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NEWTON'S FORMULA FOR $\mathfrak{gl}_n$

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Dedicated to Professor Reiji Takahashi on the occasion of his seventieth birthday

Abstract. This paper presents an explicit relation between the two sets which are well-known generators of the center of the universal enveloping algebra $U(\mathfrak{gl}_n)$ of the Lie algebra $\mathfrak{gl}_n$: one by Capelli (1890) and the other by Gelfand (1950). Our formula is motivated to give an exact analogy for the classical Newton's formula connecting the elementary symmetric functions and the power sum symmetric functions. The formula itself can be deduced from a more general result on Yangians obtained by Nazarov. Our proof is elementary and has an advantage in its direct accessibility.

Introduction

For the universal enveloping algebra $U(\mathfrak{gl}_n)$ of the Lie algebra $\mathfrak{gl}_n$, we have two well-known sets as generators of its center: one given by Capelli [Ca](1890) and the other by Gelfand [Ge](1950). In this paper, we present an explicit relation between them. The counterparts in symmetric functions for these are respectively the elementary symmetric functions and the power sum symmetric functions, for which we know a classical explicit relation called Newton's formula. Our formula is aimed at giving an exact analogy for this.

We start with recalling definitions. Denoting by E_{ij} the standard generators of $\mathfrak{gl}_n$, we consider a non-commutative determinant with parameter λ

$$C(\lambda) = \det(E_{ij} + (n - i - \lambda)\delta_{ij}),$$

where the determinant for a matrix $A = (A_{ij})_{i,j=1}^n$ is in general defined by an alternating sum

$$\det A = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sign}(\sigma)\, A_{\sigma(1)1} A_{\sigma(2)2} \cdots A_{\sigma(n)n}.$$

This $C(\lambda)$ is known to be in the center $ZU(\mathfrak{gl}_n)$ of $U(\mathfrak{gl}_n)$ for any λ, so that the coefficients are all in $ZU(\mathfrak{gl}_n)$ when $C(\lambda)$ is developed in some way in λ. (See, e.g., [H], [HU], [NUW].) More specifically, we define

$$C(\lambda) = \sum_{k=0}^{n} (-)^k \lambda^{(k)} C_{n-k}$$

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with $\lambda^{(k)} = \lambda(\lambda-1)\cdots(\lambda-k+1)$, and call C_k the k-th Capelli element. It is known that these $C_1, \cdots, C_n$ generate the center of $U(\mathfrak{gl}_n)$.

On the other hand, we put

$$P_k = \sum_{i_1,\cdots,i_k} E_{i_1 i_2} E_{i_2 i_3} \cdots E_{i_k i_1},$$

where the indices $i_1, \cdots, i_k$ run over all the integers in between 1 to n. Using the matrix $\boldsymbol{E} = (E_{ij})_{i,j=1}^n$, we can write this as $P_k = \mathrm{Tr}(\boldsymbol{E}^k)$. It is known that all P_k's are in the center $ZU(\mathfrak{gl}_n)$ [Ge] (see also [Z]).

The main result of this paper is the following formula, which gives the relation between these C_k's and P_k's:

Theorem (Newton's formula (1))**.** *As $ZU(\mathfrak{gl}_n)$-coefficient formal power series in λ, the following formula holds:*

$$\frac{C(\lambda) - C(\lambda-1)}{C(\lambda)} = \sum_{k=0}^{\infty} \lambda^{-1-k} P_k.$$

Corollary (Newton's formula (2))**.** *Explicit relations for C_k's and P_k's are given in the following form:*

$$\sum_{i=0}^{k} (-)^i \tilde{C}_i P_{k-i} = (-)^k \sum_{j=0}^{k} \binom{n-j}{n-k-1} \tilde{C}_j,$$

where $\tilde{C}_k$ is defined by

$$C(\lambda) = \sum_{k=0}^{n} (-)^k \lambda^k \tilde{C}_{n-k}.$$

Remark. The symbol S_m^n being the Stirling number of the first kind, the relation of C_k and $\tilde{C}_k$ is given by

$$\tilde{C}_k = \sum_{i=0}^{k} S_{n-k}^{n-k+i} C_{k-i},$$

because we have

$$\lambda^{(k)} = \sum_{i=1}^{k} (-)^{k-i} S_i^k \lambda^i.$$

The first form of Newton's formula is inspired by the latter form treated by M. Itoh [I]. His proof uses the result by [PP1] (see also Ch. XI of [Z]), which gives the calculation of the eigenvalues of P_k on the irreducible representations of $\mathfrak{gl}_n$. Since the eigenvalues of $C(\lambda)$ are easy to calculate on the irreducible representations of $\mathfrak{gl}_n$, the comparison of the eigenvalues proves the formula. The proof in the present paper is a bit more direct, without depending on the calculation in [PP1], and actually derives that calculation. As a by-product, we can also prove a Hamilton-Cayley formula for $\mathfrak{gl}_n$ (see, e.g., Example 8.18 of [GKLLRT]).

Notes. After finishing the first draft of this paper, the author has been informed of several related works. First, he came to know the result [O], which gives a computation of the generating function of P_k's. Second, the author got important comments on the preceding works by Maxim Nazarov, to whom he would like to give his sincere gratitude. In the paper [Na], a much wider generalization of our

Newton's formula has been formulated and proved under the name of the Liouville theorem in the framework of the Yangians of Lie superalgebras (see also §5 of [MNO]). The Hamilton-Cayley formula seems to have a long history. We just remark here that both the classical $\mathfrak{gl}_n$ case and its q-analogue are obtained in the paper [NaTa] (see Remarks 2.5 and 4.8). For more details, see the comments in 2.24 of [MNO] and the references therein.

The technique from Yangians is powerful and provides us with new view points in the study of the universal enveloping algebras, but it still seems a little bit hard to access. The author hopes that the present article will give an occasion to pay more attention to the Yangians along with the technique developed here.

1. The cofactor for the matrix $\boldsymbol{E}$

The key in this paper is in the fact that we can find the cofactor matrix for $\boldsymbol{E}$ in an explicit form. To carry out our computation, we utilize the exterior calculus.

Let $e_1, \cdots, e_n$ be the generator of the exterior algebra Λ_n subject to the relations

$$e_i e_j + e_j e_i = 0 \qquad (1 \le i, j \le n).$$

We will work in the algebra $\mathcal{E} = \Lambda_n \otimes U(\mathfrak{gl}_n)$, where the subalgebras Λ_n and $U(\mathfrak{gl}_n)$ commute with each other. As in the commutative case, this framework fits nicely with computing the determinant even for the non-commutative case. For example, the determinant for a matrix $A = (A_{ij})_{i,j=1}^n$ whose entries are in $U(\mathfrak{gl}_n)$ is given by the formula

$$e_1 e_2 \cdots e_n \det A = \eta_1 \eta_2 \cdots \eta_n$$

with $\eta_i = \sum_{\alpha=1}^n e_\alpha A_{\alpha i}$.

We consider more specifically the following elements in $\mathcal{E}$:

$$\omega_i = \sum_{\alpha=1}^n e_\alpha E_{\alpha i}; \quad \omega_i(u) = \omega_i + u e_i = \sum_{\alpha=1}^n e_\alpha (E_{\alpha i} + u\delta_{\alpha i}),$$

where u is a scalar parameter. We have first

Lemma 1. *The commutation relations for $\omega_i(u)$ are given by*

$$\omega_i(u+1)\omega_j(u) + \omega_j(u+1)\omega_i(u) = 0.$$

In particular, we have

$$\omega_i(u+1)\omega_i(u) = 0.$$

Proof. This can be shown by an easy calculation:

$$\begin{aligned}
\omega_i(u)\omega_j(v) + \omega_j(v)\omega_i(u) &= \sum_{\alpha,\beta=1}^n e_\alpha e_\beta [E_{\alpha i} + u\delta_{\alpha i}, E_{\beta j} + v\delta_{\beta j}] \\
&= \sum_{\alpha,\beta=1}^n e_\alpha e_\beta (E_{\alpha j}\delta_{i\beta} - E_{\beta i}\delta_{j\alpha}) \\
&= \sum_{\alpha=1}^n e_\alpha e_i E_{\alpha j} - \sum_{\beta=1}^n e_j e_\beta E_{\beta i} \\
&= -e_i \omega_j - e_j \omega_i.
\end{aligned}$$

From this, we see that $\omega_i(u+1)\omega_j(v)+\omega_j(v+1)\omega_i(u) = e_i\omega_j(v)+e_j\omega_i(u)-e_i\omega_j-e_j\omega_i = (v-1)e_ie_j+(u-1)e_je_i$. Put here $u=v$. Then the last term vanishes and we come to the conclusion. □

We proceed to define the cofactor matrix of $\boldsymbol{E}$ or more generally of $\boldsymbol{E}-\lambda$ with parameter λ. Define elements $\Delta_{ij}(\lambda) \in U(\mathfrak{gl}_n)$ by

$$\begin{aligned} &e_1e_2\cdots e_n\Delta_{ij}(\lambda) = \\ &\qquad \omega_1(n-1-\lambda)\omega_2(n-2-\lambda)\cdots\omega_{i-1}(n-i+1-\lambda)e_j\omega_{i+1}(n-i-\lambda)\cdots\omega_n(1-\lambda). \end{aligned}$$

This $\Delta_{ij}(\lambda)$ is actually expressed by a determinant

$$\det \begin{array}{c} \begin{array}{cccccc} \phantom{E_{11}+n-1-\lambda} & \phantom{E_{22}+n-2-\lambda} & & \underset{\smile}{i} & & \phantom{E_{nn}+1-\lambda} \end{array} \\ \begin{bmatrix} E_{11}+n-1-\lambda & E_{12} & \cdots & 0 & \cdots & E_{1n} \\ E_{21} & E_{22}+n-2-\lambda & \cdots & 0 & \cdots & E_{2n} \\ \vdots & \vdots & \ddots & 0 & \cdots & \vdots \\ E_{j1} & E_{j2} & \cdots & 1 & \cdots & E_{jn} \\ \vdots & \vdots & \cdots & 0 & \cdots & \vdots \\ E_{n1} & E_{n2} & \cdots & 0 & \cdots & E_{nn}+1-\lambda \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \\ \\ (j \\ \\ \\ \end{array}$$

which is clearly reduced to a determinant of $(n-1)\times(n-1)$ matrix. Here note that the shifts in the diagonals except for the ith column are: $n-k-\lambda$ for kth column with $k<i$ and $n-k+1-\lambda$ for kth column with $k>i$.

Now we come to our point.

Proposition 2. *The following formula holds:*

$$\sum_{k=1}^{n} \Delta_{ik}(\lambda)(E_{kj}-\lambda\delta_{kj}) = C(\lambda)\delta_{ij}.$$

In matrix form, this can be written as $\Delta(\lambda)(\boldsymbol{E}-\lambda) = C(\lambda)$, *with* $\Delta(\lambda) = (\Delta_{ij}(\lambda))_{i,j=1}^{n}$.

Proof. By definition and from Lemma 1, we see

$$\begin{aligned} &e_1e_2\cdots e_n \sum_{k=1}^{n} \Delta_{ik}(\lambda)(E_{kj}-\lambda\delta_{kj}) \\ &\quad = \sum_{k=1}^{n} (-)^{n-i}\omega_1(n-1-\lambda)\cdots\omega_{i-1}(n-i+1-\lambda)\omega_{i+1}(n-i-\lambda)\cdots \\ &\qquad\qquad\qquad\qquad \cdots\omega_n(1-\lambda)e_k(E_{kj}-\lambda\delta_{kj}) \\ &\quad = (-)^{n-i}\omega_1(n-1-\lambda)\cdots\omega_{i-1}(n-i+1-\lambda)\omega_{i+1}(n-i-\lambda)\cdots\omega_n(1-\lambda)\omega_j(-\lambda) \\ &\quad = \omega_1(n-1-\lambda)\cdots\omega_{i-1}(n-i+1-\lambda)\omega_j(n-i-\lambda)\omega_{i+1}(n-i-1-\lambda)\cdots\omega_n(-\lambda). \end{aligned}$$

If $i=j$ here, then the last term turns out to be $e_1e_2\cdots e_n\det(E_{ij}+(n-i-\lambda)\delta_{ij}) = e_1e_2\cdots e_nC(\lambda)$. If $i\neq j$, then again by Lemma 1, we can move the factor $\omega_j(n-i-\lambda)$ next to the original $\omega_j(n-j-\lambda)$. Since the parameter $n-i-\lambda$ in $\omega_j(n-i-\lambda)$ is shifted to $n-j-1-\lambda$ with this move, we get a factor $\omega_j(n-j-\lambda)\omega_j(n-j-1-\lambda)$, which vanishes yet again by Lemma 1. This completes the proof. □

For our Newton's formula, we calculate the trace of the cofactor matrix of $\boldsymbol{E}-\lambda$ as follows.

Lemma 3. *The trace of the cofactor matrix of $\boldsymbol{E}-\lambda$ is equal to the difference of $C(\lambda)$:*

$$C(\lambda) - C(\lambda-1) = -\sum_{i=1}^{n} \Delta_{ii}(\lambda).$$

Proof. Let us define a difference operator D by $D\varphi(\lambda) = \varphi(\lambda) - \varphi(\lambda-1)$. Then we have a formula for the difference of product of functions as

$$D(\varphi\psi)(\lambda) = \varphi(\lambda) D\psi(\lambda) + D\varphi(\lambda)\psi(\lambda-1).$$

Note that this is valid even when φ and ψ are not commutative. We apply this formula repeatedly to the right-hand side of $e_1 e_2 \cdots e_n C(\lambda) = \omega_1(n-1-\lambda)\cdots\omega_n(-\lambda)$; then we obtain the result. □

2. Newton's formula for $\mathfrak{gl}_n$

With explicit calculations of the cofactor in hand, we can now prove our main goal.

Theorem 4 (Newton's formula)**.** *As $ZU(\mathfrak{gl}_n)$-coefficient formal power series in λ, the following formula holds:*

$$\frac{C(\lambda) - C(\lambda-1)}{C(\lambda)} = \sum_{k=0}^{\infty} \lambda^{-1-k} P_k.$$

Proof. By Proposition 2, we have $\Delta(\lambda)(\boldsymbol{E}-\lambda) = C(\lambda)$, so that we see as a formal series in λ

$$\Delta(\lambda) = -C(\lambda)(\lambda - \boldsymbol{E})^{-1} = -C(\lambda)\sum_{k=0}^{\infty} \lambda^{-1-k}\boldsymbol{E}^k.$$

Taking the trace of both sides of this, we come to our formula, because $P_k = \operatorname{Tr}(\boldsymbol{E}^k)$ and $C(\lambda) - C(\lambda-1) = -\operatorname{Tr}(\Delta(\lambda))$. □

Corollary 5. *Explicit relations for C_k's and P_k's are given in the following form:*

$$\sum_{i=0}^{k} (-)^i \tilde{C}_i P_{k-i} = (-)^k \sum_{j=0}^{k} \binom{n-j}{n-k-1} \tilde{C}_j,$$

where $\tilde{C}_k$ is defined by

$$C(\lambda) = \sum_{k=0}^{n} (-)^k \lambda^k \tilde{C}_{n-k}.$$

The proof of this is just done by comparing the coefficients of the powers in λ as in the classical case.

Remark. We also have some modified formula in terms of $P^{(k)} = \operatorname{Tr}(\boldsymbol{E}^{(k)})$ instead of $P_k = \operatorname{Tr}(\boldsymbol{E}^k)$ (see [I]).

3. Hamilton-Cayley Theorem for $\mathfrak{gl}_n$

As another application of our computation of the cofactor, we show that a variant of Hamilton-Cayley Theorem holds for the matrix $\boldsymbol{E}$. The result itself has already appeared in several papers (see, e.g., [NaTa], [MNO], [GKLLRT]).

Theorem 6 (Hamilton-Cayley Theorem for $\mathfrak{gl}_n$)**.** *If we substitute the matrix $\boldsymbol{E}$ for λ in the $ZU(\mathfrak{gl}_n)$-coefficient polynomial $C(\lambda)$, then it vanishes. In other words, the following equality holds:*

$$C(\boldsymbol{E}) = \sum_{k=0}^{n} (-)^k \boldsymbol{E}^{(k)} C_{n-k} = \sum_{k=0}^{n} (-)^k \boldsymbol{E}^k \tilde{C}_{n-k} = 0.$$

Proof. The proof is parallel to the classical case. Let the expansion of the cofactor matrix $\Delta(\lambda)$ in λ be

$$\Delta(\lambda) = \sum_{k=0}^{n-1} B_{n-k-1} \lambda^k$$

with matrices B_k whose entries are in $U(\mathfrak{gl}_n)$. Then comparing the coefficients in the equality $\Delta(\lambda)(\boldsymbol{E} - \lambda) = C(\lambda)$, we see for $k = 0, \cdots, n$

$$B_{n-k-1}\boldsymbol{E} - B_{n-k} = (-)^k \tilde{C}_{n-k}$$

with the convention $B_{-1} = 0$. Multiply this with $\boldsymbol{E}^k$ and sum up over $k = 0$ to n. Then the left-hand side vanishes, and the resulting right-hand side is just $C(\boldsymbol{E})$. Hence the proof. □

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