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ON THE PROJECTIVITY OF MODULE COALGEBRAS

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ABSTRACT. In this paper, we derive some criteria for the projectivity of a module coalgebra over a finite dimensional Hopf algebra. In particular, we show that any Hopf algebra over a field of characteristic zero is faithfully flat over its group-like subHopf algebra. Finally, we prove that if B is a finite dimensional subHopf algebra of a Hopf algebra A, then B is normal in A if and only if $AB^+ = B^+A$. This improves a result by S. Montgomery (1993).

1. PRELIMINARY

Let B be a Hopf algebra over the field k. For any left B-module M, let M^B denote the space of invariants of M, that is

$$M^B = \{x \in M \mid bx = \varepsilon(b)x \text{ for } b \in B\},$$

where ε is the counit of B. If B is finite dimensional, B^B is just the space of left integrals $\int_B^l$. Similarly, we write $\int_B^r$ for the space of right integrals of B. We let ${}_MI$ denote the left B-submodule which consists of the elements $x \in M$ such that $\int_B^r x = 0$. Similarly, if M is a right B-module, I_M denotes the B-submodule which annihilates the left integrals of B.

For any left B-modules M and N, the space $\operatorname{Hom}(M, N)$ of k-linear homomorphisms admits a natural left B-module structure given by

$$(h \cdot f)(x) = \sum_{(h)} h_1 f(S(h_2)x)$$

for $f \in \operatorname{Hom}(M, N)$, $x \in M$, $h \in B$, where S is the antipode of B. In particular, if N is the trivial B-module k, the B-module action on $M^* = \operatorname{Hom}(M, k)$ is given by

$$(b \rightharpoonup f)(x) = f(S(b)x)$$

for $f \in M^*$, $x \in M$ and $b \in B$.

A left (right) B-module C is called a left (right) B-module coalgebra if C is a coalgebra such that the diagonal map $\Delta_C : C \longrightarrow C \otimes C$ and the counit $\varepsilon_C : C \longrightarrow k$ are left (right) B-module maps, where k is considered as a trivial B-module. We write C^+ for the coideal $C \cap \ker \varepsilon_C$. We will call $D \subseteq C$ a B-submodule coalgebra if D is a subcoalgebra of C and is invariant under the B-action. If X, $Y \subseteq C$, recall [8] that the "wedge" $X \wedge Y$ is defined as

$$X \wedge Y = \Delta_C^{-1}(X \otimes C + C \otimes Y).$$

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We define $\bigwedge^1 X = X$ and $\bigwedge^{n+1} X = X \wedge (\bigwedge^n X)$. In particular, if $C_0 \subseteq X$,

$$\sum_{n\geq 1} \bigwedge^n X = C\,,$$

where C_0 is the coradical of C (cf. [4],[8]).

Let C be a B-module coalgebra. M is a called a left (C,B)-Hopf module if M is a left C-comodule and a left B-module such that the comodule structure map $\rho_M : M \longrightarrow C \otimes M$ is a left B-module map. If C is also projective as a B-module, we simply call C a projective B-module coalgebra. The category ${}_B^C\mathcal{M}$ of all left (C,B)-Hopf modules is an abelian category.

2. Projectivity of module coalgebra

Proposition 1. *Let B be a finite dimensional Hopf algebra.*

(i) *If C is a left B-module coalgebra, then C is a projective B-module if and only if ${}_C I \subseteq \ker \varepsilon_C$.*
(ii) *If C is a right B-module coalgebra, then C is a projective B-module if and only if $I_C \subseteq \ker \varepsilon_C$.*

Proof. (i) Let us consider the B-module $\operatorname{Hom}(C,B)$ of linear homomorphisms from C to B. As a B-module, $\operatorname{Hom}(C,B)$ is isomorphic to $B\otimes C^*$ under the identification

$$(b \otimes f)(x) = f(x)b$$

for $f \in C^*$, $x \in C$ and $b \in B$. Let C_0^* be the trivial B-module with the same underlying space as C^*. It is easy to see that $B \otimes C_0^*$ is isomorphic to $B \otimes C^*$ under the B-module isomorphism $\phi : B \otimes C_0^* \longrightarrow B \otimes C^*$ given by

$$\phi(b \otimes f) = \sum_{(b)} b_1 \otimes (b_2 \rightharpoonup f)$$

for $b \in B$ and $f \in C^*$. Therefore, $\phi((B \otimes C_0^*)^B) = (B \otimes C^*)^B = \operatorname{Hom}_B(C,B)$. Since $(B \otimes C_0^*)^B = \int_B^l \otimes C^*$,

$$\operatorname{Hom}_B(C,B) = \sum_{(\Lambda)} \Lambda_1 \otimes \Lambda_2 \rightharpoonup C^*\,.$$

where Λ is a non-zero left integral of B. Let $f \in C^*$ and $x \in C$.

$$\begin{aligned} \varepsilon_B \circ \sum_{(\Lambda)} (\Lambda_1 \otimes \Lambda_2 \rightharpoonup f)(x) &= \varepsilon_B(\Lambda_1) f(S(\Lambda_2)x) \\ &= f(S(\Lambda)x)\,. \end{aligned}$$

Note that $S(\Lambda)$ is a right non-zero integral of B. Therefore, ${}_C I \subseteq \ker \varepsilon_C$ iff there exists $f \in C^*$ such that $f(S(\Lambda)x) = \varepsilon_C(x)$ for $x \in C$ which is equivalent to $\varepsilon_B \circ \sum_{(\Lambda)} (\Lambda_1 \otimes \Lambda_2 \rightharpoonup f) = \varepsilon_C$. By virtue of Doi's Theorem ([2], Corollary 1), the proof is completed. (ii) can be proved similarly. $\square$

A particular case of a result of Takeuchi ([9], Corollary 3.5) is then an immediate consequence of the above proposition.

Corollary 2. *Let $B \subseteq A$ be a Hopf algebras, with B finite dimensional. Then A is a left projective B-module iff A is a right projective B-module.*

Proof. If A is not projective as a left B-module, then by Proposition 1 there exists $a \in A$ such that $\int_B^r a = 0$ and $\varepsilon_A(a) \neq 0$. Let S be the antipode of A. Then $S(a)S(\int_B^r) = 0$ and $\varepsilon_A(S(a)) = \varepsilon_A(a) \neq 0$. Since $S(\int_B^r) = \int_B^l$, A is not a right projective B-module by Proposition 1. □

3. Projectivity for Hopf algebras over group-like subalgebras

Let A be a Hopf algebra and B a subHopf algebra of A. Following [9], we use "B-projective" to mean "projective B-module".

Lemma 3. *If the antipode of B is bijective, then the following statements are equivalent :*

(a) *A is left B faithfully flat;*
(b) *A is right B faithfully flat;*
(c) *A is left B-projective;*
(d) *A is right B-projective;*
(e) *A is a left B-projective generator;*
(f) *A is a right B-projective generator;*
(g) *for any simple subcoalgebra C of A, BC is a projective B-module;*
(h) *if $M \in_B^A \mathcal{M}$ and $M = BV$ for some simple left A-comodule V, then M is a projective B-module;*
(i) *for $M \in_B^A \mathcal{M}$, M is left B-flat.*

Proof. By [9], Corollary 3.5, (a) to (f) are all equivalent statements. (e) $\Rightarrow$ (g) and (g) $\Rightarrow$ (h) are consequences of [2], Theorem 4.

(h) $\Rightarrow$ (i) Let $M \in_B^A \mathcal{M}$ and $\mathcal{S}$ be the set of all left (A, B)-subcomodules J of M such that J is a flat left B-module. The assumption (h) assures that $\mathcal{S} \neq \emptyset$. Since flatness is preserved under direct limit, by Zorn's Lemma there is a maximal element $J_0 \in \mathcal{S}$. We claim that $J_0 = M$. If not, there exists a simple left A-subcomodule $\overline{V}$ of M/J_0. Then $B\overline{V}$ is a flat B-module. Let $V \supset J_0$ be the left (A, B)-submodule of M such that $V/J_0 = B\overline{V}$. Then $V/J_0 = B\overline{V}$ and we have the exact sequence in ${}_B^A\mathcal{M}$

$$0 \longrightarrow J_0 \longrightarrow V \longrightarrow B\overline{V} \longrightarrow 0\,.$$

As flatness is preserved under extension, V is flat and hence $V \in \mathcal{S}$. This contradicts the maximality of J_0. Therefore $J_0 = M$, and hence M is left B flat.

(i) $\Rightarrow$ (a) Let N be a non-zero right B-module. By applying the functor $N \otimes_B ?$ to the exact sequence of left B-modules :

$$0 \longrightarrow B \longrightarrow A \longrightarrow A/B \longrightarrow 0$$

we have the long exact sequence

$$\cdots \longrightarrow \operatorname{Tor}_1(N, A/B) \longrightarrow N \otimes_B B \longrightarrow N \otimes_B A \longrightarrow N \otimes_B (A/B) \longrightarrow 0\,.$$

By assumption (i), A and A/B are left B-flat since they are left (A, B)-modules. Therefore $\operatorname{Tor}_1(N, A/B) = 0$, and so $N \otimes_B A \neq 0$. Hence A is left B faithfully flat. □

Remark. If B is a Hopf subalgebra of A with bijective antipode, then by virtue of the above lemma, the adjectives "left" and "right" can be dropped. For example, we will simply say A is faithfully flat over B instead of A is left (or right) B faithfully flat.

Let $G(A)$ denote the set of all group-like elements of A. Let $G \subseteq G(A)$ be a subgroup of $G(A)$ and $B = k[G]$.

Lemma 4 ([6], Proposition 2)**.** *If C is a simple subcoalgebra of A, then*

(i) *$G_C = \{g \in G \mid gC = C\}$ is a finite subgroup of G, and*
(ii) *$BC = \bigoplus_{g\in S} gC$, where S is a set of left coset representatives of G_C in G.*

Proposition 5. *The following statements are equivalent :*

(i) *A is $k[G]$-projective;*
(ii) *for any subgroup H of G, A is projective over $k[H]$;*
(iii) *for any finite subgroup H of G, A is projective over $k[H]$.*

Proof. (i) $\Rightarrow$ (ii) Suppose A is left $k[G]$-projective. Then A is a direct summand of a free $k[G]$-module F. Since $k[G]$ is a free left $k[H]$-module, F is a free left $k[H]$-module. Hence A is left $k[H]$-projective.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i) Assume that A is left $k[H]$-projective for any finite subgroup H of G. Let C be a simple subcoalgebra of A. By Lemma 4, $H = G_C$ is a finite subgroup of G. Consider the map $\mu : B \otimes_{k[H]} C \longrightarrow BC$, $\mu : b \otimes c \mapsto bc$. Clearly, μ is left B-linear and surjective. By Lemma 4 (ii), μ is also injective and hence

$$B \otimes_{k[H]} C \cong BC$$

as B-modules. By Lemma 3, C is a projective $k[H]$-module. Since the functor $B \otimes_{k[H]} ?$ preserves projective objects, BC is B-projective. It follows from Lemma 3 that A is B-projective. $\square$

Theorem 6. *Let A be a Hopf algebra over a field k and G be a subgroup of $G(A)$.*

(i) *If $\operatorname{char} k = 0$, A is faithfully flat over $k[G]$.*
(ii) *If $\operatorname{char} k = p > 0$, A is faithfully flat over $k[G]$ if and only if A is projective over $k[H]$ for any finite p-subgroup of G.*

Proof. (i) For any finite subgroup H of G, $k[H]$ is semisimple by Maschke's Theorem. Hence, every left $k[H]$-module is projective and, in particular, A is projective over $k[H]$. By Proposition 5, A is faithfully flat over $k[G]$.

(ii) Let H be a finite subgroup of G and N a p-Sylow subgroup of H. Notice that every $k[H]$-module M can be embedded into a free $k[H]$-module F. If M is $k[N]$-projective, then M is also $k[N]$-injective and so M is a summand of F as $k[N]$-module. Therefore, M is a summand of F as $k[H]$-module (cf. [1], 63.7). Hence, the result follows from Proposition 5. $\square$

4. The existence of a unique maximal projective module subcoalgebra

Proposition 7. *Let B be a finite dimensional Hopf algebra and C a left B-module coalgebra.*

(i) *If C_1 and C_2 are projective B-submodule coalgebras, then $C_1 \wedge C_2$ and $C_1 + C_2$ are B-projective.*
(ii) *Let $\{C_i\}_{i\in J}$ be a family of projective B-submodule coalgebras of C. Then $\sum_{i\in J} C_i$ is also a projective B-module.*

Proof. Since B is a Frobenius algebra, C_1 is injective as a B-module. Let C_1' be a B-submodule of C such that $C = C_1 \oplus C_1'$. Then $C_1 \otimes C + C \otimes C_2 = C_1 \otimes C \oplus C_1' \otimes C_2$ as a B-module. Let $x \in C_1 \wedge C_2$ such that $\int_B^r x = 0$. Then $\Delta(x) = \sum_i a_i \otimes b_i + \sum_i c_i \otimes d_i$ where $\sum_i a_i \otimes b_i \in C_1 \otimes C$ and $\sum_i c_i \otimes d_i \in C_1' \otimes C_2$. For $\Lambda \in \int_B^r$, $\Delta(\Lambda x) = 0$. Therefore,

$$\sum_{i,j} \Lambda_j a_i \otimes \Lambda_j' b_i + \sum_{i,j} \Lambda_j c_i \otimes \Lambda_j' d_i = 0\,,$$

where $\Delta(\Lambda) = \sum_j \Lambda_i \otimes \Lambda_i'$. Thus,

$$\sum_{i,j} \Lambda_j a_i \otimes \Lambda_j' b_i \quad = \quad 0, \tag{1}$$

$$\sum_{i,j} \Lambda_j c_i \otimes \Lambda_j' d_i \quad = \quad 0\,. \tag{2}$$

Applying $id \otimes \varepsilon$ to equation (1) and $\varepsilon \otimes id$ to equation (2), we have

$$\begin{aligned} \Lambda(\sum_i \varepsilon(b_i) a_i) &= 0, \\ \Lambda(\sum_i \varepsilon(c_i) d_i) &= 0\,. \end{aligned}$$

By Proposition 1, $\varepsilon(\sum_i \varepsilon(b_i) a_i) = 0$ and $\varepsilon(\sum_i \varepsilon(c_i) d_i) = 0$. Thus, $\varepsilon(x) = 0$. Since $C_1 \wedge C_2$ is obviously a B-submodule coalgebra, $C_1 \wedge C_2$ is projective by Proposition 1. Obviously, C_1, C_2 are left $(C_1 \wedge C_2, B)$-Hopf modules and so is $C_1 + C_2$. By [2](Theorem 4), $C_1 + C_2$ is a projective B-module.

(ii) Let $x \in \sum_{i \in I} C_i$ such that $\Lambda x = 0$. Then there exist $C_{i_1}, \cdots, C_{i_n}$ such that $x \in \sum_{k=1}^n C_{i_k}$. By (i), $\sum_{k=1}^n C_{i_k}$ is projective and so $\varepsilon(x) = 0$ by Proposition 1. Hence $\sum_{i \in I} C_i$ is B-projective by Proposition 1. □

Corollary 8. *Let B be a finite dimensional Hopf algebra* [1]. *For any left B-module coalgebra C, there exists a unique maximal projective B-submodule coalgebra $P(C)$. Moreover, $P(C)$ is co-idempotent, i.e. $P(C) \wedge P(C) = P(C)$.*

Proof. Let $\mathcal{S}$ be the set of all projective B-submodule coalgebras of C. By Proposition 7, $P(C) = \sum_{D \in \mathcal{S}} D$ is then the largest projective B-submodule coalgebra of C. $P(C) \wedge P(C) = P(C)$ is a direct consequence of Proposition 7 (i) and the maximality of $P(C)$. □

Corollary 9. *Let B be a finite dimensional Hopf algebra and C a B-module coalgebra.* (i) *If D is a B-submodule coalgebra of C, $P(D) = P(C) \cap D$.* (ii) *If C is a direct sum of B-submodule coalgebras $\{C_i\}$, then $P(C) = \bigoplus_i P(C_i)$.*

Proof. (i) By Corollary 8, $P(D) \subseteq P(C)$ and hence $P(D) \subseteq P(C) \cap D$. Conversely, by Theorem 4 of [2], $P(C) \cap D$ is a projective B-module. Then, we have $P(C) \cap D \subseteq P(D)$.

(ii) If $C = \bigoplus_i C_i$ as B-module coalgebra, it follows by Theorem 3 of [3] that $P(C) = \bigoplus_i (P(C) \cap C_i)$. Hence, by (i), $P(C) = \bigoplus_i P(C_i)$. □

Corollary 10. *Let B be a finite dimensional Hopf algebra and C a B-module coalgebra. The following statements are equivalent :*

[1] After this paper was written, the author was able to eliminate the finite dimension hypothesis on B using different techniques.

(i) *C is a projective B-module,*
(ii) *BC_0 is B-projective, where C_0 is the coradical of C,*
(iii) *BD is B-projective for any simple subcoalgebra D of C.*

Proof. (i) $\Rightarrow$ (iii) is a direct consequence of Theorem 4 in [2].
(iii) $\Rightarrow$ (ii) follows from Proposition 7 (ii).
(ii) $\Rightarrow$ (i) Since BC_0 is B-projective, by Proposition 7 (i), $\bigwedge^n BC_0$ is B-projective for any $n \geq 1$. Hence, by Proposition 7 (ii), $\sum_{n\geq 1} \bigwedge^n BC_0$ is B-projective. The result follows from the fact that $C = \sum_{n\geq 1} \bigwedge^n BC_0$. $\square$

5. Normal subHopf algebras

Definition 11. Let A be any Hopf algebra, B a subHopf algebra of A and S the antipode of A.

(1) B is *left normal* if
$$a \triangleright b = \sum a_1 b S(a_2) \in B$$
for $a \in A$ and $b \in B$.

(2) B is *right normal* if
$$b \triangleleft a = \sum S(a_1) b a_2 \in B$$
for $a \in A$ and $b \in B$.

(3) B is *normal* if B is left normal and right normal.

It is well known that if B is a normal subHopf algebra of A, then $AB^+ = B^+A$ (cf. [4], 3.4.2). However, the converse is open. If A is left or right faithfully flat over B, the converse is known to be true (cf. [4], 3.4.3 and [9], 4.4). In this section, we will show that the converse statement holds if B is finite dimensional which enhances the result in [4], 3.4.4.

Lemma 12. *Let B be a finite dimensional Hopf algebra and C a left B-module coalgebra. Let $\eta_C : C \longrightarrow C/(B^+C)$ be the canonical B-module coalgebra homomorphism. Then, $\eta_C({}_CI)$ is a subcoalgebra of $C/(B^+C)$. In particular, if B is a subHopf algebra of a Hopf algebra A, then $\eta_A({}_AI)$ is a right A-submodule coalgebra of $\eta_A(A)$.*

Proof. To simplify, we write $\overline{C}$ for $C/(B^+C)$. Clearly, $\overline{C}$ is a left B-module coalgebra and C admits a natural left and right $\overline{C}$-comodule structure. Let Λ be a nonzero element in $\int_B^r$. Consider the map $f_C : \overline{C} \longrightarrow C$ defined by
$$f_C(\eta_C(x)) = \Lambda x$$
for $x \in C$ (see [7], p3348). Clearly, the map is well-defined and is a left and right $\overline{C}$-comodule map. Therefore, $\ker f_C$ is a subcoalgebra of $\overline{C}$. Notice that $\ker f_C = \eta_C({}_CI)$ and hence the result follows. If B is a subHopf algebra of A, then η_A is a right A-module map. Since ${}_AI$ is a right A-submodule of A, $\eta_A({}_AI)$ is a right A-submodule of $\eta_A(A)$. $\square$

Corollary 13. *Let B be a finite dimensional Hopf algebra and C a left B-module coalgebra. If C is a projective B-module, ${}_CI = B^+C$.*

Proof. Clearly, $B^+C \subseteq {}_CI$. It suffices to show that ${}_CI \subseteq B^+C$. By Proposition 1, ${}_CI \subseteq \ker \varepsilon_C$. Therefore, $\eta_C({}_CI) \subseteq \ker \varepsilon_{\overline{C}}$. By Lemma 12 $\eta_C({}_CI)$ is also a subcoalgebra of $\overline{C}$. Therefore, $\eta_C({}_CI) = 0$ and so ${}_CI \subseteq B^+C$. $\square$

Lemma 14. *Let A be a Hopf algebra and B a finite dimensional subHopf algebra of A. If A is not a left projective B-module, then*

$$ {}_AII_A + AB^+A = A\,. $$

Proof. By Lemma 12, ${}_AI/(B^+A)$ is a right A-submodule coalgebra of $A/(B^+A)$. Note that AB^+A is a Hopf ideal of A. Thus $AB^+A/(B^+A)$ is also a coideal of A/B^+A. Therefore, ${}_A\overline{I} = ({}_AI + AB^+A)/AB^+A$ is a right A-submodule coalgebra of A/AB^+A. In particular, ${}_A\overline{I}$ is a right ideal of A/AB^+A. Since A is not a projective B-module, ${}_AI \not\subseteq A^+$ by Proposition 1. Hence ${}_AI \not\subseteq AB^+A$, and so ${}_A\overline{I}$ is a non-zero right ideal as well as coideal of A/AB^+A. Therefore, ${}_A\overline{I} = A/AB^+A$ (see [8], p. 108, Exercise 5). Thus, we have

$$ {}_AI + AB^+A = A\,. $$

By Corollary 2, A is not a projective right B-module. By virtue of Proposition 1 (ii), we similarly obtain

$$ I_A + AB^+A = A\,. $$

Hence, we have ${}_AII_A + AB^+A = A$. □

Theorem 15. *Let A be a Hopf algebra and B a finite dimensional subHopf algebra of A. If $AB^+ \subseteq B^+A$ or $B^+A \subseteq AB^+$, then A is a free left (right) B-module.*

Proof. By the Nichols-Zoeller Theorem [5], it suffices to consider the case when A is infinite dimensional. By Schneider's Theorem ([7], Theorem 2.4), it suffices to show that A is a left projective B-module. Suppose A is not a projective left B-module. By Lemma 14, we have

$$ {}_AII_A + AB^+A = A\,. $$

If $AB^+ \subseteq B^+A$, then $AB^+A = B^+A$. Thus, ${}_AII_A + B^+A = A$ and so $\int_B^r A = 0$, contradiction ! Similarly, if $B^+A \subseteq AB^+$, then ${}_AII_A + AB^+ = A$ and hence $A\int_B^l = 0$, contradiction ! Therefore, A is a projective B-module. □

Corollary 16. *Let A be a Hopf algebra and B a finite dimensional subHopf algebra of A. Then*

(i) *B is left normal iff $AB^+ \subseteq B^+A$;*
(ii) *B is right normal iff $B^+A \subseteq AB^+$;*
(iii) *B is normal iff $B^+A = AB^+$.*
(iv) *If the antipode of A is bijective, B is left normal iff B is right normal.*

Proof. (i) If B is left normal, it is obvious that $AB^+ \subseteq B^+A$ (see [9], 1.4). Conversely, assume $AB^+ \subseteq B^+A$. By Theorem 15, A is a free right B-module and hence faithfully flat. By Theorem 4.4 of [9], B is left normal. Similarly, we can prove (ii).

(iii) An immediate consequence of (i) and (ii).

(iv) A direct consequence of Theorem 15 and Corollary 4.5 of [9]. □

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