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SPECTRAL CONDITIONS GUARANTEEING A NONTRIVIAL SOLUTION OF THE ABSTRACT CAUCHY PROBLEM

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Abstract. We characterize subsets, Ω, of the complex plane, with the following property: If A has spectrum contained in Ω, with polynomially bounded resolvent outside Ω, then the abstract Cauchy problem corresponding to A has a nontrivial solution.

I. Introduction and preliminaries

Many major physical problems, including initial-value problems, mixed initial-boundary-value problems, and other integrodifferential equations, may be modelled as an *abstract Cauchy problem*:

$$\text{(ACP)} \qquad \frac{d}{dt}u(t) = A(u(t)) \ (t \geq 0), \ u(0) = x.$$

See, for example, [5], [6] or [7]. This reduces the original problem to a problem in operator theory: what conditions on the operator A will guarantee that (ACP) has a unique solution?

One of the most natural ways to study a linear operator is by looking at its spectrum. The goal is to think of A as a complex number, or a set of complex numbers.

In this paper, we will discuss what conditions on the spectrum of A will guarantee a nontrivial solution of (ACP). We will not discuss uniqueness. Relatively weak conditions on A will guarantee uniqueness; see, for example, [6, Theorem 4.1.2].

For unbounded operators, it is not sufficient to specify the location of the spectrum; one must also control the rate of growth of the norm of the resolvent, $\|(z-A)^{-1}\|$, as $|z| \to \infty$. We will assume polynomial growth of the resolvent, outside a given set.

It is convenient to introduce terminology and hypotheses before proceeding.

Terminology and Hypotheses 1.0. All operators are linear, on a Banach space X. We will write $\mathcal{D}(A)$ for the domain of the operator A, $Im(A)$ for the image of A, $\sigma(A)$ for the spectrum, and $\rho(A)$ for the resolvent. $B(X)$ will be the space of bounded operators from X into itself.

Throughout this paper, Ω and O are open subsets of the complex plane, whose complement contains a half-line and whose boundaries, $\partial\Omega$ and ∂O, are positively

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oriented countable systems of piecewise-smooth, mutually nonintersecting (possibly unbounded) arcs, and α is a real number greater than or equal to -1.

By a *solution* of (ACP) we mean a strong solution; that is, $u \in C^1([0,\infty), X) \cap C([0,\infty), [\mathcal{D}(A)])$ and satisfies (ACP), where $\|x\|_{[\mathcal{D}(A)]} \equiv \|x\| + \|Ax\|$.

Definition 1.1 ([2, Definition 20.1])**.** The operator B is of *α-type* Ω if $\sigma(B) \subseteq \Omega$ and there exists a constant M so that

$$\|(\lambda - B)^{-1}\| \leq M(1 + |\lambda|)^{\alpha}, \ \ \forall \lambda \notin \Omega.$$

Note that an operator of α-type Ω may or may not be densely defined.

For any nonnegative integer m, let $H^{\infty}_m(\Omega)$ be the set of all functions f from Ω into the complex plane such that

$$z \mapsto (\lambda - z)^m f(z) \in H^{\infty}(\Omega),$$

for any $\lambda \notin \overline{\Omega}$.

Throughout this paper, we will use the following generalized Riesz-Dunford functional calculus.

Definition 1.2. If A is of α-type Ω, $m \equiv [\alpha] + 2$ and $h \in H^{\infty}_m(\Omega)$, define $h(A) \in B(X)$ by

$$h(A) \equiv \int_{\partial O} h(w)(w - A)^{-1} \, \frac{dw}{2\pi i},$$

where O is chosen so that $\overline{O} \subseteq \Omega$ and A is of α-type O (see [2, Lemma 22.4]).

Note that $h \in H^{\infty}_m(\Omega)$ need not be analytic at ∞.

See [4, Example 4.1] and [2, Definition 22.6] for equivalent definitions of $h(A)$ that make sense with the growth conditions on h removed.

Definition 1.3. The strongly continuous family $\{W(t)\}_{t\geq 0} \subseteq B(X)$ is a *C-regularized semigroup for A* if

(1) $W(0) = C$;
(2) $W(t)W(s) = CW(t + s)$, for all $s, t \geq 0$;
(3) $W(t)A \subseteq AW(t)$, for all $t \geq 0$; and
(4) for any $x \in X, t \geq 0$, $\int_0^t W(s)x \, ds \in \mathcal{D}(A)$, with

$$A\left(\int_0^t W(s)x \, ds\right) = W(t)x - Cx.$$

C-regularized semigroups may be used to deal with ill-posed abstract Cauchy problems in the same way that strongly continuous semigroups deal with well-posed problems. In particular, when there exists a C-regularized semigroup $\{W(t)\}_{t\geq 0}$ for A and A is closed, we are guaranteed solutions of (ACP) for all x in $C\left(\mathcal{D}(A)\right)$,

$$u(t) \equiv W(t)y \ \ (Cy = x),$$

and well-posedness on a Fréchet space (a Banach space, if the regularized semigroup is exponentially bounded) continuously embedded between $Im(C)$ and X. See [3], and, for C injective, [2].

We will characterize those subsets of the complex plane with the following property: When A is of α-type Ω, then (ACP) has a nontrivial solution. We shall see that it is sufficient that there exist nontrivial h such that

$$z \mapsto e^{tz} h(z) \in H^{\infty}(\Omega)$$

for all nonnegative t (Corollary 2.4). A slight weakening produces a necessary and sufficient condition (Theorem 2.7). It then follows automatically that there exists a nontrivial operator $g(A)$ such that

$$W(t) \equiv (z \mapsto e^{tz}g(z))(A) \ \ (t \geq 0)$$

defines a $g(A)$-regularized semigroup for A (Theorems 2.3 and 2.7). Thus, one nontrivial solution leads to the existence of solutions for all initial data in the image of $g(A)$, represented by a regularized semigroup constructed explicitly with a functional calculus

$$u(t) = W(t)y = \int_{\partial O} e^{tw}g(w)(w-A)^{-1}y\,\frac{dw}{2\pi i},$$

where $W(0)y = g(A)y = x$.

We mentioned earlier that the abstract Cauchy problem reduces a physical problem to a problem in operator theory; Theorem 2.7 further reduces it to a problem in complex analysis: On which subsets of the complex plane do there exist nontrivial holomorphic functions that decay more rapidly than $z \mapsto e^{tz}$, for all nonnegative t?

Intuitively, one thinks of a solution of (ACP) as $u(t) = e^{tA}x$. Spectral intuition suggests that we think of A as multiplication by z on $\sigma(A)$; a solution of (ACP) is then thought of, very informally, as

$$t \mapsto (z \mapsto e^{tz}g(z)) \ \ (z \in \sigma(A)),$$

for some function g. Our results are consistent with this intuition.

One way to regard these results is as indicating the omnipresence of regularized semigroups. If spectral conditions guarantee a nontrivial solution of (ACP), then the solution is accessible via a regularized semigroup for A.

II. Main results

When considering operators of type Ω, we will focus on sets Ω that have no extraneous connected components; that is, all connected components have nontrivial intersection with $\sigma(A)$ (see Example 2.9(e)).

Definition 2.1. We will say that Ω is an *α-spectral solution set* if, whenever A is of α-type Ω, and $\sigma(A) \cap \Omega_j$ is nontrivial when Ω_j is a connected component of Ω, then (ACP) has a nontrivial solution.

Definition 2.2 ([2, Definition 23.2])**.** We define E_Ω to be the set of all functions h from Ω into the complex plane such that

$$z \mapsto e^{tz}h(z) \in H^\infty(\Omega)$$

for all $t \geq 0$.

Theorem 2.3. *Suppose*

1. *E_Ω is nontrivial;*
2. *A is of α-type Ω; and*
3. *$\sigma(A) \cap \Omega_j$ is nontrivial whenever Ω_j is a connected component of Ω.*

Then there exists $g \in E_\Omega \cap H^\infty_m(\Omega)$ $(m \equiv [\alpha] + 2)$ such that $g(A)$ is nontrivial and

$$W(t) \equiv \left[(z \mapsto e^{tz}g(z))(A)\right] \ \ (t \geq 0)$$

defines a $g(A)$-regularized semigroup for A.

Corollary 2.4. *If E_Ω is nontrivial, then Ω is an α-spectral solution set.*

Remark 2.5. With some additional conditions on $g \in E_\Omega$, $g(A) = W(0)$, in Theorem 2.3, will be injective, so that the solutions of (ACP) guaranteed by Corollary 2.4 will also be unique; see [4, Theorem 5.2].

To characterize α-spectral solution sets, we need one more definition.

Definition 2.6 ([2, Definition 23.4])**.** If $\overline{O} \subseteq \Omega$, then O is an *α-interior subset of* Ω if there exists a constant $c > 0$ such that

$$d(O, \{z \notin \Omega \mid |z| \le R\}) \ge c(1+R)^{-\alpha},$$

for all $R > 0$.

Theorem 2.7. *The following are equivalent.*

(a) Ω *is an α-spectral solution set.*
(b) *Whenever A is of α-type Ω, and $\sigma(A) \cap \Omega_j$ is nontrivial when Ω_j is a connected component of Ω, then there exists a nontrivial C such that A has a C-regularized semigroup.*
(c) *For all α-interior subsets O, of Ω, E_O is nontrivial.*

We then have

$$u(t) = W(t)y = \left[(z \mapsto e^{tz} g(z))(A)\right] y,$$

where u is a nontrivial solution of (ACP), $\{W(t)\}_{t \ge 0}$ is a C-regularized semigroup for A, $Cy = g(A)y \neq 0$, and g is an appropriate member of E_O.

Remarks 2.8. In [2, Definition 23.1], Ω is defined to be an α-spectral dense solution set if, whenever A is of α-type Ω and densely defined, then (ACP) has a solution for all initial data in a dense set. [2, Theorem 23.6] is an analogue of Theorem 2.7(a) $\Longleftrightarrow$ (c): Ω is an α-spectral dense solution set if and only if, for all α-interior subsets O, of Ω, E_O is uniformly dense in $H^\infty(O)$.

In particular, an α-spectral dense solution set is automatically an α-spectral solution set.

We reiterate that there are no assumptions about density of domain in this paper. Our conditions on Ω are weaker than in [2, Theorem 23.6], because we are characterizing a weaker condition; all we are asking for in this paper is one nontrivial solution of (ACP). The results in [2, Chapter XXIII] contain no equivalences between α-spectral dense solution sets and regularized semigroups.

Examples 2.9. (a). The sets in [2, Examples 23.13, 23.15, 23.16, 23.17, 23.18] are α-spectral dense solution sets, hence α-spectral solution sets, for any $\alpha \ge -1$.

(b). For some negative results, consider, as in [2, Example 2.7], $A \equiv 1 + \frac{d}{dx}$, with maximal domain, on $X \equiv \{f \in C[0,1] \mid f(0) = 0\}$. Since $(1 - A)$ generates a bounded strongly continuous semigroup, A is of 0-type $S_{\frac{\pi}{2}}$, and is of (-1)-type S_θ, for $\pi > \theta > \frac{\pi}{2}$, where $S_\theta \equiv \{re^{i\phi} \mid |\phi| < \theta, r > 0\}$. However it is not hard to show that (ACP) has no nontrivial solutions.

Thus $S_{\frac{\pi}{2}}$ is not a 0-spectral solution set, and for $\pi > \theta > \frac{\pi}{2}$, S_θ is not an α-spectral solution set, for any $\alpha \ge -1$. It is shown in [2, Example 2.7] that $S_{\frac{\pi}{2}}$ is a (-1)-spectral dense solution set. Thus $S_{\frac{\pi}{2}}$ is a (-1)-spectral solution set.

(c). Example 2.9(b) provides a novel way of showing the complex analysis fact that there exists no function g such that

$$z \mapsto e^{tz} g(z) \in H^\infty(S_{\frac{\pi}{2}})$$

for all $t \geq 0$; that is, $E_{S_{\frac{\pi}{2}}}$ is trivial. Note that, by [2, Example 23.15], E_{S_θ} is dense in $H^\infty(S_\theta)$, for any positive $\theta < \frac{\pi}{2}$.

(d). An example of a set Ω that, for any $\alpha \geq -1$, is an α-spectral solution set but not an α-spectral dense solution set, is

$$\Omega \equiv \Omega_1 \cup \Omega_2,$$

where $\Omega_1 \equiv S_{\frac{2\pi}{3}}, \Omega_2 \equiv -S_{\frac{\pi}{6}}$. Since $1_{\Omega_2} \in E_\Omega$, Ω is an α-spectral solution set, for any $\alpha \geq -1$. Since Ω_1 is not an α-spectral dense solution set, Ω is not an α-spectral dense solution set, for any $\alpha \geq -1$.

(e). Note that, if Ω has a bounded connected component, then E_Ω is nontrivial. Thus an immediate corollary of Corollary 2.4 is that, if A has polynomially bounded resolvent outside a set Ω that has a bounded connected component, then (ACP) has a nontrivial solution. We believe this is well-known, since, if Ω_1 is a bounded connected component, then there exists a bounded spectral projection corresponding to Ω_1 (see [1, Chapter 2.2]).

This observation also explains why, for meaningful results, the condition in Definition 2.1 on the connected components of Ω is necessary. Let Ω_1 be any set, let Ω_2 be a bounded set whose closure is disjoint from Ω_1, and let

$$\Omega \equiv \Omega_1 \cup \Omega_2.$$

Then if, for some $\alpha \geq -1$, A is of α-type Ω_1, A is also of α-type Ω. But Ω_1, hence A, could be as bad as possible, e.g., $\Omega_1 \equiv S_\theta$, for $\pi > \theta > \frac{\pi}{2}$ (see (b)), while E_Ω is nontrivial.

III. Proofs of the main results

The following may be of some independent interest.

Proposition 3.1. *If $\rho(A)$ is nonempty, and $\mathcal{A}$ is the Banach algebra generated by all resolvents of A and the identity operator, then the maximal ideal space of $\mathcal{A}$ equals*

$$\{m_\mu \,|\, \mu \in \sigma(A) \cup \{\infty\}\},$$

where, for $\mu \in \sigma(A)$,

$$(*) \qquad m_\mu((\omega - A)^{-1}) \equiv (\omega - \mu)^{-1} \;\; \forall \omega \in \rho(A),$$

and m_∞ is the complex homomorphism that annihilates all resolvents of A.

Proof. For $\omega \in \rho(A)$, let $\hat{R_\omega}$ be the Gelfand transform of the resolvent $R_\omega \equiv (\omega - A)^{-1}$. For $\omega, \nu \in \rho(A)$, the resolvent formula implies that

$$(**) \qquad \hat{R_\omega} - \hat{R_\nu} = (\nu - \omega)\hat{R_\omega}\hat{R_\nu}.$$

For $m \neq m_\infty$ in the maximal ideal space of $\mathcal{A}$, there exists ω such that $\hat{R_\omega}(m) \neq 0$; thus we may define

$$\mu \equiv \omega - \left[\hat{R_\omega}(m)\right]^{-1}.$$

By (**), μ is independent of the choice of $\omega \in \rho(A)$. Since

$$(***) \qquad \sigma(R_\omega) = \{(\omega - \mu)^{-1} \,|\, \mu \in \sigma(A)\} \cup \{0\}$$

([1, Lemma 2.11]), $\mu \in \sigma(A)$, so that $m = m_\mu$, as defined by (*).

Conversely, suppose $\mu \in \sigma(A)$. Note first that, for a fixed $\omega_0 \in \rho(A)$, if $B \equiv R_{\omega_0}$, then $\mathcal{A}$ is the Banach algebra generated by B and the identity operator. This may be seen by noting that, if

$$f_\omega(z) \equiv \frac{z}{z(\omega - \omega_0) + 1},$$

then, for any $\omega \in \rho(A)$, $f_\omega(B) = R_\omega$, and by (***), f_ω is analytic in a neighborhood of $\sigma(B)$.

Note also that (*) may be equivalently stated as

$$m_\mu(f_\omega(B)) \equiv f_\omega\left(\frac{1}{\omega_0 - \mu}\right) \ \forall \omega \in \rho(A),$$

which, by (***), extends to a complex homomorphism on $\mathcal{A}$. □

Now we may characterize those h for which the Gelfand transform of $h(A)$ (Definition 1.2) is trivial.

Proposition 3.2. *Suppose A is of α-type Ω, $m \equiv [\alpha] + 2$ and $h \in H^\infty_m(\Omega)$. Then the following are equivalent.*

(a) $\widehat{[h(A)]} \equiv 0$.
(b) $h \equiv 0$ *on* $\sigma(A)$.

In particular, if $h(A) \equiv 0$, then (b) follows.

Proof. For any $\mu \in \sigma(A)$,

$$\begin{aligned} m_\mu(h(A)) &= \int_{\partial O} h(w) m_\mu((w - A)^{-1}) \frac{dw}{2\pi i} \\ &= \int_{\partial O} h(w)(w - \mu)^{-1} \frac{dw}{2\pi i} = h(\mu). \end{aligned}$$

The same calculation shows that $m_\infty(h(A)) = 0$.

By Proposition 3.1, the equivalence of (a) and (b) follows. □

Proof of Theorem 2.3. Fix $\lambda \notin \Omega$.

First we will show that

$$W(t) \equiv \left[(z \mapsto e^{tz} g(z))(A)\right] \quad (t \geq 0)$$

defines a $g(A)$-regularized semigroup for A, whenever

$$(*) \qquad k(z) \equiv (\lambda - z)^m g(z) \in E_\Omega.$$

By considering separately $Re(z) \leq 0$ and $Re(z) > 0$, it is not hard to see that

$$t \mapsto (z \mapsto e^{tz} k(z))$$

is a locally bounded map from $[0, \infty)$ into $H^\infty(\Omega)$. Thus dominated convergence implies that $t \mapsto W(t)$ is a continuous map from $[0, \infty)$ into $B(X)$.

The fact that $h \mapsto h(A)$ is an algebra homomorphism ([2, Lemma 22.35(a)]) implies that $\{W(t)\}_{t \geq 0}$ satisfies (2) of Definition 1.3.

Since $\lambda \in \rho(A)$, (3) of Definition 1.3 is equivalent to

$$W(t)(\lambda - A)^{-1} = (\lambda - A)^{-1} W(t)$$

for all $t \geq 0$; this is clear from the construction of $W(t)$.

To verify (4) of Definition 1.3, note that

$$\int_0^t W(s)e^{-\lambda s}\,ds = \int_{\partial O}\left[\int_0^t e^{(w-\lambda)s}\,ds\right] g(w)(w-A)^{-1}\,\frac{dw}{2\pi i}$$
$$= \int_{\partial O} (e^{(w-\lambda)t}-1)\,g(w)(w-A)^{-1}\,\frac{dw}{2\pi i(w-\lambda)}.$$

By [2, Lemma 22.35(b)], $Im\left(\int_0^t W(s)e^{-\lambda s}\,ds\right) \subseteq \mathcal{D}(A)$, with

$$(**)\qquad (A-\lambda)\int_0^t W(s)e^{-\lambda s}\,ds = \int_{\partial O}(e^{(w-\lambda)t}-1)\,g(w)(w-A)^{-1}\,\frac{dw}{2\pi i}$$
$$= e^{-\lambda t}W(t) - W(0),\ \ \forall t \geq 0.$$

We translate from $(A-\lambda)$ to A as follows. Applying $(A-\lambda)^{-1}$ to both sides of (**) and differentiating gives us

$$W(t)e^{-\lambda t} = e^{-\lambda t}\left[\frac{d}{dt}(W(t)(A-\lambda)^{-1}) - \lambda W(t)(A-\lambda)^{-1}\right],$$

so that

$$W(t) = \frac{d}{dt}(W(t)(A-\lambda)^{-1}) + (A-\lambda-A)(A-\lambda)^{-1}W(t)$$
$$= \frac{d}{dt}(W(t)(A-\lambda)^{-1}) + W(t) - A(A-\lambda)^{-1}W(t).$$

Cancelling out $W(t)$ on both sides and integrating gives us

$$(W(t)-W(0))(A-\lambda)^{-1} = A(A-\lambda)^{-1}\int_0^t W(s)\,ds$$
$$= \int_0^t W(s)\,ds + \lambda(A-\lambda)^{-1}\int_0^t W(s)\,ds,$$

so that $Im\left(\int_0^t W(s)\,ds\right) \subseteq \mathcal{D}(A)$ and, applying $(A-\lambda)$ to both sides,

$$(A-\lambda)\int_0^t W(s)\,ds = W(t) - W(0) - \lambda\int_0^t W(s)\,ds,$$

giving us (4) of Definition 1.3.

It remains to show that there exists g, satisfying (*), such that $g(A)$ is nontrivial.

By hypothesis, there exists a nontrivial $k \in E_\Omega$. Define

$$h(z) \equiv \frac{k(z)}{(\lambda - z)^m}.$$

If $h(A)$ is nontrivial, let $g \equiv h$. If $h(A)$ is trivial, then note that, since h is nontrivial, there exists a connected component of Ω, call it Ω_0, such that $h(z) \neq 0$ for some $z \in \Omega_0$. By Proposition 3.2, $h \equiv 0$ on $\sigma(A)$. Thus $\sigma(A) \cap \Omega_0$ cannot have a cluster point. This implies that there exists an isolated point λ_0 in $\sigma(A)\cap\Omega_0$. Since h is nontrivial on Ω_0, the order of λ_0 as a zero of h is finite; that is, there exists a natural number n such that

$$g(z) \equiv \frac{h(z)}{(\lambda_0 - z)^n}\ \ (z \in \Omega, z \neq \lambda_0),\quad g(\lambda_0) \equiv \frac{1}{n!}h^{(n)}(\lambda_0)$$

is analytic in Ω and $g(\lambda_0) \neq 0$. Clearly g satisfies (*). Again by Proposition 3.2, $g(A)$ cannot be trivial. Thus g is the desired function. □

Proof of Theorem 2.7. (a) → (c) is the same as the proof of [2, Theorem 23.6 (a) → (c)]. If O is an α-interior subset of Ω, then $(Af)(z) \equiv zf(z)$, with maximal domain, on $X \equiv H^\infty(O)$, is an operator of α-type Ω. If u is a guaranteed non-trivial solution of (ACP), then

$$[u(t)]\,(z) = e^{tz}\,[u(0)]\,(z);$$

thus $u(0)$ is a nontrivial member of E_O.

(b) → (a) is clear, since $u(t) \equiv W(t)y$ is a solution of (ACP), with initial data $x = Cy$, for any $y \in \mathcal{D}(A)$.

For (c) → (b), note first that by [2, Lemma 23.12], there exists an α-interior subset O of Ω such that A is of α-type O. (b) now follows from Theorem 2.3. ☐

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