

Qppi GROUPS AND QUASI-EQUIVALENCE

H. P. GOETERS AND W. J. WICKLESS

(Communicated by Ronald M. Solomon)

ABSTRACT. A torsion-free abelian group G is *qpi* if every map from a pure subgroup K of G into G lifts to an endomorphism of G . The class of *qpi* groups has been extensively studied, resulting in a number of nice characterizations. We obtain some characterizations for the class of homogeneous *Qppi* groups, those homogeneous groups G such that, for K pure in G , every $\theta : K \rightarrow G$ has a lifting to a quasi-endomorphism of G . An irreducible group is *Qppi* if and only if every pure subgroup of each of its strongly indecomposable quasi-summands is strongly indecomposable. A *Qppi* group G is *qpi* if and only if every endomorphism of G is an integral multiple of an automorphism. A group G has minimal test for quasi-equivalence (*mtqe*) if whenever K and L are quasi-isomorphic pure subgroups of G then K and L are equivalent via a quasi-automorphism of G . For homogeneous groups, we show that in almost all cases the *Qppi* and *mtqe* properties coincide.

All groups considered in the paper will be torsion-free abelian of finite rank. We assume familiarity with the standard tools used in studying these groups, such as types, pure subgroups, p -rank and quasi-isomorphism. We use \doteq for quasi-equality and H^n for a direct sum of n copies of a group H . For W a (torsion-free abelian) group or ring, we write QW for the divisible hull of W . We start by defining the class of groups that will be the object of our attention.

Definition 1. A group G is *Qppi* if whenever $0 \rightarrow K \rightarrow G$ is pure exact then the induced sequence $QE(G) \rightarrow QHom(K, G) \rightarrow 0$ is exact.

What we are saying, simply, is that for any homomorphism θ mapping a pure subgroup K into G there is a $\phi \in QE(G)$ such that $\phi|_K = \theta$. Thus, there is a positive integer t such that $t\theta$ can be lifted to an endomorphism of G .

Plainly any *qpi* group (where we require that θ itself be lifted to an endomorphism of G) is *Qppi*. For a complete characterization of the class of *qpi* groups see [A-O'B-R] together with [R-2]. We provide examples to show that the *Qppi* groups form a considerably larger class.

It follows directly from the definitions that a homogeneous *Qppi* group G must be irreducible, that is, $QE(G)$ will act transitively on the rank one subspaces of QG . The precise relation between homogeneous *Qppi* groups and irreducible groups is given in our first theorem.

Theorem 2. *Let G be irreducible and write $G \doteq H^n$, where H is strongly indecomposable and irreducible ([R-1], Th. 5.5). Then: (a) H is *Qppi* if and only if*

Received by the editors March 21, 1997.

1991 *Mathematics Subject Classification.* Primary 20K15.

each pure subgroup of H is strongly indecomposable, and (b) G is $Qppi$ if and only if H is $Qppi$.

Proof. (a) Since H is irreducible strongly indecomposable then $QE(H) = \Gamma$, a division ring ([R-1]). Thus, if K is pure in H with a nontrivial quasi-decomposition $K \doteq U \oplus V$ then $i \oplus 0 : K \rightarrow H$ ($i : U \rightarrow H$ is inclusion) cannot have a lifting to an element of $QE(H)$. It follows that if H is $Qppi$ then every pure subgroup of H must be strongly indecomposable. Conversely, assume that every pure subgroup of H is strongly indecomposable. Suppose that $K \subset H$ is pure and $\theta : K \rightarrow H$. Let X be a rank one pure subgroup of K . Since H is irreducible, $\theta|_X$ has a lifting to $\phi \in QE(H)$. If $X = K$ we're done. Otherwise we show that $\phi|_K = \theta$ by showing that $\phi|_L = \theta|_L$ for each rank two pure subgroup $L \subset K$ with $X \subset L$. To see this, note that, since L is homogeneous strongly indecomposable, then by Baer's Lemma ([A], Lemma 1.12) $\text{type } L/X > \text{type } L = \text{type } H$. Hence $\text{Hom}(L/X, H) = 0$. It follows that $(\phi - \theta)|_L = 0$ and the proof of (a) is complete.

(b) It is simple to check that the class of $Qppi$ groups is closed under taking quasi-summands. Hence, if G as above is $Qppi$ then so is H .

Conversely, assume that H is $Qppi$ and let $\theta : K \rightarrow G$ with K pure in G . We show that θ can be lifted to a quasi-endomorphism of G by induction on rank K . Since G is irreducible, θ can be quasi-lifted when $\text{rank } K = 1$. Suppose that $\text{rank } K > 1$ and that, for any pure subgroup $L \subset G$ of smaller rank, any homomorphism from L to G can be lifted to an element of $QE(G)$. Let $L \subset K$ be a pure subgroup with $\text{rank } K/L = 1$. By the inductive assumption there exists $\phi \in QE(G)$ such that $\phi|_L = \theta|_L$. Let $\psi = (\phi|_K - \theta) : K \rightarrow G$. If $\psi = 0$ we're done. If $\psi \neq 0$, since K is homogeneous of the same type as G , we can apply Baer's Lemma to obtain a quasi-decomposition $K \doteq L \oplus X$ with X a rank one pure subgroup of G .

Recall that $\Gamma = QE(H)$ can be identified with the centralizer of the simple $QE(G)$ module QG ([R-1]). We claim that we cannot have $\Gamma X \subset \Gamma L$ (as Γ -subspaces of QG). To prove the claim suppose $0 \neq x \in X$ can be written $x = \gamma l$ for some $\gamma \in \Gamma, l \in L$. Let $QG = \Gamma l \oplus Y$, where Y is any complementary Γ -subspace of QG , and let ρ be the associated projection of QG onto Γl . Then $\rho \in \text{Hom}_\Gamma(QG, QG) = QE(G)$, the equality by the double centralizer theorem. Using ρ we obtain a quasi-decomposition $G \doteq G' \oplus G''$ with $G' = \Gamma l \cap G$. Note that G' is quasi-isomorphic to H , since G' is a quasi-summand of G with $\text{rank } G' = \text{rank } \Gamma l \cap G = \text{rank } \Gamma = \text{rank } H$. But, since $L \oplus X \doteq K$, a pure subgroup of G , the pure subgroup of G' generated by x and l will be rank two completely decomposable, contradicting part (a).

Thus $\Gamma X \cap \Gamma L = 0$. Arguing as in the previous paragraph, we obtain a quasi-decomposition $G \doteq G' \oplus G''$ with $L \subset G', X \subset G''$. Let $\phi' \in QE(G)$ be such that $\phi'|_X = \theta|_X$. Then $\phi|_{G'} \oplus \phi'|_{G''}$ will be the desired quasi-lifting of θ .

Example 3. Let F be an algebraic number field of dimension 3 over Q such that in its ring of integers J there is a decomposition of an integral prime $p, pJ = PP'$, with $\dim_{Z/pZ}(J/P) = 2$. (Such examples are easy to construct, e.g. see [A-O'B-R].) Let $R = J_p$. Since R is a full subring of the field F , R will be irreducible as an additive group. Furthermore, since $\text{rank } R = 3$ and p -rank $R = 2$, in the quasi-decomposition $R \doteq H^n$ we must have $n = 1, R = H$. Thus R is strongly indecomposable. Note that $qR = R$ for all integral primes $q \neq p$, so that $Z_p J \subset R$. Denote $J_p = Z_p J$. Since R has p -rank two, $R/J_p \cong Z(p^\infty)$. Let K be a rank two pure subgroup of R . Since R is strongly indecomposable we cannot have $K + J_p = R$.

Hence $(K + J_p)/J_p$ is finite. It follows that any rank two pure subgroup of R must be isomorphic to $Z_p \oplus Z_p$. By Theorem 2 (a), R is not *Qqpi*.

We prove for future reference that the additive group R satisfies the following property, which is weaker than the *Qqpi* property: if U, V are quasi-isomorphic pure subgroups of R (which by our above discussion just means that $\text{rank } U = \text{rank } V$) then $\phi U \doteq V$ for ϕ a quasi-automorphism of R (which in this case is simply a nonzero element of $QE(R) \cong F$). Since R is irreducible, if U, V are rank one pure subgroups of R , then $\phi U \doteq V$ for $\phi \in QE(R)$. Write $F = Q(r)$ for $r \in R$ and let $U_0 = Z_p 1 \oplus Z_p r$. We show that for each pure subgroup $V \subset R$ of rank two there exists $\phi \in QE(R)$ with $\phi U_0 \doteq V$. The fact that arbitrary rank two pure subgroups $U, V \subset R$ are equivalent via a quasi-automorphism of R will follow immediately. By rank considerations $rV \cap V \neq 0$. Take $0 \neq v \in V$ such that $rv \in V$. Then, if ϕ is multiplication by v , we have $\phi U_0 \doteq V$.

The characterization of homogeneous strongly indecomposable *qpi* groups G from [A-O'B-R] and [R-2] is that $G \cong R \otimes A$, where $\text{rank } A = 1$ and $R = E(G)$ is a strongly homogeneous ring of p -rank one for some integral prime p . A strongly homogeneous ring is a full subring R of an algebraic number field such that every element of R is an integral multiple of a unit in R . Since the class of *Qqpi* groups is closed under quasi-isomorphism and the class of groups of the form $R \otimes A$ as above is not, we have some immediate simple examples of strongly indecomposable homogeneous *Qqpi* groups which fail to be *qpi*. We also note, in connection with Theorem 2 (b), that [A-O'B-R] has an example of a strongly indecomposable homogeneous *qpi* group H such that $H \oplus H$ is not *qpi*.

Our G constructed in the next example shows that the endomorphism ring of a strongly indecomposable homogeneous *Qqpi* group need not be a strongly homogeneous ring and need not have p -rank one for any prime p .

Example 4. There is a rank 4 homogeneous *Qqpi* group G such that $E(G)$ is the ring of integral quaternions.

Construction of the example. Let R denote the ring of integral quaternions and let $S = \{0 \neq s = ai + bj + ck \in R \mid \text{the first nonzero coefficient of } s \text{ is positive and } \gcd(a, b, c) = 1\}$. Enumerate the elements of S , say $S = \{s_m\}$. For $s_m = a_m i + b_m j + c_m k$, put $N(s_m) = a_m^2 + b_m^2 + c_m^2$. Let P_m be the set of integral primes p such that $-N(s_m)$ is a nonzero square mod p . Since each $-N(s_m) \neq 0$, it is well known that each P_m will be an infinite set of primes. For each m , choose an infinite subset $P'_m \subset P_m$ so that $\{P'_m \mid 1 \leq m < \infty\}$ will be a collection of disjoint sets. For each $p \in P'_m$ choose $d_p \in Z$ with $0 < d_p < p$ such that $-N(s_m) \equiv d_p^2 \pmod p$. Let G be the R -submodule of QR generated by R and $\{(d_p - s_m)/p \mid 1 \leq m < \infty, p \in P'_m\}$.

Plainly $R \subset E(G)$, so that $QR \subset QE(G)$. Since QR is a division algebra and $QR = QG$ it follows that G is irreducible, hence homogeneous. We show that $\text{type } G = \bar{0}$ ($= \text{type } Z$) by considering the divisibility of the element $1 \in G$. Clearly, 1 is divisible in G by no prime in the complement of $\bigcup_m P'_m$. Suppose that $1 = pg$ for $g \in G, p \in P'_m$. Writing g as a finite R -combination of the generators of G produces the equation $1 = r(d_p - s_m) + pr'$ for some $r, r' \in R$. Multiplying this equation by $(d_p + s_m)$ yields the equation $d_p + s_m = r[d_p^2 + N(s_m)] + pr'(d_p + s_m)$. Since $d_p^2 \equiv -N(s_m) \pmod p$ we have $(d_p + s_m) \in pR$, so that d_p is divisible by p , a contradiction. Hence, for all primes $p, 1 \notin pG$. Thus $\text{type } G = \text{type}_G(1) = \bar{0}$.

Suppose that $K \subset G$ is a pure subgroup with a nontrivial quasi-decomposition $K \doteq U \oplus V$. Since G/R is torsion, $U \cap R$ and $V \cap R$ are both nonzero. Take nonzero elements $u = (e + ai + bj + ck) \in U \cap R$, $v \in V \cap R$ and denote $\bar{u} = e - ai - bj - ck$. The element $\bar{u}v \in R$ can be written in the form $t + ls_m$ for some integers t, l and positive integer m . It follows that, for each $p \in P'_m$, $[(ld_p + t) - \bar{u}v] = l(d_p - s_m) \in pG$. Multiplication by u yields that $[(ld_p + t)u - N(u)v] \in pG$ for all $p \in P'_m$. This is impossible, since $N(u) \neq 0$ and u, v are elements lying in different quasi-summands of a pure subgroup of G , a homogeneous group of type $\bar{0}$. The resulting contradiction shows that each pure subgroup of G is strongly indecomposable. By Theorem 2 (a), G is $Qppi$.

Since the quaternion algebra QR is contained in $QE(G)$ and $QR = QG$, G is irreducible. Moreover, G itself is strongly indecomposable, so that $\text{rank } QE(G) = \text{rank } QG$. Hence, $QR = QE(G)$. We have already noted that $R \subset E(G)$, so $R \subset E(G) \subset QR$. By considering the action of a possible endomorphism of G on the set of generators for G , it is easy to check that $E(G)$ coincides precisely with R .

If G is strongly indecomposable, homogeneous and qpi , then $QE(G)$ is a field. Since quasi-isomorphic groups have isomorphic quasi-endomorphism rings, our example additionally shows that the class of $Qppi$ groups is larger than the class of groups quasi-isomorphic to a qpi group. The following simple result gives the precise connection between the $Qppi$ and qpi properties for strongly indecomposable homogeneous groups.

Theorem 5. *Let H be strongly indecomposable homogeneous $Qppi$. Then H is qpi if and only if every element of $R = E(H)$ is an integral multiple of a unit of R .*

Proof. Let H be strongly indecomposable homogeneous $Qppi$. By Theorem 2 (a) every pure subgroup of H is strongly indecomposable. By Theorem B of [A-O'B-R], to show that H is qpi it suffices to show that for each pair of rank one pure subgroups X, Y of H there exists a unit $u \in R$ with $uX = Y$. Since H is homogeneous $Qppi$, there exists $0 \neq r \in R$ with $rX \subset Y$. If $r = tu, t \in Z, u$ a unit of R , it follows that $uX = Y$.

Definition 6. A group G has minimal test for quasi-equivalence ($mtqe$) if whenever U and V are quasi-isomorphic pure subgroups of G then there exists ϕ , a quasi-automorphism of G , with $\phi U \doteq V$.

We have already noted that the group in Example 3 is homogeneous with $mtqe$ but is not $Qppi$. For a second example, if we eliminate the element s_1 from our construction of Example 4, then we obtain a subgroup $G' \subset G$ in which $\langle 1 \rangle \oplus \langle s_1 \rangle$ is pure. However, it is not too hard to show that G' remains strongly indecomposable and that $E(G')$ will coincide with $E(G)$, the ring of integral quaternions. Thus, G' is a strongly indecomposable homogeneous non- $Qppi$ group. It also is not too hard to show that G' has $mtqe$. Since the construction of this second example is not central to our work, we omit the details.

Note that the group G of Example 3 has $QE(G)$ a number field of degree 3. The group G in the modification of Example 4 would have $QE(G)$ a division algebra of degree 2. The next result, which we feel is somewhat surprising, shows that only division algebras of degree 2 or 3 can occur as $QE(G)$ for a strongly indecomposable homogeneous group G for which the $Qppi$ and $mtqe$ properties fail to coincide.

Theorem 7. *Let G be irreducible and write $G \doteq H^n$ with H strongly indecomposable and irreducible. Suppose that $\dim_Q F > 3$, where F is a maximal subfield of the division algebra $\Gamma = QE(H)$. Then G is Qqpi if and only if G has mtqe.*

Proof. First we assume that $n = 1$, that is, $H = G$ is strongly indecomposable. Since $\Gamma = QE(H)$, every $0 \neq r \in E(H)$ is a quasi-automorphism. Hence, if H is Qqpi, it follows immediately that H has mtqe. Conversely, suppose that H has mtqe. By Theorem 2 (a), to prove that H is Qqpi we need to show that any pure subgroup of H is strongly indecomposable. It will be enough to show that any rank two pure subgroup of H is strongly indecomposable.

Let K be a rank two pure subgroup of H ; say $K = \langle x, y \rangle_*$, the pure subgroup generated by elements x, y . Since H is irreducible there exists $\gamma \in QE(H)$ with $\gamma x = y$. Extend the field $Q(\gamma)$ to a maximal subfield $F \subset QE(H)$.

Suppose that for each primitive element $\alpha \in E(H)$ with $F = Q(\alpha)$ the pure subgroup $\langle x, \alpha x \rangle_* \subset H$ is completely decomposable. By the proof of the existence of a primitive element for algebraic number fields, for α primitive we can choose a positive integer t such that $\beta = \alpha + t\alpha^2$ will also be primitive. Then the pure subgroups $\langle x, \alpha x \rangle_*$ and $\langle x, \beta x \rangle_*$ will be isomorphic (both being completely decomposable subgroups of the homogeneous group H). Because H has mtqe we have $\phi \langle x, \alpha x \rangle_* \doteq \langle x, \beta x \rangle_*$ for some $\phi \in \Gamma$. Thus $\phi(x) = (q_1 + q_2\beta)x$ for some rational numbers q_1, q_2 . Since Γ is a division algebra, $\phi = (q_1 + q_2\beta)$. But $\phi(\alpha x) = (q_1 + q_2\beta)\alpha x$ cannot be a rational combination $q_3x + q_4\beta x$, for then $(q_1 + q_2\beta)\alpha = q_3 + q_4\beta$ for some rationals q_3, q_4 . Since $\beta = \alpha + t\alpha^2$, this latter equation would contradict the fact that α is algebraic over Q of degree greater than three. It follows that, for some primitive element $\alpha \in E(H)$, the pure subgroup $\langle x, \alpha x \rangle_*$ will be strongly indecomposable.

As before, the type of the rank one factor group $\langle x, \alpha x \rangle_* / \langle x \rangle_*$ must be greater than the type of H . Thus, one of two possibilities must occur.

Case I: There is an infinite set of primes P such that for $p \in P$ there exists an integer c_p with $h_p(\alpha x - c_p x) > h_p(x)$. Here h_p denotes the p -height of an element in H . In this case an easy calculation shows that, for every integer t with $1 \leq t < \text{degree } \alpha$ and $p \in P$, we have $h(\alpha^t x - c_p^t x) > h_p(x)$. Let L be the pure subgroup of H generated by x and $\{\alpha^t x \mid 1 \leq t < \text{degree } \alpha\}$. Our set of height inequalities shows that the inner type of $L/\langle x \rangle_*$ is greater than the type of H . Since $y = \gamma x \in L$ and $K = \langle x, y \rangle_*$, then $K/\langle x \rangle_*$ is a pure subgroup of $L/\langle x \rangle_*$. Hence $\text{type}[K/\langle x \rangle_*] > \text{type } H$, so that K cannot be homogeneous and completely decomposable. Thus K must be strongly indecomposable, as desired.

Case II: For some prime p with $pH \neq H$ there is a set of integers $c_n, 1 \leq n < \infty$, with $(\alpha x - c_n x) \in p^n H$. Arguing as in Case I, we can conclude that $L/\langle x \rangle_*$ is p -divisible; hence so is $K/\langle x \rangle_*$. Again we have that K cannot be homogeneous and completely decomposable, so K is strongly indecomposable. The proof that H is Qqpi is complete.

Now suppose that $n > 1$ and $G \doteq H^n$ is as in the statement of the theorem. If G has mtqe it is immediate that H has mtqe. By what we have proved already, H is Qqpi and, by Theorem 2 (b), G is Qqpi. Conversely, let G be Qqpi. Then H is also Qqpi, so, by the remark at the beginning of the proof, H has mtqe. In view of the fact that $G \doteq H^n$ is irreducible with $QE(G) \cong (\Gamma)_{n \times n}$, it is not hard to see that, for X, Y any rank one pure subgroups of G , we can choose a quasi-automorphism

$\phi \in QE(G)$ with $\phi X = Y$. With this observation, the proof of Theorem 2 (b) goes through *mutatis mutandis* to show that G has *mtqe*.

ACKNOWLEDGMENT

We thank the referee for his or her careful reading of the paper.

REFERENCES

- [A] D. Arnold, **Finite Rank Torsion Free Abelian Groups and Rings**, Springer-Verlag LNM **931** (1982). MR **84d**:20002
- [A-O'B-R] D. Arnold, B. O'Brien and J. Reid, Quasipure injective and projective torsion-free abelian groups of finite rank, *Proceedings of the London Math. Soc.* **38** (1979), 532-44. MR **84f**:20060
- [R-1] J. Reid, On the ring of quasi-endomorphisms of a torsion-free group, **Topics in Abelian Groups**, Scott Foresman, 1963, 51-58. MR **30**:158
- [R-2] ———, Abelian groups cyclic over their endomorphism rings, **Abelian Group Theory**, Springer-Verlag LNM **1006** (1983), 190-203. MR **85e**:16053

DEPARTMENT OF MATHEMATICS, AUBURN UNIVERSITY, AUBURN, ALABAMA 36849
E-mail address: goetehp@mail.auburn.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CONNECTICUT, STORRS, CONNECTICUT 06269
E-mail address: wjwick@uconnvm.uconn.edu