x consisting of all $y = \left((a_n)_n, y_1, y_2, \ldots\right) \in P$ such that for $1 \leq n \leq p$ we have $|a_n - \lambda_n| \leq \epsilon$ and $y_n - x_n \in \frac{1}{3}V$. Thus $y \in U$ implies that

$$\begin{aligned} f(y) - f(x) &= \sum_{n \leq p} (a_n y_n - \lambda_n x_n) + \sum_{n > p} (a_n y_n - \lambda_n x_n) \\ &= \sum_{n \leq p} (a_n - \lambda_n) y_n + \sum_{n \leq p} \lambda_n (y_n - x_n) + \sum_{n > p} (a_n y_n - \lambda_n x_n) \end{aligned}$$

is a member of $p\left(\frac{1}{3p}V\right) + \frac{1}{3}V + \frac{1}{6}V - \frac{1}{6}V = V$; i.e., f is continuous. $\square$

A space E is *C-barrelled* [5] if $U = \bigcap_n U_n$ is a neighborhood of 0 in E whenever $\{U_n\}_n$ is a sequence of barrels that are neighborhoods of 0 such that each $x \in E$ is in U_n for all but finitely many n.

Theorem 3.2. *Each Mackey dual locally complete space E is C-barrelled.*

Proof. Using the definition's notation and taking polars with respect to the usual pairing $\langle E, E' \rangle$, we have each $\{x\}^\circ \supset U_n^\circ$ for almost all n, so that almost all $C_n = U_n^\circ$ are contained in any given $\sigma(E', E)$-neighborhood of 0. Dual local completeness and the lemma ensure that $\overline{\Gamma}\left(\bigcup_n C_n\right)$ is weakly compact. The Mackey topology and the bipolar theorem imply that $\left(\overline{\Gamma}\left(\bigcup_n C_n\right)\right)^\circ = \left(\bigcup_n C_n\right)^\circ = \bigcap_n C_n^\circ = \bigcap_n U_n^{\circ\circ} = U$ is a neighborhood of 0 in E. $\square$

Corollary 3.3. *If E is a Mackey space, the following assertions are equivalent:*

1. *E is dual locally complete.*
2. *E is c_0-barrelled.*
3. *E has property (L).*
4. *E is C-barrelled.*

Proof. It is well-known that, in general, C-barrelled $\Rightarrow$ [c_0-barrelled $\wedge$ property (L)] and [c_0-barrelled $\vee$ property (L)] $\Rightarrow$ dlc (cf. [5, 8.2.23(a), (b) and 8.2.7], [8]). □

As a special case we have 5.1.33 and 8.1.29(i) of [5], the equivalence of (1)-(3).

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Department of Mathematics, University of Florida, P.O. Box 118105, Gainesville, Florida 32611-8105

E-mail address: `saxon@math.ufl.edu`

EUITI-Departamento de Matemática Aplicada, Universidad Politécnica de Valencia, E-46022 Valencia, Spain

E-mail address: `lmsr@mat.upv.es`