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Qqpi GROUPS AND QUASI-EQUIVALENCE

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ABSTRACT. A torsion-free abelian group G is *qpi* if every map from a pure subgroup K of G into G lifts to an endomorphism of G. The class of *qpi* groups has been extensively studied, resulting in a number of nice characterizations. We obtain some characterizations for the class of homogeneous *Qqpi* groups, those homogeneous groups G such that, for K pure in G, every $\theta : K \to G$ has a lifting to a quasi-endomorphism of G. An irreducible group is *Qqpi* if and only if every pure subgroup of each of its strongly indecomposable quasi-summands is strongly indecomposable. A *Qqpi* group G is *qpi* if and only if every endomorphism of G is an integral multiple of an automorphism. A group G has minimal test for quasi-equivalence (*mtqe*) if whenever K and L are quasi-isomorphic pure subgroups of G then K and L are equivalent via a quasi-automorphism of G. For homogeneous groups, we show that in almost all cases the *Qqpi* and *mtqe* properties coincide.

All groups considered in the paper will be torsion-free abelian of finite rank. We assume familiarity with the standard tools used in studying these groups, such as types, pure subgroups, p-rank and quasi-isomorphism. We use $\doteq$ for quasi-equality and H^n for a direct sum of n copies of a group H. For W a (torsion-free abelian) group or ring, we write QW for the divisible hull of W. We start by defining the class of groups that will be the object of our attention.

Definition 1. A group G is *Qqpi* if whenever $0 \to K \to G$ is pure exact then the induced sequence $QE(G) \to QHom(K, G) \to 0$ is exact.

What we are saying, simply, is that for any homomorphism θ mapping a pure subgroup K into G there is a $\phi \in QE(G)$ such that $\phi\mid_K = \theta$. Thus, there is a positive integer t such that $t\theta$ can be lifted to an endomorphism of G.

Plainly any *qpi* group (where we require that θ itself be lifted to an endomorphism of G) is *Qqpi*. For a complete characterization of the class of *qpi* groups see [A-O'B-R] together with [R-2]. We provide examples to show that the *Qqpi* groups form a considerably larger class.

It follows directly from the definitions that a homogeneous *Qqpi* group G must be irreducible, that is, $QE(G)$ will act transitively on the rank one subspaces of QG. The precise relation between homogeneous *Qqpi* groups and irreducible groups is given in our first theorem.

Theorem 2. *Let G be irreducible and write $G \doteq H^n$, where H is strongly indecomposable and irreducible (*[R-1]*, Th. 5.5). Then: (a) H is Qqpi if and only if*

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each pure subgroup of H is strongly indecomposable, and (b) G is Qqpi if and only if H is Qqpi.

Proof. (a) Since H is irreducible strongly indecomposable then $QE(H) = \Gamma$, a division ring ([R-1]). Thus, if K is pure in H with a nontrivial quasi-decomposition $K \doteq U \oplus V$ then $i \oplus 0 : K \to H$ ($i : U \to H$ is inclusion) cannot have a lifting to an element of $QE(H)$. It follows that if H is *Qqpi* then every pure subgroup of H must be strongly indecomposable. Conversely, assume that every pure subgroup of H is strongly indecomposable. Suppose that $K \subset H$ is pure and $\theta : K \to H$. Let X be a rank one pure subgroup of K. Since H is irreducible, $\theta \mid_X$ has a lifting to $\phi \in QE(H)$. If $X = K$ we're done. Otherwise we show that $\phi \mid_K = \theta$ by showing that $\phi \mid_L = \theta \mid_L$ for each rank two pure subgroup $L \subset K$ with $X \subset L$. To see this, note that, since L is homogeneous strongly indecomposable, then by Baer's Lemma ([A], Lemma 1.12) type $L/X >$ type $L =$ type H. Hence $Hom(L/X, H) = 0$. It follows that $(\phi - \theta) \mid_L = 0$ and the proof of (a) is complete.

(b) It is simple to check that the class of *Qqpi* groups is closed under taking quasi-summands. Hence, if G as above is *Qqpi* then so is H.

Conversely, assume that H is *Qqpi* and let $\theta : K \to G$ with K pure in G. We show that θ can be lifted to a quasi-endomorphism of G by induction on rank K. Since G is irreducible, θ can be quasi-lifted when rank $K = 1$. Suppose that rank $K > 1$ and that, for any pure subgroup $L \subset G$ of smaller rank, any homomorphism from L to G can be lifted to an element of $QE(G)$. Let $L \subset K$ be a pure subgroup with rank $K/L = 1$. By the inductive assumption there exists $\phi \in QE(G)$ such that $\phi \mid_L = \theta \mid_L$. Let $\psi = (\phi \mid_K - \theta) : K \to G$. If $\psi = 0$ we're done. If $\psi \neq 0$, since K is homogeneous of the same type as G, we can apply Baer's Lemma to obtain a quasi-decomposition $K \doteq L \oplus X$ with X a rank one pure subgroup of G.

Recall that $\Gamma = QE(H)$ can be identified with the centralizer of the simple $QE(G)$ module QG ([R-1]). We claim that we cannot have $\Gamma X \subset \Gamma L$ (as Γ-subspaces of QG). To prove the claim suppose $0 \neq x \in X$ can be written $x = \gamma l$ for some $\gamma \in \Gamma, l \in L$. Let $QG = \Gamma l \oplus Y$, where Y is any complementary Γ-subspace of QG, and let ρ be the associated projection of QG onto Γl. Then $\rho \in Hom_\Gamma(QG, QG) = QE(G)$, the equality by the double centralizer theorem. Using ρ we obtain a quasi-decomposition $G \doteq G' \oplus G''$ with $G' = \Gamma l \cap G$. Note that G' is quasi-isomorphic to H, since G' is a quasi-summand of G with rank $G' =$ rank $\Gamma l \cap G =$ rank $\Gamma =$ rank H. But, since $L \oplus X \doteq K$, a pure subgroup of G, the pure subgroup of G' generated by x and l will be rank two completely decomposable, contradicting part (a).

Thus $\Gamma X \cap \Gamma L = 0$. Arguing as in the previous paragraph, we obtain a quasi-decomposition $G \doteq G' \oplus G''$ with $L \subset G', X \subset G''$. Let $\phi' \in QE(G)$ be such that $\phi' \mid_X = \theta \mid_X$. Then $\phi \mid_{G'} \oplus \phi' \mid_{G''}$ will be the desired quasi-lifting of θ.

Example 3. Let F be an algebraic number field of dimension 3 over Q such that in its ring of integers J there is a decomposition of an integral prime p, $pJ = PP'$, with $\dim_{Z/pZ}(J/P) = 2$. (Such examples are easy to construct, e.g. see [A-O'B-R].) Let $R = J_P$. Since R is a full subring of the field F, R will be irreducible as an additive group. Furthermore, since rank $R = 3$ and p-rank $R = 2$, in the quasi-decomposition $R \doteq H^n$ we must have $n = 1, R = H$. Thus R is strongly indecomposable. Note that $qR = R$ for all integral primes $q \neq p$, so that $Z_pJ \subset R$. Denote $J_p = Z_pJ$. Since R has p-rank two, $R/J_p \cong Z(p^\infty)$. Let K be a rank two pure subgroup of R. Since R is strongly indecomposable we cannot have $K + J_p = R$.

Hence $(K + J_p)/J_p$ is finite. It follows that any rank two pure subgroup of R must be isomorphic to $Z_p \oplus Z_p$. By Theorem 2 (a), R is not *Qqpi*.

We prove for future reference that the additive group R satisfies the following property, which is weaker than the *Qqpi* property: if U, V are quasi-isomorphic pure subgroups of R (which by our above discussion just means that rank U = rank V) then $\phi U \doteq V$ for ϕ a quasi-automorphism of R (which in this case is simply a nonzero element of $QE(R) \cong F$). Since R is irreducible, if U, V are rank one pure subgroups of R, then $\phi U \doteq V$ for $\phi \in QE(R)$. Write $F = Q(r)$ for $r \in R$ and let $U_0 = Z_p 1 \oplus Z_p r$. We show that for each pure subgroup $V \subset R$ of rank two there exists $\phi \in QE(R)$ with $\phi U_0 \doteq V$. The fact that arbitrary rank two pure subgroups $U, V \subset R$ are equivalent via a quasi-automorphism of R will follow immediately. By rank considerations $rV \cap V \neq 0$. Take $0 \neq v \in V$ such that $rv \in V$. Then, if ϕ is multiplication by v, we have $\phi U_0 \doteq V$.

The characterization of homogeneous strongly indecomposable *qpi* groups G from [A-O'B-R] and [R-2] is that $G \cong R \otimes A$, where rank $A = 1$ and $R = E(G)$ is a strongly homogeneous ring of p-rank one for some integral prime p. A strongly homogeneous ring is a full subring R of an algebraic number field such that every element of R is an integral multiple of a unit in R. Since the class of *Qqpi* groups is closed under quasi-isomorphism and the class of groups of the form $R \otimes A$ as above is not, we have some immediate simple examples of strongly indecomposable homogeneous *Qqpi* groups which fail to be *qpi*. We also note, in connection with Theorem 2 (b), that [A-O'B-R] has an example of a strongly indecomposable homogeneous *qpi* group H such that $H \oplus H$ is not *qpi*.

Our G constructed in the next example shows that the endomorphism ring of a strongly indecomposable homogeneous *Qqpi* group need not be a strongly homogeneous ring and need not have p-rank one for any prime p.

Example 4. There is a rank 4 homogeneous *Qqpi* group G such that $E(G)$ is the ring of integral quaternions.

Construction of the example. Let R denote the ring of integral quaternions and let $S = \{0 \neq s = ai + bj + ck \in R \mid$ the first nonzero coefficient of s is positive and $\gcd(a, b, c) = 1\}$. Enumerate the elements of S, say $S = \{s_m\}$. For $s_m = a_m i + b_m j + c_m k$, put $N(s_m) = a_m^2 + b_m^2 + c_m^2$. Let P_m be the set of integral primes p such that $-N(s_m)$ is a nonzero square mod p. Since each $-N(s_m) \neq 0$, it is well known that each P_m will be an infinite set of primes. For each m, choose an infinite subset $P_m' \subset P_m$ so that $\{P_m' \mid 1 \leq m < \infty\}$ will be a collection of disjoint sets. For each $p \in P_m'$ choose $d_p \in Z$ with $0 < d_p < p$ such that $-N(s_m) \equiv d_p^2 \bmod p$. Let G be the R-submodule of QR generated by R and $\{(d_p - s_m)/p \mid 1 \leq m < \infty, p \in P_m'\}$.

Plainly $R \subset E(G)$, so that $QR \subset QE(G)$. Since QR is a division algebra and $QR = QG$ it follows that G is irreducible, hence homogeneous. We show that type $G = \bar{0}$ (= type Z) by considering the divisibility of the element $1 \in G$. Clearly, 1 is divisible in G by no prime in the complement of $\bigcup_m P_m'$. Suppose that $1 = pg$ for $g \in G, p \in P_m'$. Writing g as a finite R-combination of the generators of G produces the equation $1 = r(d_p - s_m) + pr'$ for some $r, r' \in R$. Multiplying this equation by $(d_p + s_m)$ yields the equation $d_p + s_m = r[d_p^2 + N(s_m)] + pr'(d_p + s_m)$. Since $d_p^2 \equiv -N(s_m) \bmod p$ we have $(d_p + s_m) \in pR$, so that d_p is divisible by p, a contradiction. Hence, for all primes $p, 1 \notin pG$. Thus type $G = \text{type}_G(1) = \bar{0}$.

Suppose that $K \subset G$ is a pure subgroup with a nontrivial quasi-decomposition $K \doteq U \oplus V$. Since G/R is torsion, $U \cap R$ and $V \cap R$ are both nonzero. Take nonzero elements $u = (e + ai + bj + ck) \in U \cap R$, $v \in V \cap R$ and denote $\bar{u} = e - ai - bj - ck$. The element $\overline{u}v \in R$ can be written in the form $t + ls_m$ for some integers t, l and positive integer m. It follows that, for each $p \in P_m^{'}$, $[(ld_p + t) - \overline{u}v] = l(d_p - s_m) \in pG$. Multiplication by u yields that $[(ld_p + t)u - N(u)v] \in pG$ for all $p \in P_m^{'}$. This is impossible, since $N(u) \neq 0$ and u, v are elements lying in different quasi-summands of a pure subgroup of G, a homogeneous group of type $\bar{0}$. The resulting contradiction shows that each pure subgroup of G is strongly indecomposable. By Theorem 2 (a), G is *Qqpi*.

Since the quaternion algebra QR is contained in $QE(G)$ and $QR = QG, G$ is irreducible. Moreover, G itself is strongly indecomposable, so that rank $QE(G) =$ rank QG. Hence, $QR = QE(G)$. We have already noted that $R \subset E(G)$, so $R \subset E(G) \subset QR$. By considering the action of a possible endomorphism of G on the set of generators for G, it is easy to check that $E(G)$ coincides precisely with R.

If G is strongly indecomposable, homogeneous and *qpi*, then $QE(G)$ is a field. Since quasi-isomorphic groups have isomorphic quasi-endomorphism rings, our example additionally shows that the class of *Qqpi* groups is larger than the class of groups quasi-isomorphic to a *qpi* group. The following simple result gives the precise connection between the *Qqpi* and *qpi* properties for strongly indecomposable homogeneous groups.

Theorem 5. *Let H be strongly indecomposable homogeneous Qqpi. Then H is qpi if and only if every element of $R = E(H)$ is an integral multiple of a unit of R.*

Proof. Let H be strongly indecomposable homogeneous *Qqpi*. By Theorem 2 (a) every pure subgroup of H is strongly indecomposable. By Theorem B of [A-O'B-R], to show that H is *qpi* it suffices to show that for each pair of rank one pure subgroups X, Y of H there exists a unit $u \in R$ with $uX = Y$. Since H is homogeneous *Qqpi*, there exists $0 \neq r \in R$ with $rX \subset Y$. If $r = tu, t \in Z, u$ a unit of R, it follows that $uX = Y$.

Definition 6. A group G has minimal test for quasi-equivalence (*mtqe*) if whenever U and V are quasi-isomorphic pure subgroups of G then there exists ϕ, a quasi-automorphism of G, with $\phi U \doteq V$.

We have already noted that the group in Example 3 is homogeneous with *mtqe* but is not *Qqpi*. For a second example, if we eliminate the element s_1 from our construction of Example 4, then we obtain a subgroup $G' \subset G$ in which $\langle 1 \rangle \oplus \langle s_1 \rangle$ is pure. However, it is not too hard to show that G' remains strongly indecomposable and that $E(G')$ will coincide with $E(G)$, the ring of integral quaternions. Thus, G' is a strongly indecomposable homogeneous non-*Qqpi* group. It also is not too hard to show that G' has *mtqe*. Since the construction of this second example is not central to our work, we omit the details.

Note that the group G of Example 3 has $QE(G)$ a number field of degree 3. The group G in the modification of Example 4 would have $QE(G)$ a division algebra of degree 2. The next result, which we feel is somewhat surprising, shows that only division algebras of degree 2 or 3 can occur as $QE(G)$ for a strongly indecomposable homogeneous group G for which the *Qqpi* and *mtqe* properties fail to coincide.

Theorem 7. *Let G be irreducible and write $G \doteq H^n$ with H strongly indecomposable and irreducible. Suppose that $dim_Q F > 3$, where F is a maximal subfield of the division algebra $\Gamma = QE(H)$. Then G is Qqpi if and only if G has mtqe.*

Proof. First we assume that $n = 1$, that is, $H = G$ is strongly indecomposable. Since $\Gamma = QE(H)$, every $0 \neq r \in E(H)$ is a quasi-automorphism. Hence, if H is *Qqpi*, it follows immediately that H has *mtqe*. Conversely, suppose that H has *mtqe*. By Theorem 2 (a), to prove that H is *Qqpi* we need to show that any pure subgroup of H is strongly indecomposable. It will be enough to show that any rank two pure subgroup of H is strongly indecomposable.

Let K be a rank two pure subgroup of H; say $K = \langle x, y\rangle_*$, the pure subgroup generated by elements x, y. Since H is irreducible there exists $\gamma \in QE(H)$ with $\gamma x = y$. Extend the field $Q(\gamma)$ to a maximal subfield $F \subset QE(H)$.

Suppose that for each primitive element $\alpha \in E(H)$ with $F = Q(\alpha)$ the pure subgroup $\langle x, \alpha x\rangle_* \subset H$ is completely decomposable. By the proof of the existence of a primitive element for algebraic number fields, for α primitive we can choose a positive integer t such that $\beta = \alpha + t\alpha^2$ will also be primitive. Then the pure subgroups $\langle x, \alpha x\rangle_*$ and $\langle x, \beta x\rangle_*$ will be isomorphic (both being completely decomposable subgroups of the homogeneous group H). Because H has *mtqe* we have $\phi\langle x, \alpha x\rangle_* \doteq \langle x, \beta x\rangle_*$ for some $\phi \in \Gamma$. Thus $\phi(x) = (q_1 + q_2\beta)x$ for some rational numbers q_1, q_2. Since Γ is a division algebra, $\phi = (q_1 + q_2\beta)$. But $\phi(\alpha x) = (q_1 + q_2\beta)\alpha x$ cannot be a rational combination $q_3 x + q_4\beta x$, for then $(q_1 + q_2\beta)\alpha = q_3 + q_4\beta$ for some rationals q_3, q_4. Since $\beta = \alpha + t\alpha^2$, this latter equation would contradict the fact that α is algebraic over Q of degree greater than three. It follows that, for some primitive element $\alpha \in E(H)$, the pure subgroup $\langle x, \alpha x\rangle_*$ will be strongly indecomposable.

As before, the type of the rank one factor group $\langle x, \alpha x\rangle_* / \langle x\rangle_*$ must be greater than the type of H. Thus, one of two possibilities must occur.

Case I: There is an infinite set of primes P such that for $p \in P$ there exists an integer c_p with $h_p(\alpha x - c_p x) > h_p(x)$. Here h_p denotes the p-height of an element in H. In this case an easy calculation shows that, for every integer t with $1 \leq t <$ degree α and $p \in P$, we have $h(\alpha^t x - c_p^t x) > h_p(x)$. Let L be the pure subgroup of H generated by x and $\{\alpha^t x \mid 1 \leq t < \text{degree } \alpha\}$. Our set of height inequalities shows that the inner type of $L/\langle x\rangle_*$ is greater than the type of H. Since $y = \gamma x \in L$ and $K = \langle x, y\rangle_*$, then $K/\langle x\rangle_*$ is a pure subgroup of $L/\langle x\rangle_*$. Hence type $[K/\langle x\rangle_*] >$ type H, so that K cannot be homogeneous and completely decomposable. Thus K must be strongly indecomposable, as desired.

Case II: For some prime p with $pH \neq H$ there is a set of integers $c_n, 1 \leq n < \infty$, with $(\alpha x - c_n x) \in p^n H$. Arguing as in Case I, we can conclude that $L/\langle x\rangle_*$ is p-divisible; hence so is $K/\langle x\rangle_*$. Again we have that K cannot be homogeneous and completely decomposable, so K is strongly indecomposable. The proof that H is *Qqpi* is complete.

Now suppose that $n > 1$ and $G \doteq H^n$ is as in the statement of the theorem. If G has *mtqe* it is immediate that H has *mtqe*. By what we have proved already, H is *Qqpi* and, by Theorem 2 (b), G is *Qqpi*. Conversely, let G be *Qqpi*. Then H is also *Qqpi*, so, by the remark at the beginning of the proof, H has *mtqe*. In view of the fact that $G \doteq H^n$ is irreducible with $QE(G) \cong (\Gamma)_{n\times n}$, it is not hard to see that, for X,Y any rank one pure subgroups of G, we can choose a quasi-automorphism

$\phi \in QE(G)$ with $\phi X = Y$. With this observation, the proof of Theorem 2 (b) goes through *mutatis mutandis* to show that G has *mtqe*.

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